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IDEALWISE ALGEBRAIC INDEPENDENCE FOR ELEMENTS OF THE COMPLETION OF A LOCAL DOMAIN

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1. Introduction

Over the past forty years many examples in commutative algebra have been constructed using the following principle: Let k be a field, let $S = k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$ be a localized polynomial ring over k, and let $\mathbf{a}$ be an ideal in the completion $\widehat{S}$ of S such that the associated primes of $\mathbf{a}$ are in the generic formal fiber of S; that is, $\mathbf{p} \cap S = (0)$ for each $\mathbf{p} \in \operatorname{Ass}(\widehat{S}/\mathbf{a})$. Then S embeds in $\widehat{S}/\mathbf{a}$, the fraction field $Q(S)$ of S embeds in the fraction ring of $\widehat{S}/\mathbf{a}$, and for certain choices of $\mathbf{a}$, the intersection $D = Q(S) \cap (\widehat{S}/\mathbf{a})$ is a local Noetherian domain with completion $\widehat{D} = \widehat{S}/\mathbf{a}$.

Examples constructed by this method include Nagata's first examples of non-excellent rings [N], Ogoma's celebrated counterexample to Nagata's catenary conjecture [O1], [O2], examples of Rotthaus and Brodmann [R1], [R2], [BR1], [BR2], and examples of Nishimura and Weston [Ni], [W]. In fact all examples we know of local Noetherian reduced rings which contain and are of finite transcendence degree over a coefficient field may be realized using this principle.[1]

The key to these examples is usually the behavior of the formal fibers of the domain D. A major problem in this setting is to identify and classify ideals in the formal fiber of S according to the properties of the intersection domain $D = Q(S) \cap (\widehat{S}/\mathbf{a})$. The goal of this paper is to study the significance of the choice of the ideal $\mathbf{a}$ in this construction.

In many of the examples mentioned above, the expression $D = Q(S) \cap (\widehat{S}/\mathbf{a})$ may be interpreted so that D is an intersection of the completion of a local Noetherian domain R with a subfield. In this paper we consider this latter form. More precisely we use the following setting throughout this paper.

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[1] We conjecture that all local Noetherian reduced rings D which contain a coefficient field k and which are of finite transcendence degree over k relate to an ideal $\mathbf{a}$ in the generic formal fiber of the localization of a polynomial ring $S = k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$, in such a way that D is a direct limit of étale extensions of such an intersection ring $Q(S) \cap (\widehat{S}/\mathbf{a})$ as above.

Setting. Let $(R, \mathbf{m})$ be an excellent normal local domain with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}})$. Suppose that $\tau_1, \dots, \tau_n$ are elements of $\widehat{\mathbf{m}}$ which are algebraically independent over R and that $t_1, \dots, t_n$ are indeterminates over K. For $S = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}$ and L the fraction field of S, we consider in this paper intermediate rings of the form $A = L \cap \widehat{R}$. It is immediate that:

(1) The completion of $R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}$ is isomorphic to $\widehat{R}[[t_1, \dots, t_n]]$.
(2) For $\mathbf{a} = (t_1 - \tau_1, t_2 - \tau_2, \dots, t_n - \tau_n)$ in $\widehat{R}[[t_1, \dots, t_n]]$, $R \cap \mathbf{a} = 0$.
(3) $\widehat{R}[[t_1, \dots, t_n]]/\mathbf{a} \cong \widehat{R}$.

Thus, with $S_n = R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)}$ and $\mathbf{a} = (t_1 - \tau_1, t_2 - \tau_2, \dots, t_n - \tau_n)$ in $\widehat{R}[[t_1, \dots, t_n]]$, we have $Q(S_n) \cap (\widehat{S_n}/\mathbf{a}) = L \cap \widehat{R}$. That is, the expression from the above paragraphs now has the form $L \cap \widehat{R}$, where L is a field between K and the fraction field of $\widehat{R}$.

Before we proceed with our summary of this paper, we give questions, motivation and background information on the study of $L \cap \widehat{R}$, where L is a field between K and the fraction field of $\widehat{R}$.

Background. Suppose A is a local Noetherian intermediate ring dominating R (in the sense that the maximal ideal $\mathbf{n}$ of A intersects R in $\mathbf{m}$) and dominated by $\widehat{R}$. The local injective morphisms $R \hookrightarrow A \hookrightarrow \widehat{R}$ imply the existence of a canonical surjection $\pi\colon \widehat{A} \twoheadrightarrow \widehat{R}$, where $\widehat{A}$ is the $\mathbf{n}$-adic completion of A. In this setting it is well known that A is a topological subspace of $\widehat{R}$; i.e., π is an isomorphism, if and only if every ideal of A is closed in the topology on A defined by the powers of $\widehat{\mathbf{m}}$. Since A is assumed to be Noetherian, π an isomorphism implies $\widehat{R}$ is faithfully flat over A, and hence $a\widehat{R} \cap A = aA$ for each principal ideal aA of A, so $A = L \cap \widehat{R}$ where L is the field of fractions of A. Thus $\widehat{A} = \widehat{R}$ implies $A = L \cap \widehat{R}$ and there can be at most one Noetherian[2] A with $\widehat{A} = \widehat{R}$ for each intermediate field L between K and the fraction field $\widehat{K}$ of $\widehat{R}$.

On the other hand, if L is any intermediate field between K and $\widehat{K}$, then the ring $A = L \cap \widehat{R}$ is a quasilocal domain dominating R and dominated by $\widehat{R}$. It is easily seen that such a ring A is Hausdorff in the topology defined by the powers of its maximal ideal, and again the injective local morphisms $R \hookrightarrow A \hookrightarrow \widehat{R}$ imply the existence of a canonical surjection $\pi\colon \widehat{A} \twoheadrightarrow \widehat{R}$, where $\widehat{A}$ is the completion of A. This leads us to the question:

> What subdomains A of $\widehat{R}$ have the form $L \cap \widehat{R}$, where L is an intermediate field between K and $\widehat{K}$?

In considering this question, we have come to realize that it is quite broad, and that the explicit determination of $L \cap \widehat{R}$ is computationally challenging even for relatively

[2]Without the assumption that A is Noetherian there are examples where $\widehat{A} = \widehat{R}$ and A is non-Noetherian with the same fraction field as $\widehat{R}$, see for example B, [Chap. III, pages 119–120, Ex. 14].

simple examples of R and L. We have also discovered that for many excellent normal local domains R of dimension at least two, there exist intermediate fields L between K and $\widehat{K}$ such that $A = L \cap \widehat{R}$ fails to be a subspace of $\widehat{R}$. It can happen that $A = L \cap \widehat{R}$ is an excellent normal local domain of dimension greater than that of R, or even that A fails to be Noetherian. In order to exhibit such examples we concentrate in this paper on elements $a_1, \dots a_n \in \widehat{\mathfrak{m}}$ which are algebraically independent over K and satisfy certain additional independence conditions. We plan to continue the study of domains of the form $L \cap \widehat{R}$ in [HRW].

Here are some specific results related to the general question of the structure of $L \cap \widehat{R}$.

(1.1) For $a_1, \dots a_n$ arbitrary elements of $\widehat{R}$ and $L = K(a_1, \dots, a_n)$, the subdomain $A = L \cap \widehat{R}$ is a normal quasilocal birational extension of $R[a_1, \dots, a_n]$. If the a_i are algebraic over R, then the structure of A is well understood; A is an étale extension of R with completion $\widehat{A} = \widehat{R}$ (see [R4]). But if the a_i are not algebraic over R, the situation is more complicated and the structure of A depends on the residual behavior of the a_i modulo various prime ideals of $\widehat{R}$.

(1.2) For a specific example of an excellent normal local domain to illustrate these ideas, we refer to $R = k[x, y]_{(x,y)}$, the localization of the polynomial ring over a field k at the maximal ideal generated by the indeterminates x and y. For this example $\widehat{R} = k[[x, y]]$. A result of Valabrega [V, Proposition 3] implies that, for $R = k[x, y]_{(x,y)}$, if L is a field between K and the fraction field F of $k[x][[y]]$ or if L is a field between F and $\widehat{K}$, then $A = L \cap \widehat{R}$ is a two-dimensional regular local domain with completion $\widehat{R}$.

(1.3) On the other hand, again considering $R = k[x, y]_{(x,y)}$, it is well known that, for each positive integer n, the formal power series ring in n variables over k can be embedded in $\widehat{R} = k[[x, y]]$; in fact $k[[x, y]]$ contains infinitely many analytically independent elements [A], [AM], [AHW]. However, if a formal power series ring $\widehat{S_n}$ in n variables over k is embedded in $\widehat{R} = k[[x, y]]$, and if $R \subseteq \widehat{S_n}$ (so, in particular, if $\widehat{S_n} = L \cap \widehat{R}$ for some field L between K and $\widehat{K}$), then $\widehat{S_n} = \widehat{R}$, so that $n = 2$. More generally, if A is a local Noetherian ring with completion $\widehat{A}$ and B is a complete local ring such that B dominates A and $\widehat{A}$ dominates B, then a well-known theorem of Cohen (cf. [M1, (8.4)]) implies that $B = \widehat{A}$.

(1.4) An example of Nagata shows the existence of a 3-dimensional regular local domain D with completion $\widehat{D}$ a formal power series ring in 3 variables over a field k of characteristic $p > 0$ for which there exists an intermediate field L between the fraction fields of D and $\widehat{D}$ such that $A = \widehat{D} \cap L$ is non-Noetherian. In this example, D is not excellent and L is a finite purely inseparable extension of the fraction field of D. A discussion of this example is given on pages 31–32 of [HRS].

(1.5) An example of Ogoma shows the existence of a four-dimensional excellent regular local domain, indeed a domain D obtained as a localization of a polynomial ring in 4 variables over a countable field, for which there exists a field L contained in the fraction field of $\widehat{D}$ and generated by two elements over the fraction field of D

such that $A = \widehat{D} \cap L$ is not Noetherian. A discussion of this example is given on pages 32–34 of [HRS].

(1.6) In [HR], an excellent normal local domain R is said to have *the Noetherian intermediate rings property*, (NIR), if for each subfield L of $\widehat{K}$ containing K, the quasilocal ring $L \cap \widehat{R}$ is Noetherian and has completion $\widehat{R}$. Interesting examples of excellent regular local rings satisfying (NIR) are constructed in [R3] and by Shelburne in [S].[3]

(1.7) For a subfield L of $\widehat{K}$ containing K, the following construction considered in [HRS] is sometimes useful for obtaining information about $A = L \cap \widehat{R}$. Suppose $(S, \mathbf{n})$ is an excellent normal local domain dominating R and $\mathbf{q}$ is a prime ideal of $\widehat{S}$ such that $\widehat{S}/\mathbf{q} \cong \widehat{R}$ and $\mathbf{q} \cap S = (0)$. Then L, the fraction field of S, embeds in the fraction field of $\widehat{S}/\mathbf{q}$ and with this identification, $A = L \cap (\widehat{S}/\mathbf{q}) = L \cap \widehat{R}$ is a quasilocal domain birationally dominating S. In §3 we present examples of this type where $A = S$.

(1.8) Recent work of Heitmann in [H1] and [H2] and Loepp in [L] shows the richness of the structure of the local domains with a given completion. In [H1], Heitmann shows that a complete local ring $\widehat{T}$ is the completion of a local unique factorization domain (UFD) if (i) $\widehat{T}$ has depth at least 2, and (ii) no nonzero element of the prime subring of $\widehat{T}$ is a zero divisor of $\widehat{T}$. In [H2], Heitmann proves that very often a complete local ring is the completion of a local ring having an isolated singularity. In particular, for $\widehat{T}$ satisfying (i) and (ii) he shows the existence of a local UFD all of whose proper localizations are regular that has completion $\widehat{T}$. His construction is adopted by his student Loepp to obtain more examples of strange phenomena which can occur in passing from a local (Noetherian) ring to its completion. A central step in Heitmann's construction involves passing from a subring D of $\widehat{T}$ to a bigger ring D' by adjoining a kind of independent element, similar to the residually algebraically independent elements defined below and studied in §4 of this article. These residually algebraically independent elements play an important role in his construction. Certain relations from $\widehat{T}$ become satisfied in D' (defined as a limit), but by using the residually algebraically independent elements, he is able to control the correspondence between the height-one prime ideals of D and those of D'.

We now summarize the results of the present paper.

Summary of this paper. In this paper we consider three concepts of independence over R for elements $\tau_1, \dots, \tau_n$ of $\widehat{\mathbf{m}}$ which are algebraically independent over K (as in the setting above). We relate these three concepts of independence to flatness conditions of extensions of Krull domains, establish implications among them, and draw some conclusions concerning their existence and equivalence in special situations.

[3]We remark that Shelburne in [S] has provided examples answering Question 2.8 of [HR]. He shows existence for each positive integer $d \geq 3$ of an excellent local domain R containing a field of characteristic $p > 0$ such that $\dim(R) = d$, the dimension of the generic formal fiber of R is 0, and R is properly contained in its completion $\widehat{R}$. In his examples, $\widehat{R}$ has, in fact, infinite transcendence degree over R.

We also investigate their stability under change of base ring.

We begin our analysis in §2 with the definition of the first independence condition: the elements $\tau_1, \dots, \tau_n$ are *idealwise independent* if $K(\tau_1, \dots, \tau_n) \cap \hat{R}$ equals the localized polynomial ring $R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}$. We observe that $\tau_1, \dots, \tau_n \in \mathbf{m}\widehat{R}$ are idealwise independent over R if and only if the extension $R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ is weakly flat in the sense of the definition given in §2. We also show in §2 that a sufficient condition for $\tau_1, \dots, \tau_n$ to be idealwise independent over R is that the extension $R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE ("pas d'éclatement", or in English "no blowing up"). At the end of §2 we display in a schematic diagram the relationships between these concepts and some others, for extensions of Krull domains.

In §3 and §4 we present two methods for obtaining idealwise independent elements over a countable ring R. The method in §3 is to find elements $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$, the maximal ideal of $\widehat{R}$, so that (1) $\tau_1, \dots, \tau_n$ are algebraically independent over the fraction field of R, and (2) for every prime ideal P of $S = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}$ with $\dim(S/P) = n$, the ideal $P\widehat{R}$ is $\widehat{\mathbf{m}}$-primary. If (1) and (2) hold, we say that $\tau_1, \dots, \tau_n$ are *primarily independent* over R; we show in (3.4) that primarily independent elements are idealwise independent. If R is countable and $\dim(R) > 2$, we show in (4.5) the existence over R of idealwise independent elements that fail to be primarily independent.

For every countable excellent normal local domain R of dimension at least two, we prove in Theorem 3.9 the existence of an infinite sequence $\tau_1, \tau_2, \dots$ of elements of $\widehat{R}$ which are primarily independent over R. It follows that $A = K(\tau_1, \tau_2, \dots) \cap \widehat{R}$ is an infinite-dimensional non-Noetherian quasilocal domain. Thus, for the example $R = k[x, y]_{(x,y)}$ with k a countable field, and for every positive integer n or $n = \infty$, there exists an extension $A_n = L_n \cap \widehat{R}$ of R such that $\dim(A_n) = \dim(R) + n$. In particular, the canonical surjection $\widehat{A_n} \to \widehat{R}$ has a nonzero kernel.

In §4 we define $\tau \in \mathbf{m}\widehat{R}$ to be *residually algebraically independent* over R if τ is algebraically independent over the fraction field of R and for each height-one prime ideal P of $\widehat{R}$ such that $P \cap R \neq 0$, the image of τ in $\widehat{R}/P$ is algebraically independent over $R/(P \cap R)$. We extend the concept of residual algebraic independence to a finite or infinite number of elements $\tau_1, \dots, \tau_n \in \mathbf{m}\widehat{R}$ and observe the equivalence of residual algebraic independence to the extension $R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ satisfying PDE.

We show that primary independence $\Longrightarrow$ residual algebraic independence $\Longrightarrow$ idealwise independence. For R of dimension two, we show that primary independence is equivalent to residual algebraic independence, but as remarked above, if R has dimension greater than two, then primary independence is stronger than residual algebraic independence. We show in (4.7) and (4.9) the existence of idealwise independent elements that fail to be residually algebraically independent.

In §5 we describe the three concepts of idealwise independence, residual algebraic independence, and primary independence in terms of certain flatness conditions on the embedding ϕ: $R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \hookrightarrow \widehat{R}$. In §6 we investigate the stability of these independence concepts under base change, composition and polynomial extension. We prove in (6.10) the existence of uncountable excellent

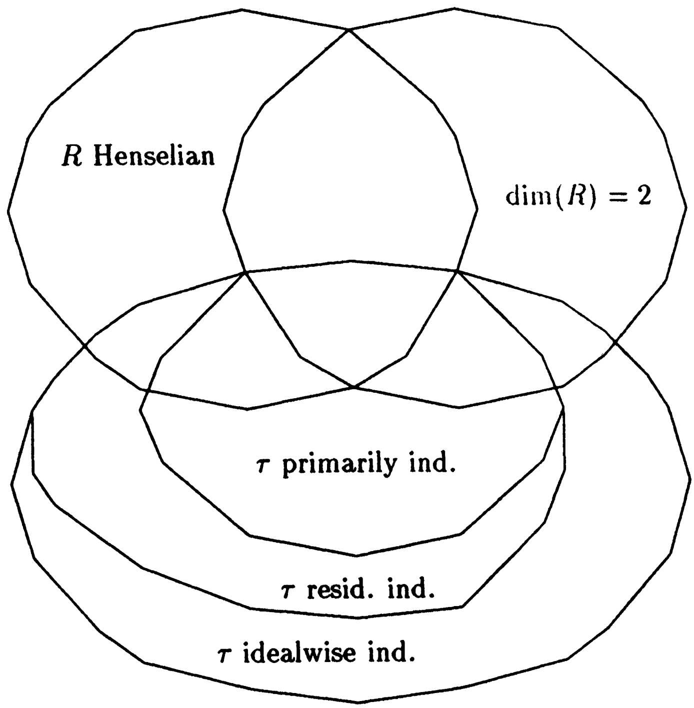


Figure 1

normal local domains R such that $\widehat{R}$ contains infinite sets of primarily independent elements.

We show in §7 that both residual algebraic independence and primary independence hold for elements over the original ring R exactly when they hold over the Henselization R^h of R (7.2). Also idealwise independence descends from the Henselization to the ring R. If R is Henselian of dimension two, then all three concepts of independence are equivalent for one element $\tau \in \widehat{\mathbf{m}}$ (Corollary 7.6).

Fig. 1 summarizes some relationships between the independence concepts for one element τ of $\widehat{\mathbf{m}}$, over a local normal excellent domain $(R, \mathbf{m})$. In the diagram we use "ind." and "resid." to abbreviate "independent" and "residually algebraic".

In §8 we include a diagram which displays many more relationships among the independence concepts and other related properties.

2. Idealwise independence, weakly flat and PDE extensions

First we describe the setting of the idealwise independent concept and we establish notation to be used throughout the paper.

2.1. *Setting and notation.* Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension d with field of fractions K and completion $(\widehat{R}, \widehat{\mathbf{m}} = \mathbf{m}\widehat{R})$, and let $t_1, \dots, t_n, \dots$ be indeterminates over R. Suppose that $\tau_1 = \tau, \tau_2, \dots, \tau_n, \dots \in \widehat{\mathbf{m}}$ are algebraically independent over K. For each $n \geq 0$, we consider the following localized polynomial rings:

$$\begin{aligned} S_n &= R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)}, \\ R_n &= R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}, \\ S_\infty &= R[t_1, \dots, t_n, \dots]_{(\mathbf{m}, t_1, \dots, t_n, \dots)} \text{ and} \\ R_\infty &= R[\tau_1, \dots, \tau_n, \dots]_{(\mathbf{m}, \tau_1, \dots, \tau_n, \dots)}. \end{aligned}$$

Of course, S_n is R-isomorphic to R_n and S_∞ is R-isomorphic to R_∞ with respect to the R-algebra homomorphism taking $t_i \to \tau_i$ for each i. When working with a particular n or ∞, we sometimes define S to be R_n or R_∞. If $n = 0$, take $R_n = R = S_n$.

The completion $\widehat{S_n}$ of S_n is $\widehat{R}[[t_1, \dots, t_n]]$, and we have the following commutative diagram:

$$\begin{array}{ccccc} & & S_n = R[t_1, \dots, t_n]_{(\mathbf{m}, t_1, \dots, t_n)} & \xrightarrow{\subseteq} & \widehat{S_n} = \widehat{R}[[t_1, \dots, t_n]] \\ & & \cong \downarrow & & \lambda \downarrow \\ R & \xrightarrow{\subseteq} & S = R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} & \xrightarrow{\varphi} & \widehat{R}. \end{array}$$

Here the first vertical isomorphism is the R-algebra map taking $t_i \to \tau_i$, the restriction of the R-algebra surjection λ: $\widehat{S_n} \to \widehat{R}$ where $\ker(\lambda) = (t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S_n} = \widehat{p}$; note that $\widehat{p} \cap S_n = (0)$.

The central definition of this paper is the following:

2.2. *Definition.* Let $(R, \mathbf{m})$ and $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ be as in the setting of (2.1). We say that $\tau_1, \dots, \tau_n$ are *idealwise independent over* R provided $\widehat{R} \cap K(\tau_1, \dots, \tau_n) = R_n$. Similarly, an infinite sequence $\{\tau_i\}_{i=1}^\infty$ of algebraically independent elements of $\mathbf{m}\widehat{R}$ is *idealwise independent over* R if $\widehat{R} \cap K(\{\tau_i\}_{i=1}^\infty) = R_\infty$.

2.3. *Remarks.* (1) A subset of an idealwise independent set $\{\tau_1, \dots, \tau_n\}$ over R is also idealwise independent over R. For example, to see that $\tau_1, \dots, \tau_m$ are idealwise independent over R for $m \leq n$, let K denote the quotient field of R and observe that

$$\begin{aligned} \widehat{R} \cap K(\tau_1, \dots, \tau_m) &= \widehat{R} \cap K(\tau_1, \dots, \tau_n) \cap K(\tau_1, \dots, \tau_m) \\ &= R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \cap K(\tau_1, \dots, \tau_m) = R[\tau_1, \dots, \tau_m]_{(\mathbf{m}, \tau_1, \dots, \tau_m)}. \end{aligned}$$

(2) Idealwise independence is a strong property of the elements $\tau_1, \dots, \tau_n$ and of the embedding morphism φ: $R_n \hookrightarrow \widehat{R}$. As we stated in the introduction, it is often difficult to compute $\widehat{R} \cap L$ when L is an intermediate field between the quotient fields

of R and $\widehat{R}$. In order for $\widehat{R} \cap L$ to be an intermediate localized polynomial ring R_n, there can be no new quotients in $\widehat{R}$ other than those in $\varphi(R_n)$; that is, if $f/g \in \widehat{R}$ and $f, g \in R_n$, then $f/g \in R_n$. This does not happen, for example, if one of the τ_i is in the completion of R with respect to a principal ideal; in particular, if $\dim(R) = 1$, then there do not exist idealwise independent elements over R.

The following example illustrates Remark 2.3.2:

2.4 *Example.* Let $R = \mathbb{Q}[x, y]_{(x,y)}$, the localized ring of polynomials in two variables over the rational numbers. The elements $\tau_1 = e^x - 1$, $\tau_2 = e^y - 1$, and $\rho = e^x - e^y = \tau_1 - \tau_2$ of $\widehat{R} = \mathbb{Q}[[x, y]]$ belong to completions of R with respect to principal ideals (and so are not idealwise independent). If $S = R_2 = \mathbb{Q}[x, y, \tau_1, \tau_2]_{(x,y,\tau_1,\tau_2)}$ and L is the quotient field of S, then the elements $(e^x - 1)/x$, $(e^y - 1)/y$, and $(e^x - e^y)/(x - y)$ are certainly in $L \cap \widehat{R}$ but not in S. A result of Valabrega [V, Proposition 3] implies that $L \cap \widehat{R}$ is a two-dimensional regular local ring with completion $\widehat{R}$.

In the remainder of this section we discuss some properties of extensions of Krull domains related to idealwise independence. (A diagram near the end of this section displays the relationships among these properties.) We start by defining a property which we prove in (2.7) is satisfied by the extension $\phi\colon R_n \longrightarrow \widehat{R}$:

2.5. *Definition.* Let $A \hookrightarrow B$ be an extension of Krull domains. We say that B is a *height-one preserving* extension of A if for every height-one prime ideal P of A with $PB \neq B$ there exists a height-one prime ideal Q of B with $PB \subseteq Q$.

2.6. *Remark.* If $A \hookrightarrow B$ is an extension of Krull domains, and if A is factorial, or more generally, if every height-one prime ideal of A is the radical of a principal ideal, then B is a height-one preserving extension of A. This is clear from the fact that every minimal prime divisor of a principal ideal in a Krull domain is of height one.

2.7. PROPOSITION. *Let $(R, \mathbf{m})$ and $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ be as in the setting of* (2.1). *Then the embedding*

$$\phi\colon R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)} \longrightarrow \widehat{R}$$

is a height-one preserving extension.

Proof. Consider the commutative diagram of (2.1):

$$\begin{array}{ccccc} & & S_n = R[t_1, \dots, t_n]_{(\mathbf{m},t_1,\dots,t_n)} & \xrightarrow{\subseteq} & \widehat{S_n} = \widehat{R}[[t_1, \dots, t_n]] \\ & & \cong\big\downarrow & & \lambda\big\downarrow \\ R & \xrightarrow{\subseteq} & S = R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)} & \xrightarrow{\varphi} & \widehat{R}. \end{array}$$

Let $P \subseteq R_n$ be a prime ideal of height one. Under the above isomorphism of R_n with S_n, P correponds to a height-one prime ideal P_0 of S_n. The extended ideal $P_0\widehat{S_n}$ is reduced and each of its minimal prime divisors is of height one.

The minimal prime divisors $\widehat{Q}$ in $\widehat{R}$ of the ideal $P\widehat{R}$ are in 1-1 correspondence with the minimal prime divisors $\widehat{Q_0}$ in $\widehat{S_n}$ of the ideal

$$\widehat{J} = (P_0, t_1 - \tau_1, \ldots, t_n - \tau_n)\widehat{S_n} = (P_0, \widehat{p})\widehat{S_n}$$

via $\widehat{Q_0}/\widehat{p} = \widehat{Q}$. Since $\widehat{p} = \ker(\lambda)$ is a prime ideal of height n with $\widehat{p} \cap S_n = (0)$, each $\widehat{Q_0}$ is of height $n+1$ and each $\widehat{Q}$ is of height one. Therefore $R_n \to \widehat{R}$ is a height-one preserving extension. □

The concept of idealwise independence is naturally related to other ideal-theoretic properties. If $A \hookrightarrow B$ is an extension of domains and F is the fraction field of A, then it is well known and easily seen that $A = B \cap F \iff$ each principal ideal of A is contracted from B. For an extension $A \hookrightarrow B$ of Krull domains, the condition that $A = B \cap F$, where F is the fraction field of A, is related to the following concepts.

2.8. *Definition.* Let $A \hookrightarrow B$ be an extension of Krull domains.

(a) We say that B is *weakly flat* over A if every height-one prime ideal P of A with $PB \neq B$ satisfies $PB \cap A = P$.

(b) The extension $A \hookrightarrow B$ is said to satisfy PDE ("pas d'éclatement", or "no blowing up") if for every height-one prime ideal Q in B, the height of $Q \cap A$ is at most one (cf. [F, page 30]).

2.9. *Remarks.* Let $A \hookrightarrow B$ be an extension of Krull domains and let F be the fraction field of A.

(a) We have $B \cap F = A \iff$ each height-one prime of A is the contraction of a height-one prime of B. If this holds, then B is height-one preserving and weakly flat over A (cf. [N, (33.5) and (33.6)]).

(b) If $A \hookrightarrow B$ is flat, then $A \hookrightarrow B$ is height-one preserving, weakly flat and satisfies PDE (cf. [B, Chapitre 7, Proposition 15, page 19]).

(c) If S is a multiplicative system in A consisting of units of B, then $A \hookrightarrow B$ is height-one preserving (respectively weakly flat, respectively satisfies PDE) $\iff$ $S^{-1}A \hookrightarrow B$ is height-one preserving (respectively weakly flat, respectively satisfies PDE).

2.10. PROPOSITION. *If ϕ: $A \hookrightarrow B$ is a weakly flat extension of Krull domains, then ϕ is height-one preserving. Moreover, for every height-one prime ideal P of A with $PB \neq B$ there is a height-one prime ideal Q_0 of B with $Q_0 \cap A = P$.*

Proof. Let $P \in \operatorname{Spec}(A)$ with $\operatorname{ht}(P) = 1$. By assumption $PB \cap A = P$. Therefore the ideal PB of B is contained in an ideal Q of B that is maximal with respect to not meeting the multiplicative system $A - P$. It follows that Q is a prime ideal of B and $Q \cap A = P$. Let $a \in P - (0)$ and let $Q_0 \subseteq Q$ be a minimal prime divisor of aB. Then Q_0 has height one and $(0) \neq Q_0 \cap A \subseteq P$; thus $Q_0 \cap A = P$. □

2.11. PROPOSITION. *Let $A \hookrightarrow B$ be an extension of Krull domains which is height-one preserving and satisfies PDE. Then B is weakly flat over A.*

Proof. Let $P \in \text{Spec}(A)$ with $\text{ht}(P) = 1$. Then PB is contained in a prime ideal Q of B of height one. The PDE hypothesis on $A \hookrightarrow B$ implies that $Q \cap A$ has height one. It follows that $Q \cap A = P$ and thus $PB \cap A = P$. □

2.12. COROLLARY. *Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \mathbf{m}\widehat{R}$ be as in the setting of (2.1). Let $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$. If $S \hookrightarrow \widehat{R}$ satisfies PDE, then $\widehat{R}$ is weakly flat over S.*

Proof. This is immediate from (2.7) and (2.11). □

2.13. *Example.* Without assuming that the extension is height-one preserving, it can happen that an extension $A \hookrightarrow B$ of Krull domains satisfies PDE and yet B fails to be weakly flat over A. This is the case, for example, if $A = k[x, y, z, w] = k[X, Y, Z, W]/(XY - ZW)$, where k is a field and X, Y, Z, W are indeterminates over k, and $B = A[x/z]$. Since $x/z = w/y$, $B = k[y, z, x/z]$ is a polynomial ring in three variables over k and the height-one prime ideal $P = (y, z)A$ extends in B to a prime ideal of height two. Another way to describe this example is to let r, s, t be indeterminates over a field k, and let $A = k[r, s, rt, st] \subset k[r, s, t] = B$. Then $A \hookrightarrow B$ satisfies PDE since B is an intersection of localizations of A, but $P = (r, s)A$ is a height-one prime of A such that PB is a height-two prime of B, so B is not weakly flat over A.

2.14. PROPOSITION. *Let $A \hookrightarrow B$ be an extension of Krull domains with $PB \neq B$ for every height-one prime ideal P of A and let F denote the fraction field of A. Then B is weakly flat over $A \iff A = F \cap B$. Moreover, in this setting, these equivalent conditions imply that $A \hookrightarrow B$ is height-one preserving.*

Proof. The assertion that $A = F \cap B$ implies B is weakly flat over A is Remark (2.9)(a). A direct proof of this assertion involving primary decomposition of principal ideals goes as follows: Let P be a height-one prime ideal of A, let $a \in P - (0)$, and consider an irredundent primary decomposition

$$aB = Q_1 \cap \cdots \cap Q_s$$

of the principal ideal aB in the Krull domain B. Since B is a Krull domain, each Q_i is primary for a height-one prime ideal P_i of B. The fact that $A = F \cap B$ implies

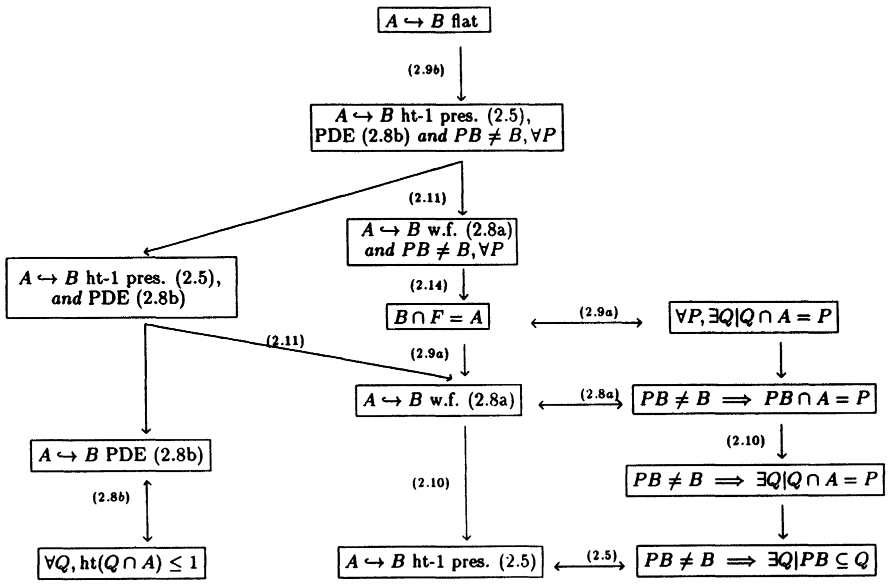


Figure 2. The relationships between properties for extensions of Krull domains

that $aA = F \cap aB$. Thus, after renumbering, there is an integer $t \in \{1, \dots, s\}$ such that the ideal

$$Q_1 \cap \cdots \cap Q_t \cap A$$

is the P-primary component of the ideal aA. Hence for at least one integer $i \in \{1, \dots, t\}$ we must have $P_i \cap A = P$. Therefore B is weakly flat over A.

Conversely, if B is weakly flat over A, then since $PB \neq B$, we have $PB \cap A = P$ for each height-one prime ideal P of A. It follows that $A \hookrightarrow D = F \cap B$ and $PD \cap A = P$, so $A_P = (A - P)^{-1}D$, and $D \subseteq A_P$ for each height-one prime ideal P of A. Since $A = \bigcap\{A_P\colon P \text{ is a height-one prime of } A\}$, we have $A = D$.

The last assertion follows by (2.9a) or (2.10). □

Let $A \hookrightarrow B$ be an extension of Krull domains, F the quotient field of A, $Q \in \operatorname{Spec}(B)$, $\operatorname{ht}(Q) = 1$, $P \in \operatorname{Spec}(A)$, $\operatorname{ht}(P) = 1$. Fig. 2 illustrates (2.5)–(2.14):

2.15. *Remark.* The condition in (2.14) that $PB \neq B$ for all height-one prime ideals P of A holds if $A \hookrightarrow B$ are quasilocal Krull domains with B dominating A, and so it holds for $R_n \hookrightarrow \widehat{R}$ as in (2.1).

Summarizing from (2.12) and (2.14), we have the following implications among the concepts of weakly flat, PDE and idealwise independence in the setting of (2.1):

2.16. THEOREM. *Let* $(R, \mathbf{m})$ *and* $\tau_1, \ldots, \tau_n \in \mathbf{m}\widehat{R}$ *be as in the setting of* (2.1). *Then*:

(1) $\tau_1, \ldots, \tau_n$ *are idealwise independent over* $R \iff R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ *is weakly flat.*
(2) $R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ *satisfies PDE* $\Longrightarrow R[\tau_1, \ldots, \tau_n] \hookrightarrow \widehat{R}$ *is weakly flat.*

Moreover, in view of part (c) *of* (2.9), *these assertions also hold with* $R[\tau_1, \ldots, \tau_n]$ *replaced by its localization* $R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$.

In order to demonstrate idealwise independence we develop in the next two sections the concepts of primary independence and residual algebraic independence, each of which implies idealwise independence.

3. Primary independence

In this section we introduce primary independence, a concept we show to be stronger than idealwise independence (in (3.4) and (4.5)). We construct infinitely many primarily independent elements over any countable excellent normal local domain of dimension at least two (in (3.9)).

3.1. *Definition.* Let $(R, \mathbf{m})$ be an excellent normal local domain. We say that $\tau_1, \ldots, \tau_n \in \mathbf{m}\widehat{R}$, which are algebraically independent over the fraction field of R, are *primarily independent over* R, provided that, for every prime ideal P of $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ such that $\dim(S/P) \leq n$, the ideal $P\widehat{R}$ is $\mathbf{m}\widehat{R}$-primary. A countably infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of elements of $\mathbf{m}\widehat{R}$ is *primarily independent over* R if, for each n, $\tau_1, \ldots, \tau_n$ are primarily independent over R.

3.2. *Remarks.* (1) Referring to the diagram, notation and setting of (2.1), primary independence of $\tau_1, \ldots, \tau_n$ as defined in (3.1) is equivalent to the statement that for every prime ideal P of S with $\dim(S/P) \leq n$, the ideal $\lambda^{-1}(P\widehat{R}) = P\widehat{S_n} + \ker(\lambda)$ is primary for the maximal ideal of $\widehat{S_n}$.

(2) A subset of a primarily independent set is again primarily independent. For example, if $\tau_1, \ldots, \tau_n$ are primarily independent over R, to see that $\tau_1, \ldots, \tau_{n-1}$ are primarily independent, let P be a prime ideal of R_{n-1} with $\dim(R_{n-1}/P) \leq n - 1$. Then PR_n is a prime ideal of R_n with $\dim(R_n/PR_n) \leq n$, and so $P\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$.

3.3. LEMMA. *Let* $(R, \mathbf{m})$ *be an excellent normal local domain of dimension at least* 2, *let* n *be a positive integer, and let* $S = R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$, *where* $\tau_1, \ldots, \tau_n$ *are primarily independent over* R. *Let* P *be a prime ideal of* S *such that* $\dim(S/P) \geq n + 1$. *Then* (1) $P\widehat{R}$ *is not* $\mathbf{m}\widehat{R}$*-primary, and* (2) $P\widehat{R} \cap S = P$.

Proof. For the first statement, suppose that $\dim(S/P) \geq n+1$ and that $P\widehat{R}$ is primary for $\mathbf{m}\widehat{R}$. Then, referring to the diagram in (2.1), $\lambda^{-1}(P\widehat{R}) = P\widehat{S_n} + \ker(\lambda)$ is primary for the maximal ideal of $\widehat{S}$, and hence the maximal ideal of $\widehat{S}/P\widehat{S}$ is the radical of an n-generated ideal, a contradiction because $\widehat{S_n}/P\widehat{S_n} \cong \widehat{(S/P)}$ is the completion of S/P, and $\dim(S/P) \geq n+1$ implies that $\dim(\widehat{S/P}) \geq n+1$.

For the second assertion, note that if $\dim(S/P) = n+1$, and $P < (P\widehat{R} \cap S)$, then $\dim(S/(P\widehat{R} \cap S)) \leq n$, which implies that $P\widehat{R} = (P\widehat{R} \cap S)\widehat{R}$ is primary for $\mathbf{m}\widehat{R}$, a contradiction to the first assertion of the lemma. Thus we have $P\widehat{R} \cap S = P$ for each P such that $\dim(S/P) = n+1$.

If $\dim(S/P) > n+1$, then P is an intersection of prime ideals P' of S such that $\dim(S/P') = n+1$, say $P = \cap_{P' \in \mathcal{I}} P'$. Using the result for P', we have

$$P \subseteq P\widehat{R} \cap S = (\cap_{P' \in \mathcal{I}} P')\widehat{R} \cap S \subseteq \cap_{P' \in \mathcal{I}}(P'\widehat{R} \cap S) = \cap_{P' \in \mathcal{I}} P' = P. \qquad \square$$

3.4. PROPOSITION. *Let $(R, \mathbf{m})$ be an excellent normal local domain of dimension at least 2, let n be a positive integer, and let $S = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$, where $\tau_1, \ldots, \tau_n$ are primarily independent over R. Then $S = L \cap \widehat{R}$, where L is the fraction field of S. Thus $\tau_1, \ldots, \tau_n$ are idealwise independent elements of $\widehat{R}$ over R. If $\{\tau_i\}_{i=1}^{\infty}$ is a countably infinite sequence of primarily independent elements of $\mathbf{m}\widehat{R}$ over R, then $\{\tau_i\}_{i=1}^{\infty}$ are idealwise independent over R.*

Proof. Let P be a height-one prime of S. Since S is catenary, $\dim(S/P) \geq n+1$. By (3.3.2), $P\widehat{R} \cap S = P$. Therefore $\widehat{R}$ is weakly flat over S and by (2.16) we have $S = L \cap \widehat{R}$. $\square$

3.5. PROPOSITION. *Let $(R, \mathbf{m})$ and $\tau_1, \ldots, \tau_n \in \mathbf{m}\widehat{R}$ be as in (2.1). Let $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \cong S_n = R[t_1, \ldots, t_n]_{(\mathbf{m}, t_1, \ldots, t_n)}$, where $t_1, \ldots, t_n$ are indeterminates over R. Then $\tau_1, \ldots, \tau_n$ are primarily independent over R if and only if one of the equivalent statements (1), (2) or (3) holds:*

1. *For each prime ideal P of S_n such that $\dim(S_n/P) \geq n$ and each prime ideal $\widehat{P}$ of $\widehat{S_n}$ minimal over $P\widehat{S_n}$, the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/\widehat{P}$ generate an ideal of height n in $\widehat{S_n}/\widehat{P}$.*
2. *For each prime ideal P of S_n with $\dim(S_n/P) \geq n$ and each nonnegative integer $i \leq n$, every prime ideal $\widehat{Q}$ of $\widehat{S_n}$ minimal over $(P, t_1 - \tau_1, \ldots, t_{i-1} - \tau_{i-1})\widehat{S_n}$ fails to contain $t_i - \tau_i$.*
3. *For each ideal P of S_n such that $\dim(S_n/P) = n$, the images of $t_1 - \tau_1, \ldots, t_n - \tau_n$ in $\widehat{S_n}/P\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S_n}/P\widehat{S_n}$.*

Proof. It is clear that (1) and (2) are equivalent, that (1) and (2) imply (3) and that (3) is equivalent to the primary independence of $\tau_1, \ldots, \tau_n$ over R. It remains to observe that (3) implies (1). For this, let P be a prime ideal of S_n such that $\dim(S_n/P) = n + h$, where $h \geq 0$. There exist $s_1, \ldots, s_h \in S_n$ such that if

$I = (P, s_1, \dots, s_h)S_n$, then for each minimal prime Q of I we have $\dim(S_n/Q) = n$. Item (3) implies that the images of $t_1 - \tau_1, \dots, t_n - \tau_n$ in $\widehat{S_n}/Q\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S_n}/Q\widehat{S_n}$. It follows that the images of $t_1 - \tau_1, \dots, t_n - \tau_n$ in $\widehat{S_n}/I\widehat{S_n}$ generate an ideal primary for the maximal ideal of $\widehat{S_n}/I\widehat{S_n}$, and therefore that the images of $s_1, \dots, s_h, t_1 - \tau_1, \dots, t_n - \tau_n$ in $\widehat{S_n}/P\widehat{S_n}$ are a system of parameters for the $(n+h)$-dimensional local ring $\widehat{S_n}/P\widehat{S_n}$. Let $\widehat{P}$ be a minimal prime of $P\widehat{S_n}$. Then $\dim(\widehat{S_n}/\widehat{P}) = n + h$, and the images of $s_1, \dots, s_h, t_1 - \tau_1, \dots, t_n - \tau_n$ in the complete local domain $\widehat{S_n}/\widehat{P}$ are a system of parameters. It follows that the images of $t_1 - \tau_1, \dots, t_n - \tau_n$ in $\widehat{S_n}/\widehat{P}$ generate an ideal of height n in $\widehat{S_n}/\widehat{P}$. □

3.6. COROLLARY. *With the notation of* (2.1) *and* (3.5) *assume that* $\tau_1, \dots, \tau_n$ *are primarily independent over* R.

(1) *Let* I *be an ideal of* S_n *such that* $\dim(S/I) = n$. *Then the ideal* $(I, t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S_n}$ *is primary to the maximal ideal of* S_n.
(2) *Let* $P \in \operatorname{Spec}(S_n)$ *be a prime ideal with* $\dim(S_n/P) > n$. *Then the ideal* $\widehat{W} = (P, t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S_n}$ *has* $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$ *and* $\widehat{W} \cap S_n = P$.

Proof. Part (1) is an immediate corollary of (3.5.3) and it follows from (3.5.1) that $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$. Let λ_n be the restriction to S_n of the canonical homomorphism $\lambda\colon \widehat{S_n} \to \widehat{R}$ from (2.1) so that $\lambda_n\colon S_n \xrightarrow{\cong} R_n$. Then $\dim(R_n/\lambda_n(P)) > n$, and so by (3.3.2), $\lambda_n(P)\widehat{R} \cap R_n = \lambda_n(P)$. Now

$$\begin{aligned}\widehat{W} \cap S_n &= \lambda^{-1}(\lambda_n(P)\widehat{R}) \cap \lambda_n^{-1}(R_n)\\ &= \lambda_n^{-1}(\lambda_n(P)\widehat{R} \cap R_n) = \lambda_n^{-1}(\lambda_n(P)) = P.\end{aligned}$$ □

To prove the existence of primarily independent elements, we use the following prime avoidence lemma over a complete local ring (cf. [Bu, Lemma 3], [WW, Lemma 10]). We also use this result in two constructions given in Section 4.

3.7. LEMMA. *Let* $(T, \mathbf{n})$ *be a complete local ring of dimension at least* 2, *and let* $t \in \mathbf{n} - \mathbf{n}^2$. *Assume that* I *is an ideal of* T *containing* t, *and that* $\mathcal{U}$ *is a countable set of prime ideals of* T *each of which fails to contain* I. *Then there exists an element* $a \in I \cap \mathbf{n}^2$ *such that* $t - a \notin \bigcup\{Q\colon Q \in \mathcal{U}\}$.

Proof. Let $\{P_i\}_{i=1}^{\infty}$ be an enumeration of the prime ideals of $\mathcal{U}$. We may assume that there are no containment relations between the primes of $\mathcal{U}$. Choose $f_1 \in \mathbf{n}^2 \cap I$ so that $t - f_1 \notin P_1$. Then choose $f_2 \in P_1 \cap \mathbf{n}^3 \cap I$ so that $t - f_1 - f_2 \notin P_2$. Note that $f_2 \in P_1$ implies $t - f_1 - f_2 \notin P_1$. Successively, by induction, choose

$$f_n \in P_1 \cap P_2 \cap \cdots \cap P_{n-1} \cap \mathbf{n}^{n+1} \cap I$$

so that $t - f_1 - \cdots - f_n \notin \bigcup_{i=1}^{n} P_i$ for each positive integer n. Then $\{f_1 + \cdots + f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in T which converges to an element $a \in \mathbf{n}^2$. Now

$$t - a = (t - f_1 - \cdots - f_n) + (f_{n+1} + \cdots),$$

where

$$(t - f_1 - \cdots - f_n) \notin P_n, (f_{n+1} + \cdots) \in P_n.$$

Therefore $t - a \notin P_n$, for all n, and $t - a \in I$. □

3.8. *Remark.* Let $A \hookrightarrow B$ be an extension of Krull domains. If α is a nonzero nonunit of B and α is outside every height-one prime Q of B such that $Q \cap A \neq (0)$, then $\alpha B \cap A = (0)$. In particular, such an element α is algebraically independent over the fraction field of A.

3.9. THEOREM. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension at least* 2. *Then*:

(1) *There exists $\tau \in \mathbf{m}\widehat{R}$ which is primarily independent over R.*
(2) *If $\tau_1, \ldots, \tau_{n-1} \in \mathbf{m}\widehat{R}$ are primarily independent over R, then there exists $\tau_n \in \mathbf{m}\widehat{R}$ such that $\tau_1, \ldots, \tau_{n-1}, \tau_n$ are primarily independent over R.*
(3) *Thus there exists an infinite sequence $\tau_1, \ldots, \tau_n, \ldots \in \mathbf{m}\widehat{R}$ of elements which are primarily independent over R.*

Proof. The proof for part (2) also establishes part (1) and part (3). To prove (2), let $t_1, \ldots, t_n$ be indeterminates over R, and let the notation be as in the setting of (2.1). Thus we have $S_{n-1} \cong R_{n-1}$, under the R-algebra isomorphism taking $t_i \to \tau_i$. Let $\widehat{\mathbf{n}}$ denote the maximal ideal of $\widehat{S_n}$. We show the existence of $a \in \widehat{\mathbf{n}}^2$ such that, if λ denotes the $\widehat{R}$-algebra surjection $\widehat{S_n} \to \widehat{R}$ with kernel $(t_1 - \tau_1, \ldots, t_{n-1} - \tau_{n-1}, t_n - a)\widehat{S_n}$, then $\tau_1, \ldots, \tau_{n-1}$ together with the image τ_n of t_n under the map λ are primarily independent over R.

Since S_n is countable and Noetherian we can enumerate as $\{P_j\}_{j=1}^{\infty}$ the prime ideals of S_n such that $\dim(S_n/P_j) \geq n$. Let $\widehat{I} = (t_1 - \tau_1, \ldots, t_{n-1} - \tau_{n-1})\widehat{S_{n-1}}$, and let $\mathcal{U}$ be the set of all prime ideals of $\widehat{S_n} = \widehat{R}[[t_1, \ldots, t_n]]$ minimal over ideals of the form $(P_j, \widehat{I})\widehat{S_n}$ for some P_j; then $\mathcal{U}$ is countable and $\widehat{\mathbf{n}} \notin \mathcal{U}$ since $(P_j, \widehat{I})\widehat{S_n}$ is generated by $n - 1$ elements over $P_j\widehat{S_n}$ and $\dim(\widehat{S_n}/P_j\widehat{S_n}) \geq n$. By Lemma 3.7 with the ideal I of that lemma taken to be $\widehat{\mathbf{n}}$, there exists an element $a \in \widehat{\mathbf{n}}^2$ so that $t_n - a$ is outside $\widehat{Q}$, for every prime ideal $\widehat{Q} \in \mathcal{U}$. Let $\tau_n \in \widehat{R}$ denote the image of t_n under the $\widehat{R}$-algebra surjection λ: $\widehat{S_n} \to \widehat{R}$ with kernel $(\widehat{I}, t_n - a)\widehat{S_n}$. The kernel of λ is also generated by $(\widehat{I}, t_n - \tau_n)\widehat{S_n}$. Therefore the setting will be as in the diagram of (2.1) after we establish Claim 1.

Claim 1. $(\widehat{I}, t_n - \tau_n)\widehat{S_n} \cap S_n = (0)$.

Proof of Claim 1. Since $\tau_1, \ldots, \tau_{n-1}$ are primarily independent, $\widehat{I} \cap S_{n-1} = (0)$ and $\widehat{I\widehat{S_n}} \cap S_n = (0)$. Let $R_n' = R_{n-1}[t_n]_{(\max(R_{n-1})t_n))}$. Consider the diagram

$$\begin{array}{ccc} S_n = S_{n-1}[t_n]_{(\max(S_{n-1}),t_n)} & \xrightarrow{\subset} & \widehat{S_n} = \widehat{S_{n-1}}[[t_n]] \\ \cong\Big\downarrow & & \lambda_1\Big\downarrow \\ R_n' = R_{n-1}[t_n]_{(\max(R_{n-1}),t_n)} & \xrightarrow{\subset} & \widehat{R}[[t_n]] \cong (\widehat{S_{n-1}}/\widehat{I})[[t_n]], \end{array}$$

where $\lambda_1\colon \widehat{S_n} \to \widehat{S_n}/(\widehat{I\widehat{S_n}})$ is the canonical projection.

For $\widehat{Q}$ a prime ideal of $\widehat{S_n}$, we have $\widehat{Q} \in \mathcal{U} \iff \lambda_1(\widehat{Q}) = \widehat{P}$, where $\widehat{P}$ is a prime ideal of $\widehat{R}[[t_n]] \cong (\widehat{S_{n-1}}/\widehat{I})[[t_n]]$ minimal over $\lambda_1(P_j)\widehat{R}[[t_n]]$ for some prime ideal P_j of S_n such that $\dim(S_n/P_j) \le n$. Since $t_n - a$ is outside every $\widehat{Q} \in \mathcal{U}$, $t_n - \lambda_1(a) = \lambda_1(t_n - a)$ is outside every prime ideal $\widehat{P}$ of $\widehat{R}[[t_n]]$, such that $\widehat{P}$ is minimal over $\lambda_1(P_j)\widehat{R}[[t_n]]$. Since S_n is catenary and $\dim(S_n) = n+\dim(R)$, a prime ideal P_j of S_n is such that $\dim(S_n/P_j) \ge n \iff \operatorname{ht}(P_j) \le \dim(R)$. Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}[[t_n]]$ such that $\widehat{P} \cap R_n' = P \ne (0)$. Then $\widehat{P}$ is a minimal prime ideal of $P\widehat{R}[[t_n]]$. But also $P = \lambda_1(Q)$, where Q is a height-one prime of S_n and $\dim(S_n/Q) = n+\dim(R)-1 \ge n$. Therefore $Q \in \{P_j\}_{j=1}^{\infty}$. Hence by choice of a, we have $t_n - \lambda_1(a) \notin \widehat{P}$. By Remark 3.8, $(t_n - \lambda_1(a))\widehat{R}[[t_n]]) \cap R_n' = (0)$. Hence $(\widehat{I}, t_n - \tau_n)\widehat{S_n} \cap S_n = (0)$.

Claim 2. Let P be a prime ideal of S_n such that $\dim(S_n/P) = n$. Then the ideal $(P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathfrak{n}}$-primary.

Proof of Claim 2. Let $Q = P \cap S_{n-1}$. Either $QS_n = P$, or $QS_n < P$. If $QS_n = P$, then $\dim(S_{n-1}/Q) = n-1$ and the primary independence of $\tau_1, \ldots, \tau_{n-1}$ implies that $(Q, \widehat{I})\widehat{S_{n-1}}$ is primary for the maximal ideal of $\widehat{S_{n-1}}$. Therefore $(Q, \widehat{I}, t_n - \tau_n)\widehat{S_n} = (P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathfrak{n}}$-primary in this case. On the other hand, if $QS_n < P$, then $\dim(S_{n-1}/Q) = n$. Let $\widehat{Q}'$ be a minimal prime of $(Q, \widehat{I})\widehat{S_{n-1}}$. By (3.5), $\dim(\widehat{S_{n-1}}/\widehat{Q}') = 1$, and hence $\dim(\widehat{S_n}/\widehat{Q}'\widehat{S_n}) = 2$. The primary independence of $\tau_1, \ldots \tau_{n-1}$ implies that $\widehat{Q}' \cap S_{n-1} = Q$. Therefore $\widehat{Q}'\widehat{S_{n-1}}[[t_n]] \cap S_n = QS_n < P$, so P is not contained in $\widehat{Q}'\widehat{S_n}$. Therefore $\dim(\widehat{S_n}/(P, \widehat{I})\widehat{S_n})) = 1$ and our choice of a implies that $(P, \widehat{I}, t_n - \tau_n)\widehat{S_n}$ is $\widehat{\mathfrak{n}}$-primary.

This completes the proof of Theorem 3.9. □

3.10. COROLLARY. *Let $(R, \mathbf{m})$ be a countable excellent normal local domain of dimension at least 2, and let K denote the fraction field of R. Then there exist $\tau_1, \ldots, \tau_n, \ldots \in \mathbf{m}\widehat{R}$ such that $A = K(\tau_1, \tau_2, \ldots) \cap \widehat{R}$ is an infinite-dimensional quasilocal (non-Noetherian) domain. In particular, for k a countable field, the localized polynomial ring $R = k[x, y]_{(x,y)}$ has such extensions inside $\widehat{R} = k[[x, y]]$.*

Proof. By (3.9.3), there exist $\tau_1, \dots, \tau_n, \dots \in \mathbf{m}\widehat{R}$ which are primarily independent over R. It follows that $A = K(\tau_1, \tau_2, \dots) \cap \widehat{R}$ is an infinite-dimensional quasilocal domain. In particular, A is not Noetherian. □

4. Residual algebraic independence

We introduce in this section a third concept, that of residual algebraic independence. Residual algebraic independence is a stronger notion than idealwise independence, but is weaker than primary independence. In (4.5) we show that over every countable normal excellent local domain $(R, \mathbf{m})$ of dimension at least three there exists an element residually algebraically independent over R, but not primarily independent over R. In (4.7) and (4.9) we show the existence of idealwise independent elements that fail to be residually algebraically independent.

4.1. *Definition.* Let $(\widehat{R}, \widehat{\mathbf{m}})$ be a complete normal local domain and let A be a Krull subdomain of $\widehat{R}$ such that $A \hookrightarrow \widehat{R}$ satisfies PDE.

(1) An element $\tau \in \widehat{\mathbf{m}}$ is *residually algebraically independent with respect to* $\widehat{R}$ over A provided that τ is algebraically independent over the fraction field of A and for each height-one prime $\widehat{P}$ of $\widehat{R}$ such that $\widehat{P} \cap A \neq (0)$, the image of τ in $\widehat{R}/\widehat{P}$ is algebraically independent over the fraction field of $A/(\widehat{P} \cap A)$.

(2) Elements $\tau_1, \dots \tau_n \in \widehat{\mathbf{m}}$ are said to be *residually algebraically independent* over A if for each $0 \leq i < n$, τ_{i+1} is residually algebraically independent over $A[\tau_1, \dots, \tau_i]$.

(3) An infinite sequence $\{\tau_i\}_{i=1}^{\infty}$ of elements of $\widehat{\mathbf{m}}$ is *residually algebraically independent* over A, if $\tau_1, \dots \tau_n$ are residually algebraically independent over A for each positive integer n.

The following result shows the equivalence of residual algebraic independence for τ over A to the PDE property for $A[\tau] \hookrightarrow \widehat{R}$.

4.2. Proposition. *Let* $(R, \mathbf{m}), \tau \in \mathbf{m}\widehat{R}$ *be as in the setting of* (2.1) *and let* A *be a Krull subdomain of* $\widehat{R}$ *such that* $A \hookrightarrow \widehat{R}$ *satisfies PDE. Then* τ *is residually algebraically independent with respect to* $\widehat{R}$ *over* $A \iff A[\tau] \hookrightarrow \widehat{R}$ *satisfies PDE.*

Proof. Assume $A[\tau] \hookrightarrow \widehat{R}$ does not satisfy PDE. Then there exists a prime ideal $\widehat{P}$ of $\widehat{R}$ of height one such that $\mathrm{ht}(\widehat{P} \cap A[\tau]) \geq 2$. Now $\mathrm{ht}(\widehat{P} \cap A) = 1$, since PDE holds for $A \hookrightarrow \widehat{R}$. Thus, with $\mathbf{p} = \widehat{P} \cap A$, we have $\mathbf{p}A[\tau] < \widehat{P} \cap A[\tau]$; that is, there exists $f(\tau) \in \widehat{P} \cap A[\tau] - \mathbf{p}A[\tau]$, or equivalently there is a nonzero polynomial $\bar{f}(x) \in (A/(\widehat{P} \cap A))[x]$ so that $\bar{f}(\bar{\tau}) = \bar{0}$ in $A[\tau]/(\widehat{P} \cap A[\tau])$, where $\bar{\tau}$ denotes the image of τ in $\widehat{R}/\widehat{P}$. This means that $\bar{\tau}$ is algebraic over the quotient field of $A/(\widehat{P} \cap A)$. Hence τ is not residually algebraically independent with respect to $\widehat{R}$ over A.

For the converse, assume that $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE and let $\widehat{P}$ be a height-one prime of $\widehat{R}$ such that $\widehat{P} \cap A = \mathbf{p} \neq 0$. Since $A[\tau] \hookrightarrow \widehat{R}$ satisfies PDE, $\widehat{P} \cap A[\tau] = \mathbf{p}A[\tau]$ and $A[\tau]/(\mathbf{p}A[\tau])$ canonically embeds in $\widehat{R}/\widehat{P}$. Since the image of τ in $A[\tau]/\mathbf{p}A[\tau]$ is algebraically independent over $A/\mathbf{p}$, it follows that τ is residually algebraically independent with respect to $\widehat{R}$ over A. □

4.3. THEOREM. *Let $(R, \mathbf{m})$ and $\tau_1, \dots, \tau_n \in \mathbf{m}\widehat{R}$ be as in the setting of* (2.1). *The following statements are equivalent*:

(1) *The elements $\tau_1, \dots, \tau_n$ are residually algebraically independent with respect to $\widehat{R}$ over R.*
(2) *For each $1 \le i \le n$, if $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ such that $\widehat{P} \cap R[\tau_1, \dots, \tau_{i-1}] \neq 0$, then* $\operatorname{ht}(\widehat{P} \cap R[\tau_1, \dots, \tau_i]) = 1$.
(3) *$R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE and is weakly flat.*
(4) *$R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE.*

Proof. The equivalence of (1) and (2) and of (1) and (4) follows from (4.2). By (2.16) and part (c) of (2.9), (3) and (4) are equivalent. □

4.4. THEOREM. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in the setting of* (2.1), *where* $\dim(R) \ge 2$ *and m is either a positive integer or $m = \infty$.*

(1) *If $\{\tau_i\}_{i=1}^m$ is primarily independent over R, then $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R.*
(2) *If $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*
(3) *If* $\dim(R) = 2$, *then $\{\tau_i\}_{i=1}^m$ is primarily independent over R if and only if it is residually algebraically independent over R.*

Proof. To prove (1), it suffices by (4.3) to show that for each positive integer $n \le m$, if $\tau_1, \dots, \tau_n$ are primarily independent over R, then $R[\tau_1, \dots, \tau_n] \hookrightarrow \widehat{R}$ satisfies PDE. Let $S = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}$ and let the notation be as in the diagram of (2.1).

Let $\widehat{P}$ be a height-one prime ideal of $\widehat{R}$ with $P = \widehat{P} \cap R \neq (0)$. Consider the ideal $\widehat{W} = (P, t_1 - \tau_1, \dots, t_n - \tau_n)\widehat{S_n}$. Using the diagram in (2.1) we see that $\lambda(\widehat{W}) = P\widehat{R} \subseteq \widehat{P}$. By Corollary 3.6.2, $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(P) + n$. But $\widehat{W} \subseteq (\widehat{P}, t_1 - \tau_1, \dots, t_n - \tau_n) = \lambda^{-1}(\widehat{P})$ and thus

$$1 + n \le \operatorname{ht}(P) + n = \operatorname{ht}(\widehat{W}) \le \operatorname{ht}(\lambda^{-1}(P)) \le \operatorname{ht}(\widehat{P}) + n = 1 + n.$$

Therefore $\operatorname{ht}(P) = 1$.

The proof of (2) follows from (4.3) and (2.16).

In view of (1), to prove (3), we assume that $\dim(R) = 2$ and $n \le m$ is a positive integer such that $\tau_1, \dots, \tau_n$ are residually algebraically independent over R. Let

$S = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)}$. By (4.2), $S \hookrightarrow \widehat{R}$ satisfies PDE. Let P be a prime ideal of S such that $\dim(S/P) \leq n$. Since $\dim(S) = n+2$ and S is catenary, it follows that $\operatorname{ht}(P) \geq 2$. To show $\tau_1, \dots, \tau_n$ are primarily independent over R, it suffices to show that $P\widehat{R}$ is primary for the maximal ideal of $\widehat{R}$. Since $\dim(\widehat{R}) = 2$, this is equivalent to showing P is not contained in a height-one prime of $\widehat{R}$, and this last statement holds since $S \hookrightarrow \widehat{R}$ satisfies PDE. $\square$

4.5. Proposition. *If $(R, \mathbf{m})$ is a countable excellent normal local domain of dimension at least 3, then there exists an element $\tau \in \mathbf{m}\widehat{R}$ which is residually algebraically independent over R, but not primarily independent over R.*

Proof. We modify the proof of (3.9). Let t be an indeterminate over R and set $S_1 = R[t]_{(\mathbf{m},t)}$ so that $\widehat{S_1} = \widehat{R}[[t]]$. Let $\widehat{Q_0}$ be a height-three prime of $\widehat{S_1}$ that contains t and is such that $Q_0 = \widehat{Q_0} \cap S_1$ also has height three. Using Lemma 3.7 with $I = \widehat{Q_0}$, there exists $a \in \widehat{Q_0} \cap \widehat{\mathbf{n}_1}^2$, where $\widehat{\mathbf{n}_1}$ is the maximal ideal of $\widehat{S_1}$, so that $t - a \in \widehat{Q_0}$ but $t - a$ is not in any of the other height-three prime ideals of $\widehat{S_1}$ that are minimal over a height-three prime of S_1. Let λ be the surjection $\widehat{S_1} \to \widehat{R}$ with kernel $(t-a)\widehat{S_1}$. Then the image $\tau \in \mathbf{m}\widehat{R}$ of t under λ: $\widehat{S_1} \to \widehat{R}$ is not primarily independent because the prime ideal $\lambda(Q_0)$ in $S = R[\tau]_{(\mathbf{m},\tau)}$ is of height three and is the contraction to S of the prime ideal $\lambda(\widehat{Q_0})$ of $\widehat{R}$. Since $(t-\tau)\widehat{S_1} = (t-a)\widehat{S_1} \subseteq \widehat{Q_0}$, $\lambda(\widehat{Q_0})$ is of height two. Therefore τ is not primarily independent.

We prove that τ is residually algebraically independent over R: If $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $\widehat{P} \cap R \neq 0$, then the height of $\widehat{P} \cap R$ is 1 and so the height of $\widehat{P} \cap S$ is at most 2. Also $\lambda^{-1}(\widehat{P}) = \widehat{Q}$ in $\widehat{S_1}$ has height two—since it's generated by the inverse images of the generators of $\widehat{P}$ and $\ker(\lambda) = (t-a)\widehat{S_1}$.

Suppose that the height of $\widehat{P} \cap S = 2$. Then under the R-isomorphism of S_1 to S taking t to τ, $\widehat{P} \cap S$ corresponds to a height-two prime P of S_1. We have $P \subseteq \widehat{Q} \cap S_1$ and since $\widehat{S_1}$ is flat over S_1, the height of $\widehat{Q} \cap S_1$ is at most two, so we have $P = \widehat{Q} \cap S_1$. Let $\mathbf{n}_1$ denote the maximal ideal of S_1, and choose $b \in \mathbf{n}_1 - (P \cup Q_0)$ and a prime ideal $\widehat{Q_1}$ in $\widehat{S_1}$ minimal over $(\widehat{Q}, b)\widehat{S_1}$. Since $b \notin \widehat{Q}$, we see that $\operatorname{ht}(\widehat{Q_1}) = 3$ and $\operatorname{ht}(\widehat{Q_1} \cap S_1) = 3$, because it properly contains $P = \widehat{Q} \cap S_1$. We have

$$
\begin{array}{llcccc}
\text{In } S_1: & Q_0 \neq \widehat{Q_1} \cap S_1 \ (\text{ht } 3) & \xrightarrow{\subseteq} & \widehat{Q_1} & \xleftarrow{\min} & (b, \widehat{Q})\widehat{S_1} \ (\text{ht. } 3) \\
& \quad \cup\!\uparrow & & \cup\!\uparrow & & \\
\text{In } S_1: & P = \widehat{Q} \cap S_1 (\text{ht } 2) & \xrightarrow{\subseteq} & \widehat{Q} = \lambda^{-1}(\widehat{P}) & \xleftarrow{\supseteq} & (\widehat{P}, (t-a))\widehat{S_1} \ (\text{ht } 2) \\
& \quad \cong\!\downarrow & & \lambda\!\downarrow & & \\
\text{In } S: & \widehat{P} \cap S \ (\text{ht } 2) & \xrightarrow{\subseteq} & \widehat{P} & & (\text{ht } 1 \text{ in } \widehat{R}).
\end{array}
$$

But then $\widehat{Q_1}$ is minimal over a height-three prime $(\widehat{Q_1} \cap S_1)$ of S_1 and $t - a \in \widehat{Q_1}$. This implies that $\widehat{Q_1} = \widehat{Q_0}$, and so $\widehat{Q_1} \cap S_1 = \widehat{Q_0} \cap S_1 = Q_0$, a contradiction since

$b \notin Q_0$. We conclude that $\mathrm{ht}(\widehat{P} \cap S) = 1$ and that τ is residually algebraically independent over R. □

4.6. *Example*. The following construction, similar to that in (4.5), shows that condition (2) in Definition 4.1 is stronger than the following:

(2′) For each height-one prime ideal $\widehat{P}$ of $\widehat{R}$ with $\widehat{P} \cap R \neq 0$, the images of $\tau_1, \ldots, \tau_n$ in $\widehat{R}/\widehat{P}$ are algebraically independent over $R/(\widehat{P} \cap R)$.

Construction. Let R be a countable excellent local unique factorization domain (UFD) of dimension two, for example $R = \mathbb{Q}[x, y]_{(x,y)}$. As in (3.9), construct $\tau_1 \in \mathbf{m}\widehat{R}$ primarily independent over R (or equivalently, residually algebraically independent in this context). Let $S_2 = R[t_1, t_2]_{(\mathbf{m}, t_1, t_2)}$, let $\mathbf{n}$ denote the maximal ideal of S_2, and let $\mathcal{U} = \{$prime ideals $\widehat{Q}$ of $\widehat{S_2}$ minimal over some ideal of form $(P, t_1 - \tau_1)\widehat{S_2}$ where P is a prime ideal of S_2 with $\dim(S_2/P) \geq 2$ and $P \neq (t_1, t_2)S_2\}$. Note that all the prime ideals in $\mathcal{U}$ have height at most 3 and the ideal $I = (t_1, t_2, t_1 - \tau_1)\widehat{S_2}$ is not contained in any prime ideal in $\mathcal{U}$. By Lemma 3.7, we can choose $a \in \mathbf{n}^2 \cap I$ so that $t_2 - a \notin \bigcup\{\widehat{Q} \colon \widehat{Q} \in \mathcal{U}\}$. Let τ_2 be the image of t_2 under the R-algebra surjection $\lambda\colon \widehat{S_2} \to \widehat{R}$ with kernel $(t_1 - \tau_1, t_2 - a)\widehat{S_2}$; then $\ker(\lambda)$ has height two. As before, set $R_i = R[\tau_1, \tau_i]_{(\mathbf{m}, \tau_1, \tau_i)}$, for $i = 1, 2$.

Claim 1. τ_1, τ_2 do not satisfy (2) of Definition 4.1.

Proof of Claim 1. Let $\widehat{Q}$ be a prime ideal of $\widehat{S_2}$ which is minimal over $(t_1, t_2, t_1 - \tau_1, t_2 - a)\widehat{S_2}$. Then by the choice of a, $\widehat{Q}$ is minimal over $(t_1, t_2, t_1 - \tau_1)\widehat{S_2}$. Therefore $\mathrm{ht}(\widehat{Q}) \leq 3$ and $\widehat{Q} \supseteq \ker(\lambda)$. Let $\widehat{P} = \lambda(\widehat{Q})$ in $\widehat{R}$; then $\mathrm{ht}(\widehat{P}) \leq 1$. In fact $\mathrm{ht}(\widehat{P}) = 1$ since $0 \neq \tau_1 = \lambda(t_1) \in \widehat{P}$. Since τ_1 is residually algebraically independent over R, $\mathrm{ht}(\widehat{P} \cap R_1) \leq 1$. But $\tau_1 \in \widehat{P} \cap R_1$, so $\mathrm{ht}(\widehat{P} \cap R_1) = 1$ and $\widehat{P} \cap R = (0)$. Now also $\tau_2 = \lambda(t_2) \in \widehat{P}$; thus $\tau_1, \tau_2 \in \widehat{P} \cap R_2$, so $\mathrm{ht}(\widehat{P} \cap R_2) \geq 2$. Thus (2) fails by (4.2).

Claim 2. τ_1, τ_2 satisfy (2′) above.

Proof of Claim 2. Suppose $\widehat{P}$ is a height-one prime ideal of $\widehat{R}$ with $\widehat{P} \cap R \neq (0)$ and let $\widehat{Q} = \lambda^{-1}(\widehat{P})$. Then $\mathrm{ht}(\widehat{Q}) = 3$ and $\mathrm{ht}(\widehat{P} \cap R) = 1$. By the residual algebraic independence of τ_1 over R, $\mathrm{ht}(\widehat{P} \cap R_1) = 1$, and so $\mathrm{ht}(\widehat{P} \cap R_2) \leq 2$. If $\mathrm{ht}(\widehat{P} \cap R_2) = 1$, we are done. Suppose $\mathrm{ht}(\widehat{P} \cap R_2) = 2$. We have

$$\begin{array}{ccccccccc}
 & & \widehat{Q} \cap S_1 & \xrightarrow{\subseteq} & \widehat{Q} \cap S_2 & \xrightarrow{\subseteq} & \widehat{Q} = \lambda^{-1}(\widehat{P}) & \xrightarrow{\subseteq} & \widehat{S_2} \\
 & & \cong\downarrow & & \cong\downarrow & & \lambda\downarrow & & \lambda\downarrow \\
\widehat{P} \cap R & \xrightarrow{\subseteq} & \widehat{P} \cap R_1 & \xrightarrow{\subseteq} & \widehat{P} \cap R_2 & \xrightarrow{\subseteq} & \widehat{P} & \xrightarrow{\subseteq} & \widehat{R}.
\end{array}$$

Thus $\widehat{Q} \cap S_2 = P$ is a prime ideal of height 2, and $\mathrm{ht}(\widehat{Q} \cap S_1) = 1$. Also $P \neq (t_1, t_2)S_2$ because $(t_1, t_2)S_2 \cap R = (0)$. But this means that $\widehat{Q} \in \mathcal{U}$ since $\widehat{Q}$ is minimal over

$(P, t_1 - \tau_1)\widehat{S_2}$ where P is a prime of S_2 with $\dim(S_2/P) = 2$ and $P \neq (t_1, t_2)S_2$. This contradicts the choice of a and establishes that (2′) holds. □

Following a suggestion of the referee, we present in (4.7) a method to obtain an idealwise independent element that fails to be residually algebraically independent.

4.7. PROPOSITION. *Let $(R, \mathbf{m})$ be a countable excellent local UFD of dimension at least two. Assume there exists a height-one prime P of $\widehat{R}$ such that P is contained in at least two distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ of $\widehat{R}$. Also assume that $\widehat{P}$ is not the radical of a principal ideal in $\widehat{R}$. Then there exists $\tau \in \mathbf{m}\widehat{R}$ that is idealwise independent but not residually algebraically independent over R.*

Proof. Let t be an indeterminate over R and set $S_1 = R[t]_{(\mathbf{m},t)}$ so that $\widehat{S_1} = \widehat{R}[[t]]$. Let $\widehat{\mathbf{n}_1}$ denote the maximal ideal of $\widehat{S_1}$.

Using Lemma 3.7 with $I = (\widehat{P}, t)\widehat{S_1}$ and $\mathcal{U} = \{\mathbf{p} \in \operatorname{Spec}(\widehat{S_1}) \mid \mathbf{p} \neq I,\ \operatorname{ht}(\mathbf{p}) \leq 2,$ and $\mathbf{p}$ minimal over $\mathbf{p} \cap S_1\}$, there exists $a \in (\widehat{P}, t)\widehat{S_1} \cap \widehat{\mathbf{n}_1}^2$, such that $t - a \notin \bigcup\{\mathbf{p} \mid \mathbf{p} \in \mathcal{U}\}$, but $t - a \in (\widehat{P}, t)\widehat{S_1}$. That is, if $t - a \in \mathbf{p}$, for some prime ideal $\mathbf{p} \neq (\widehat{P}, t)\widehat{S_1}$ of $\widehat{S_1}$ with $\operatorname{ht}(\mathbf{p}) \leq 2$, then $\operatorname{ht}(\mathbf{p}) > \operatorname{ht}(\mathbf{p} \cap S_1)$. Let λ be the surjection $\widehat{S_1} \to \widehat{R}$ with kernel $(t - a)\widehat{S_1}$. By construction, $(t - a)\widehat{S_1} \cap S_1 = (0)$. Therefore the restriction of λ to S_1 maps S_1 isomorphically onto $S = R[\tau]_{(\mathbf{m},\tau)}$, where $\lambda(t) = \tau \in \mathbf{m}\widehat{R}$ is algebraically independent over the fraction field of R.

That τ is not residually algebraically independent over R follows because the prime ideal $\lambda((P, t)S_1) = (P, \tau)S$ has height two and is the contraction to S of the prime ideal $\lambda((\widehat{P}, t)\widehat{S_1}) = \widehat{P}$ of $\widehat{R}$. Since $(t - \tau)\widehat{S_1} = (t - a)\widehat{S_1} \subseteq (\widehat{P}, t)\widehat{S_1}$, $\lambda((\widehat{P}, t)\widehat{S_1})$ has height one and equals $\widehat{P}$. Therefore τ is not residually algebraically independent over R.

Our choice of $t - a$ insures that each height-one prime $\widehat{\mathbf{q}}$ other than $\widehat{P}$ of $\widehat{R}$ has the property that $\operatorname{ht}(\widehat{\mathbf{q}} \cap S) \leq 1$. We show that τ is idealwise independent over R by showing each height-one prime of S is the contraction of a height-one prime of $\widehat{R}$. Let ϕ: $S_1 \to S$ denote the restriction of λ. For $\mathbf{q}$ a height-one prime of S, let $\mathbf{q}_1 := \phi^{-1}(\mathbf{q})$ denote the corresponding height-one prime of S_1. Then $(\mathbf{q}_1, t - a)\widehat{S_1}$ is an ideal of height two. Let $\mathbf{w}_1$ be a height-two prime of $\widehat{S_1}$ containing $(\mathbf{q}_1, t - a)$. If $\mathbf{q}_1$ is not contained in $(\widehat{P}, t)\widehat{S_1}$, then by the choice of $t - a$, $\mathbf{w}_1 \cap S_1$ has height at most one. Therefore $\mathbf{w}_1 \cap S_1 = \mathbf{q}_1$. Let $\mathbf{w} = \lambda(\mathbf{w}_1)$. Then $\mathbf{w}$ is a height-one prime of $\widehat{R}$ and $\mathbf{w} \cap S = \mathbf{q}$.

Therefore each height-one prime $\mathbf{q}$ of S such that $\mathbf{q}_1 := \phi^{-1}(\mathbf{q})$ is not contained in $(\widehat{P}, t)\widehat{S_1}$ is the contraction of a height-one prime of $\widehat{R}$. Since $\lambda((\widehat{P}, t)\widehat{S_1}) \cap S = (P, \tau)S$, it remains to consider height-one primes $\mathbf{q}$ of S such that $\mathbf{q} \subseteq (P, \tau)S$. By construction we have $PS = \widehat{Q} \cap S$. Let $\mathbf{q}$ be a height-one prime of S such that $\mathbf{q} \neq PS$ and $\mathbf{q} \subseteq (P, \tau)S$. Since R is a UFD, S is a UFD and $\mathbf{q} = fS$ for an element $f \in \mathbf{q}$. Since $\widehat{P}$ is not the radical of a principal ideal, there exists a height-one prime $\widehat{\mathbf{q}} \neq \widehat{P}$ of $\widehat{R}$ such that $f \in \widehat{\mathbf{q}}$. Since $\operatorname{ht}(\widehat{\mathbf{q}} \cap S) \leq 1$, we have $\widehat{\mathbf{q}} \cap S = fS = \mathbf{q}$. Therefore τ is idealwise independent over R. □

4.8. *Remark.* A specific example of a countable excellent local UFD having a height-one prime P satisfying the conditions in (4.7) is $R = k[x, y, z]_{(x,y,z)}$, where k is the algebraic closure of the field $\mathbb{Q}$ and $z^2 = x^3 + y^7$. That R is a UFD is shown in [Sa, page 32]. Since $z - xy$ is an irreducible element of R, the ideal $P = (z - xy)R$ is a height-one prime of R. It is observed in [HL, pages 300-301] that in the completion $\widehat{R}$ of R there exist distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ lying over P. Moreover, the blowup of $\widehat{P}$ has a unique exceptional prime divisor and this exceptional prime divisor is not on the blowup of an $\widehat{\mathbf{m}}$-primary ideal. Therefore $\widehat{P}$ is not the radical of a principal ideal of $\widehat{R}$.

In (4.9) we present an alternative method to obtain idealwise independent elements that are not residually algebraically independent.

4.9. PROPOSITION. *Let $(R, \mathbf{m})$ be a countable excellent local UFD of dimension at least two. Assume there exists a height-one prime P_0 of R such that P_0 is contained in at least two distinct height-one primes $\widehat{P}$ and $\widehat{Q}$ of $\widehat{R}$. Also assume that the Henselization $(R^h, \mathbf{m}^h)$ of R is a UFD. Then there exists $\tau \in \mathbf{m}\widehat{R}$ that is idealwise independent but not residually algebraically independent over R.*

Proof. Since R is excellent, $P := \widehat{P} \cap R^h$ and $Q := \widehat{Q} \cap R^h$ are distinct height-one primes of R^h with $P\widehat{R} = \widehat{P}$, and $Q\widehat{R} = \widehat{Q}$. Let $x \in R^h$ be such that $xR^h = P$. Theorem 3.9 implies there exists $y \in \mathbf{m}\widehat{R}$ that is primarily independent and hence residually algebraically independent over R^h.

We show that $\tau = xy$ is idealwise independent but not residually algebraically independent over R. Since x is nonzero and algebraic over R, xy is algebraically independent over R. Let $S = R[xy]_{(\mathbf{m},xy)}$. Then S is a UFD and $\widehat{P} \cap S = x\widehat{R} \cap S \supseteq (P_0, xy)S$ has height at least two in S. Therefore by (4.3), xy is not residually algebraically independent over R.

Since y is idealwise independent over R^h, every height-one prime of the polynomial ring $R^h[y]$ contained in the maximal ideal $\mathbf{n} = (\mathbf{m}^h, y)R^h[y]$ is the contraction of a height-one prime of $\widehat{R}$. To show xy is idealwise independent over R, it suffices to show every prime element $w \in (\mathbf{m}, xy)R[xy]$ is such that $wR[xy]$ is the contraction of a height-one prime of $R^h[y]$ contained in $\mathbf{n}$. If $w \notin (P, xy)R^h[xy]$, then the constant term of w as a polynomial in $R^h[xy]$ is in $\mathbf{m}^h - P$. Thus $w \in \mathbf{n}$ and $w \notin xR^h[y]$. Since $R^h[xy][1/x] = R^h[y][1/x]$ and $xR^h[y] \cap R^h[xy] = (x, xy)R^h[xy]$, it follows that there is a prime factor u of w in $R^h[xy]$ such that $u \in \mathbf{n} - xR^h[y]$. Then $uR^h[y]$ is a height-one prime of $R^h[y]$ and $uR^h[x] \cap R^h[xy] = uR^h[xy]$. Since $R^h[xy]$ is faithfully flat over $R[xy]$, it follows that $uR^h[y] \cap R[xy] = wR[xy]$.

We have $QR^h[xy] = QR^h[y] \cap R^h[xy]$ and $QR^h[xy] \cap R[xy] = P_0R[xy]$. Thus it remains to show, for a prime element $w \in (\mathbf{m}, xy)R[xy]$ such that $w \in (P, xy)R^h[xy]$ and $wR[xy] \neq P_0R[xy]$, that $wR[xy]$ is the contraction of a height-one prime of R^h contained in $\mathbf{n}$. Since $(P, xy)R^h[xy] \cap R[xy] = (P_0, xy)R[xy]$, it follows that w is a nonconstant polynomial in $R[xy]$ and the constant term w_0 of w is in P_0. In the polynomial ring $R^h[y]$ we have $w = x^n v$, where $v \notin xR^h[y]$. If v_0

denotes the constant term of v as a polynomial in $R^h[y]$, then $x^n v_0 = w_0 \in P_0 \subseteq R$ implies $x^n v_0 \in Q \subseteq R^h$. Since $x \in R^h - Q$, we must have $v_0 \in Q$ and hence $v \in \mathbf{n}$. Also $v \notin xR^h[y]$ implies there is a height-one prime ideal $\mathbf{v}$ of $R^h[y]$ with $v \in \mathbf{v}$ and $x \notin \mathbf{v}$. Then, since $R^h[y]_{\mathbf{v}}$ is a localization of $R^h[xy]$, $\mathbf{v} \cap R^h[xy]$ is a height-one prime of $R^h[xy]$ that is contained in $(\mathbf{m}^h, xy)R^h[xy]$. It follows that $\mathbf{v} \cap R^h[xy] = wR^h[xy]$ which completes the proof of (4.9). □

4.10. *Remark.* For a specific example of (4.9), take R to be the localized polynomial ring in two variables over a countable field k where k has characteristic not equal to 2, say $R = k[s, t]_{(s,t)}$. Then $P_0 = (s^2 - t^2 - t^3)R$ is a height-one prime of R and $P_0\widehat{R} = (s^2 - t^2 - t^3)k[[s, t]]$ is the product of two distinct height-one primes of $\widehat{R}$.

5. Idealwise independence and flatness

This section contains more results relating idealwise independence, residual algebraic independence, and primary independence. We describe all three notions in terms of flatness of certain localizations of the canonical embedding ϕ: $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)} \hookrightarrow \widehat{R}$. We start with an easy characterization of weakly flat and PDE morphisms.

5.1. PROPOSITION. *Let ϕ: $A \longrightarrow B$ be an injective morphism of Krull domains.*

(1) *ϕ is weakly flat if and only if for every height-one prime ideal $P \in$ Spec(A) such that $PB \neq B$ there is a height-one prime ideal $Q \in$ Spec(B) with $P \subseteq Q \cap A$ such that the induced morphism on the localizations*

$$\phi_Q\colon A_{Q\cap A} \longrightarrow B_Q$$

is faithfully flat.

(2) *ϕ satisfies PDE if and only if for every $Q \in$ Spec(B) with* ht$(Q) = 1$ *the induced morphism on the localizations*

$$\phi_Q\colon A_{Q\cap A} \longrightarrow B_Q$$

is faithfully flat.

Proof. In both (1) and (2) we use the fact that for each height-one prime $P \in$ Spec(A) the induced morphism ϕ_P: $A_P \longrightarrow (A-P)^{-1}B$ is flat (a domain extension of a DVR is always flat); and ϕ_P is faithfully flat $\Longleftrightarrow$ $P(A-P)^{-1}B \neq (A-P)^{-1}B$ which is equivalent to the existence of a prime in B lying over P in A.

For the proof of (1), to see ($\Longleftarrow$), we use the fact that ϕ_Q a faithfully flat morphism implies ϕ_Q satisfies the going-down property (see (5.5.1)). Hence $Q \cap A$ is of height one, so $P = Q \cap A$, and thus $PB \cap A = P$. For ($\Longrightarrow$), suppose $P \in \operatorname{Spec}(A)$ has height one and ϕ is weakly flat. Then (2.10) implies the existence of $Q \in \operatorname{Spec}(B)$ of height one such that $Q \cap A = P$. Since B_Q is a localization of $(A - P)^{-1}B$, we see that ϕ_Q is faithfully flat.

For the proof of (2), ($\Longrightarrow$) is clear by the remark above, and ($\Longleftarrow$) follows from the fact that a faithfully flat morphism satisfies the going-down property. □

5.2. COROLLARY. *Let* $(R, \mathbf{m})$ *and* $\tau_1, \ldots, \tau_n \in \widehat{\mathbf{m}}$ *be as in the setting of* (2.1), *and let* ϕ: $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)} \hookrightarrow \widehat{R}$ *denote the canonical embedding. Then*:

(1) $\tau_1, \ldots, \tau_n$ *are idealwise independent over* R *if and only if for every height-one prime ideal* P *of* R_n *there is a prime ideal* $\widehat{Q} \subseteq \widehat{R}$ *with* $\widehat{Q} \cap R_n = P$ *such that the induced morphism of the localizations*

$$\phi_{\widehat{Q}}\colon (R_n)_P \longrightarrow \widehat{R}_{\widehat{Q}}$$

is faithfully flat.

(2) $\tau_1, \ldots, \tau_n$ *are residually algebraic independent over* R *if and only if for every height-one prime ideal* $\widehat{Q} \subseteq \widehat{R}$ *the induced morphism of the localizations*

$$\phi_{\widehat{Q}}\colon (R_n)_{\widehat{Q} \cap R_n} \longrightarrow \widehat{R}_{\widehat{Q}}$$

is faithfully flat.

In order to describe primary independence in terms of flatness of certain localizations of the embedding ϕ: $R_n \longrightarrow \widehat{R}$, we introduce the following definition:

5.3. *Definition.* Let ϕ: $A \longrightarrow B$ be an injective morphism of commutative rings and let $k \in \mathbf{N}$ be an integer with $1 \leq k \leq d = \dim(B)$ where d is an integer or $d = \infty$. Then ϕ is called *locally flat in height k—LF_k for short*—if for every prime ideal $Q \in \operatorname{Spec}(B)$ with $\operatorname{ht}(Q) \leq k$ the induced morphism on the localizations

$$\phi_Q\colon A_{Q \cap A} \longrightarrow B_Q$$

is faithfully flat.

The following proposition is an immediate consequence of (5.1):

5.4. PROPOSITION. *Let* ϕ: $A \longrightarrow B$ *be an injective morphism of Krull domains. Then* ϕ *satisfies PDE if and only if* ϕ *satisfies* LF_1.

5.5. *Remarks.* We use the following results on flatness.

(1) Let $\phi\colon A \longrightarrow B$ be an injective morphism of commutative rings. Suppose that ϕ satisfies LF_k. Then for every $Q \in \operatorname{Spec}(B)$ with $\operatorname{ht}(Q) \leq k$ we have $\operatorname{ht}(Q \cap A) \leq \operatorname{ht}(Q)$ [M3, Theorem 4, page 33].

(2) Let A be a Noetherian ring, I an ideal in A, and M an I-adically ideal-separated A-module. Then M is A-flat $\iff$ (i) M/IM is (A/I)-flat, and (ii) $I \otimes_A M \cong IM$ [M1, part (1) $\iff$ (3) of Theorem 22.3].

5.6. THEOREM. *Let* $(R, \mathbf{m})$ *and* $\tau_1, \dots, \tau_n \in \widehat{\mathbf{m}}$ *be as in the setting of* (2.1). *Suppose that* $\dim(R) = d$. *Then*:

(1) *The elements* $\tau_1, \dots, \tau_n$ *are residually algebraically independent over* R $\iff$ $\phi\colon R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \longrightarrow \widehat{R}$ *satisfies* LF_1.

(2) *The elements* $\tau_1, \dots, \tau_n$ *are primarily independent over* R $\iff$ $\phi\colon R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \longrightarrow \widehat{R}$ *satisfies* LF_{d-1}.

Proof. For (1) apply (5.4) and (4.2). To prove ($\Longrightarrow$) in (2), let $\widehat{Q} \in \operatorname{Spec}(\widehat{R})$ with $\operatorname{ht}(\widehat{Q}) \leq d-1$. Put $Q = \widehat{Q} \cap R_n$ and $P = \widehat{Q} \cap R = Q \cap R$. We show that the induced morphism

$$\phi_{\widehat{Q}}\colon (R_n)_Q \longrightarrow \widehat{R}_{\widehat{Q}}$$

is faithfully flat. By (5.5.2), we have to verify two conditions:

(a) The morphism $\bar{\phi}_{\widehat{Q}}\colon (R_n/PR_n)_Q \longrightarrow (\widehat{R}/P\widehat{R})_{\widehat{Q}}$ is faithfully flat.

(b) $P(R_n)_Q \otimes_{(R_n)_Q} \widehat{R}_{\widehat{Q}} \cong P\widehat{R}_{\widehat{Q}}$.

Proof of (*a*). We observe that the ring $(R_n/PR_n)_Q$ is a localization of the polynomial ring $k(P)[\tau_1, \dots, \tau_n]$ where $k(P) = R_P/PR_P$. Hence the ring $(R_n/PR_n)_P$ is regular and so is the ring $(\widehat{R}/P\widehat{R})_{\widehat{Q}}$, since R is excellent. In particular, the ring $(\widehat{R}/P\widehat{R})_{\widehat{Q}}$ is Cohen-Macaulay, and [M1, Theorem 23.1] applies. Therefore we only need to show the following dimension formula:

$$\dim(\widehat{R}/P\widehat{R})_{\widehat{Q}} = \dim(R_n/PR_n)_Q + \dim(\widehat{R}/Q\widehat{R})_{\widehat{Q}}.$$

Since $Q\widehat{R}$ is contained in $\widehat{Q}$ and $\operatorname{ht}(\widehat{Q}) \leq d-1$, primary independence implies that $\dim(R_n/Q) > n$. (If $\dim(R_n/Q) \leq n$, then $Q\widehat{R}$ is $\mathbf{m}\widehat{R}$-primary.)

By Corollary 3.6.2, every minimal prime divisor $\widehat{W} \in \operatorname{Spec}(\widehat{R})$ of $Q\widehat{R}$ has $\operatorname{ht}(\widehat{W}) = \operatorname{ht}(Q)$. Let $\widehat{W} \in \operatorname{Spec}(\widehat{R})$ be a minimal prime divisor of $Q\widehat{R}$ contained in $\widehat{Q}$. Then

$$\begin{aligned}
\dim(\widehat{R}/Q\widehat{R})_{\widehat{Q}} &= \dim(\widehat{R}_{\widehat{Q}}) - \operatorname{ht}(Q\widehat{R}_{\widehat{Q}}) \\
&= \dim(\widehat{R}_{\widehat{Q}}) - \operatorname{ht}(\widehat{W}) \\
&= \dim(\widehat{R}_{\widehat{Q}}) - \operatorname{ht}(Q(R_n)_Q) \\
&= \dim(\widehat{R}_{\widehat{Q}}) - \operatorname{ht}(P\widehat{R}_{\widehat{Q}}) - \big(\operatorname{ht}(Q(R_n)_Q) - \operatorname{ht}(P(R_n)_Q)\big) \\
&= \dim(\,(\widehat{R}/P\widehat{R})_{\widehat{Q}}) - \dim(\,(R_n/PR_n)_Q).
\end{aligned}$$

Proof of (*b*). Since $R_P \longrightarrow (R-P)^{-1}(R_n)$ is a flat extension we have

$$P(R_n)_Q \cong PR_P \otimes_{R_P} (R_n)_Q.$$

Therefore

$$P(R_n)_Q \otimes_{(R_n)_Q} \widehat{R}_{\widehat{Q}} \cong (PR_P \otimes_{R_P} (R_n)_Q) \otimes_{(R_n)_Q} \widehat{R}_{\widehat{Q}} \cong PR_P \otimes_{R_P} \widehat{R}_{\widehat{Q}} \cong P\widehat{R}_{\widehat{Q}}$$

where the last isomorphism is implied by the flatness of the canonical morphism $R_P \longrightarrow \widehat{R}_{\widehat{Q}}$. □

For ($\Longleftarrow$) of (2), let $P \in \operatorname{Spec}(R_n)$ be a prime ideal with $\dim(R_n/P) \leq n$. Suppose that $P\widehat{R}$ is not **m**-primary and let $\widehat{Q} \supseteq P\widehat{R}$ be a minimal prime divisor of $P\widehat{R}$. Then $\operatorname{ht}(\widehat{Q}) \leq d-1$. Put $Q = \widehat{Q} \cap R_n$. Then LF_{d-1} implies that the morphism

$$\phi_{\widehat{Q}}\colon (R_n)_Q \longrightarrow \widehat{R}_{\widehat{Q}}$$

is faithfully flat. Hence by going-down (5.5.1), $\operatorname{ht}(Q) \leq d-1$. But $P \subseteq Q$ and R_n is catenary, so $d-1 \geq \operatorname{ht}(Q) \geq \operatorname{ht}(P) \geq d$, a contradiction. □

5.7. *Remark.* The results above yield a different proof of statements (1) and (3) of Theorem 4.4, that primarily independent elements are residually algebraically independent and that in dimension two, the two concepts are equivalent. Considering again our basic setting from (2.1), with $d = \dim(R)$, Theorem 5.6 equates the LF_{d-1} condition on the extension $R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \longrightarrow \widehat{R}$, to the primary independence of the τ_i. Also Proposition 5.4 and Theorem 4.3 yield that residual algebraic independence of the τ_i is equivalent to the extension $R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m}, \tau_1, \dots, \tau_n)} \longrightarrow \widehat{R}$ satisfying LF_1. Clearly $LF_i \Longrightarrow LF_{i-1}$, for $i > 1$, and if $d = \dim(R) = 2$, then $LF_{d-1} = LF_1$.

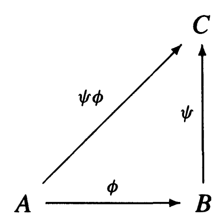
5.8. *Remark.* In the setting of (2.1), if $\tau_1, \dots, \tau_n$ are primarily independent over R and $\dim(R) = d$, then $\phi\colon R_n \longrightarrow \widehat{R}$ satisfies LF_{d-1}, but *not* LF_d; that is, ϕ fails to be faithfully flat. (Faithful flatness would imply going-down and hence $\dim(R_n) \leq d = \dim(\widehat{R})$.)

5.9. *Remark.* By a modification of Example 2.13, it is possible to obtain, for each integer $d \geq 2$, a local injective morphism $\phi\colon (A, \mathbf{m}) \longrightarrow (B, \mathbf{n})$ of normal local Noetherian domains with B essentially of finite type over A, $\phi(\mathbf{m})B = \mathbf{n}$, and $\dim(B) = d$ such that ϕ satisfies LF_{d-1}, but fails to be faithfully flat over A. Let k be a field and let $x_1, \dots, x_d, y$ be indeterminates over k. Let A be the localization of $k[x_1, \dots, x_d, x_1 y, \dots, x_d y]$ at the maximal ideal generated by $(x_1, \dots, x_d, x_1 y, \dots, x_d y)$, and let B be the localization of $A[y]$ at the prime ideal $(x_1, \dots, x_d)A[y]$. Then A is a $(d+1)$-dimensional normal local domain and B is a d-dimensional regular local domain birationally dominating A. For any nonmaximal prime Q of B we have $B_Q = A_{Q\cap A}$. Hence ϕ: $A \longrightarrow B$ satisfies LF_{d-1}, but ϕ is not faithfully flat since $\dim(B) < \dim(A)$. However, this example of a local LF_k-morphism which fails to be faithfully flat also fails to be height-one preserving. As Proposition 2.7 shows the morphisms studied in this paper are automatically height-one preserving, and we believe that this condition is central for our investigations. We do not have an example of a local algebra extension essentially of finite type which is both LF_k and height-one preserving, but fails to be faithfully flat.

6. Composition, base change and polynomial extensions

In this section we investigate idealwise independence, residual algebraic independence, and primary independence under polynomial ring extensions and localizations of these polynomial extensions.

We start with a more general situation. Consider the following commutative diagram of commutative rings and injective morphisms:



We see in (6.1) that many of the properties of injective morphisms we consider are stable under composition of morphisms.

6.1. PROPOSITION. *Let ϕ: $A \longrightarrow B$ and ψ: $B \longrightarrow C$ be injective morphisms of commutative rings.*

(1) *If ϕ and ψ satisfy LF_k, then $\psi\phi$ satisfies LF_k.*
(2) *If C is Noetherian, ψ is faithfully flat and the composite map $\psi\phi$ satisfies LF_k, then ϕ satisfies LF_k.*
(3) *Let A, B and C be Krull domains. Assume that for each height-one prime Q of B, $QC \neq C$. If ϕ and ψ are height-one preserving (respectively weakly flat), then $\psi\phi$ is height-one preserving (respectively weakly flat).*

Proof. The first statement follows from the fact that a flat morphism satisfies going-down [M3, Theorem 4, page 33]. For (2), since C is Noetherian and ψ is faithfully flat, B is Noetherian. Let $Q \in \mathrm{Spec}(B)$ with $\mathrm{ht}(Q) = d \leq k$. We show ϕ_Q: $A_{Q\cap A} \longrightarrow B_Q$ is faithfully flat. By localization of B and C at $B - Q$, we may assume that B is local with maximal ideal Q. Since C is faithfully flat over B, $QC \neq C$. Let $Q' \in \mathrm{Spec}(C)$ be a minimal prime of QC. Since C is Noetherian and

B is local with maximal ideal Q, we have $\mathrm{ht}(Q') \leq d$ and $Q' \cap B = Q$. Since the composite map $\psi\phi$ satisfies LF_k, the composite map

$$A_{Q'\cap A} = A_{Q\cap A} \xrightarrow{\phi_Q} B_Q = B_{Q'\cap B} \xrightarrow{\psi_{Q'}} C_{Q'}$$

is faithfully flat. This and the faithful flatness of $\psi_{Q'}$: $B_{Q'\cap B} \longrightarrow C_{Q'}$ implies that ϕ_Q is faithfully flat [M3, (4.B) page 27].

For (3), let P be a height-one prime of A such that $PC \neq C$. Then $PB \neq B$ so if ϕ and ψ are height-one preserving then there exists a height-one prime Q of B such that $PB \subseteq Q$. By assumption, $QC \neq C$ (and ψ is height-one preserving), so there exists a height-one prime Q' of C such that $QC \subseteq Q'$. Hence $PC \subseteq Q'$.

Now if ϕ and ψ are weakly flat, by (2.10) there exists a height-one prime Q of B such that $Q \cap A = P$. Again by assumption, $QC \neq C$, thus weakly flatness of ψ implies $QC \cap B = Q$. Now

$$P \subseteq PC \cap A \subseteq QC \cap A = QC \cap B \cap A \ = \ Q \cap A = P. \qquad \square$$

6.2. *Remark.* If in (6.1.3) the Krull domains B and C are quasilocal and ψ is a local morphism, then clearly $QC \neq C$ for each height-one prime Q of B.

If a morphism λ of Krull domains is faithfully flat, then λ is a height-one preserving, weakly flat morphism which satisfies condition LF_k for every integer $k \in \mathbf{N}$. Thus if ϕ: $A \longrightarrow B$ and ψ: $B \longrightarrow C$ are injective morphisms of Krull domains, such that one of ϕ or ψ is faithfully flat and the other is weakly flat (respectively height-one preserving or satisfies LF_k), then the composition $\psi\phi$ is again weakly flat (respectively height-one preserving or satisfies LF_k). Moreover, if the morphism ψ is faithfully flat, we also obtain the following converse to (6.1.3):

6.3. PROPOSITION. *Let ϕ: $A \longrightarrow B$ and ψ: $B \longrightarrow C$ be injective morphisms of Krull domains. Suppose that the morphism ψ is faithfully flat. If $\psi\phi$ is height-one preserving* (*respectively weakly flat*), *then ϕ is height-one preserving* (*respectively weakly flat*).

Proof. Suppose that P is a height-one prime ideal of A such that $PB \neq B$. Since ψ is faithfully flat, $PC \neq C$, so if $\psi\phi$ is height-one preserving, then there exists a height-one prime ideal Q' of C containing PC. Now $Q = Q' \cap B$ has height one by going-down for flat extensions, and $PB \subseteq Q' \cap B = Q$, so ϕ is height-one preserving. The proof of the weakly flat statement is similar, using (2.10). $\square$

Next we consider a commutative square of commutative rings and injective morphisms:

$$\begin{array}{ccc} A' & \xrightarrow{\phi'} & B' \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ A & \xrightarrow{\phi} & B. \end{array}$$

6.4. PROPOSITION. *In the diagram above, suppose that μ and ν are faithfully flat. Let $k \in \mathbf{N}$. Then:*

(1) *(Ascent) Suppose $B' = B \otimes_A A'$, or a localization of $B \otimes_A A'$, and ν is the canonical morphism associated with this tensor product. If $\phi\colon A \longrightarrow B$ satisfies LF_k, then $\phi'\colon A' \longrightarrow B'$ satisfies LF_k.*
(2) *(Descent) If B' is Noetherian and $\phi'\colon A' \longrightarrow B'$ satisfies LF_k, then $\phi\colon A \longrightarrow B$ satisfies LF_k.*
(3) *(Descent) Suppose that the rings A, A', B and B' are Krull domains. If $\phi'\colon A' \longrightarrow B'$ is height-one preserving (respectively weakly flat), then $\phi\colon A \longrightarrow B$ is height-one preserving (respectively weakly flat).*

Proof. For (1), assume that ϕ satisfies LF_k; let $Q' \in \operatorname{Spec}(B')$ with $\operatorname{ht}(Q') \leq k$. Put $Q = (\nu)^{-1}(Q')$, $P' = (\phi')^{-1}(Q')$, and $P = \mu^{-1}(P') = \phi^{-1}(Q)$ and consider the commutative diagrams

$$\begin{array}{ccc} A' & \xrightarrow{\phi'} & B' \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\nu}\uparrow \\ A & \xrightarrow{\phi} & B \end{array} \qquad\qquad \begin{array}{ccc} A'_{P'} & \xrightarrow{\phi'_{Q'}} & B'_{Q'} \\ {\scriptstyle\mu_{P'}}\uparrow & & {\scriptstyle\nu_{Q'}}\uparrow \\ A_P & \xrightarrow{\phi_Q} & B_Q. \end{array}$$

The flatness of ν implies that $\operatorname{ht}(Q) \leq k$ and so by assumption, ϕ_Q is faithfully flat. The ring $B'_{Q'}$ is a localization of $B_Q \otimes_{A_P} A'_{P'}$ and B_Q is faithfully flat over A_P implies $B'_{Q'}$ is faithfully flat over $A'_{P'}$.

For (2), by (6.1.1), $\phi'\mu = \nu\phi$ satisfies LF_k. Now by (6.1.2), ϕ satisfies LF_k.

Item (3) follows immediately from the assumption that μ and ν are faithfully flat morphisms and hence going-down holds [M3, Theorem 4, page 33]. □

Next we examine the situation for polynomial extensions.

6.5. PROPOSITION. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in the setting of* (2.1), *where m is either an integer or $m = \infty$, and the dimension of R is at least* 2. *Let z be an indeterminate over $\widehat{R}$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R \iff \{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R[z]_{(\mathbf{m},z)}$.*
(2) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over $R[z]_{(\mathbf{m},z)}$, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

Proof. Let $n \in \mathbf{N}$ be an integer with $n \leq m$ and put $R_n = R[\tau_1, \dots, \tau_n]_{(\mathbf{m},\tau_1,\dots,\tau_n)}$. Let $\phi\colon R_n \longrightarrow \widehat{R}$ and $\mu\colon R_n \longrightarrow R_n[z]$ be the inclusion maps. We have the following commutative diagram:

$$\begin{array}{ccc} R_n[z]_{(\max(R_n),z)} & \xrightarrow{\phi'} & R' = \widehat{R}[z]_{(\widehat{\mathbf{m}},z)} \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu'}\uparrow \\ R_n & \xrightarrow{\phi} & \widehat{R}. \end{array}$$

The ring R' is a localization of the tensor product $\widehat{R} \otimes_{R_n} R_n[z]$ and Proposition 6.4 applies. Thus, for (1), ϕ satisfies LF_1 if and only if ϕ' satisfies LF_1. Since the inclusion map ψ of $R' = \widehat{R}[z]_{(\widehat{\mathbf{m}},z)}$ to its completion $\widehat{R}[[z]]$ is faithfully flat, we obtain equivalences

ϕ satisfies $LF_1 \iff \phi'$ satisfies $LF_1 \iff \psi\phi'$ satisfies LF_1.

For (2), if the τ_i are idealwise independence over $R[z]_{(\mathbf{m},z)}$, the morphism $\psi\phi'$ is weakly flat. Thus ϕ' is weakly flat and the statement follows by (6.1). □

We also obtain:

6.6. PROPOSITION. *Let $A \hookrightarrow B$ be an extension of Krull domains such that for each height-one prime $P \in \operatorname{Spec}(A)$ we have $PB \neq B$, and let Z be a (possibly uncountable) set of indeterminates over A. Then $A \hookrightarrow B$ is weakly flat if and only if $A[Z] \hookrightarrow B[Z]$ is weakly flat.*

Proof. Let F denote the fraction field of A. By (2.14), the extension $A \hookrightarrow B$ is weakly flat if and only if $F \cap B = A$. Thus the assertion follows from $F \cap B = A \iff F(Z) \cap B[Z] = A[Z]$. □

6.7. *Remark.* It would be interesting to know whether the converse of (6.5.2) is true. It is unclear that a localization of a weakly flat morphism is again weakly flat. In other words: Does there exist a weakly flat morphism $\phi\colon A \longrightarrow B$ of Krull domains and a height-one prime $P \in \operatorname{Spec}(A)$ such that PB has a minimal prime divisor Q with $\operatorname{ht}(Q) > 1$?

If so, the map $A \longrightarrow B_Q$ fails to be weakly flat. Note that if P is the radical of a principal ideal, then each minimal prime divisor of PB is of height one.

6.8. *Remark.* Primary independence never lifts to polynomial rings. To see that $\tau \in \mathbf{m}\widehat{R}$ fails to be primarily independent over $R[z]_{(\mathbf{m},z)}$, observe that $\mathbf{m}R[z]_{(\mathbf{m},z)}$ is a dimension-one prime ideal that extends to $\mathbf{m}\widehat{R}[[z]]$, which also has dimension one and is *not* $(\mathbf{m}, z)$-primary in $\widehat{R}[[z]]$. Alternatively, in the language of locally flat morphisms, if the elements $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ are primarily independent over R, then (6.1) implies that the morphism

$$\phi'\colon R_n[z]_{(\max(R_n),z)} \longrightarrow \widehat{R}[[z]]$$

satisfies condition LF_{d-1}, where $d = \dim(R)$. For $\{\tau_i\}_{i=1}^m$ to be primarily independent over $R[z]_{(\mathbf{m},z)}$, however, the morphism ϕ' has to satisfy LF_d, since $\dim R[z]_{(\mathbf{m},z)} = d+1$. Using (6.1) again this forces $\phi\colon R_n \longrightarrow \widehat{R}$ to satisfy condition LF_d and thus ϕ is flat, which can happen only if $n = 0$. This is an interesting phenomenon; the construction of primarily independent elements involves all parameters of the ring R.

In the remainder of this section we consider localizations of polynomial extensions so that the dimension does not increase. Theorem 6.9 gives a method to obtain residually algebraically independent and primarily independent elements over an uncountable excellent local domain. In (6.9) we make use of the fact that if A is a Noetherian ring and Z is a set of indeterminates over A, then the ring $A(Z)$ obtained by localizing the polynomial ring $A[Z]$ at the multiplicative system of polynomials whose coefficients generate the unit ideal of A is again a Noetherian ring [GH, Theorem 6].

6.9. THEOREM. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m \subseteq \widehat{\mathbf{m}}$ be as in the setting of (2.1), where m is either an integer or $m = \infty$, and $\dim(R) = d \geq 2$. Let Z be a set (possibly uncountable) of indeterminates over $\widehat{R}$ and let $R(Z) = R[Z]_{(\mathbf{m}R[Z])}$. Then:*

(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over R $\Longleftrightarrow$ $\{\tau_i\}_{i=1}^m$ is primarily independent over $R(Z)$.*
(2) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R $\Longleftrightarrow$ $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over $R(Z)$.*
(3) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over $R(Z)$, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

Proof. Let $n \in \mathbf{N}$ be an integer with $n \leq m$, put $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m},\tau_1,\ldots,\tau_n)}$ and let $\mathbf{n}$ denote the maximal ideal of R_n. Let $\phi\colon R_n \longrightarrow \widehat{R}$ and $\mu\colon R_n \longrightarrow R_n(Z) = R_n[Z]_{\mathbf{n}R_n[Z]}$ be the inclusion maps. We have the following commutative diagram:

$$\begin{array}{ccc} R_n(Z) & \xrightarrow{\phi'} & \widehat{R}(Z) \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\mu'}\uparrow \\ R_n & \xrightarrow{\phi} & \widehat{R}. \end{array}$$

The ring $\widehat{R}(Z)$ is a localization of the tensor product $\widehat{R} \otimes_{R_n} R_n[Z]$ and Proposition 6.4 applies. Thus, for (1), ϕ satisfies LF_{d-1} if and only if ϕ' satisfies LF_{d-1}. Similarly, for (2), ϕ satisfies LF_1 if and only if ϕ' satisfies LF_1.

Since the inclusion map ψ taking $\widehat{R}(Z)$ to its completion is faithfully flat, we obtain these equivalences:

ϕ satisfies $LF_k \Longleftrightarrow \phi'$ satisfies $LF_k \Longleftrightarrow \psi\phi'$ satisfies LF_k.

Since primary independence is equivalent to LF_{d-1} by (5.6) and residual algebraic independence is equivalent to LF_1 by (5.4), statements (1) and (2) follow.

For (3), if the τ_i are idealwise independence over $R(Z)$, the morphism $\psi\phi'$ is weakly flat. Thus ϕ' is weakly flat and the statement follows by (6.1). □

6.10. COROLLARY. *Let k be a countable field, let Z be an uncountable set of indeterminates over k and let x, y be additional indeterminates. Let* $R = k(Z)[x, y]_{(x,y)}$. *Then R is an uncountable excellent normal local domain and, for m a positive integer or* $m = \infty$, *there exist m primarily independent elements (and hence also residually algebraically and idealwise independent elements) over R.*

Proof. Apply (3.9), (4.4) and (6.9). □

7. Passing to the Henselization

In this section we investigate idealwise independence, residual algebraic independence, and primary independence as we pass from R to the Henselization R^h of R. In particular, we show in Proposition (7.5) that for a single element $\tau \in \mathbf{m}\widehat{R}$ the notions of idealwise independence and residual algebraic independence coincide if $R = R^h$. This implies that for every excellent normal local Henselian domain of dimension 2 all three concepts coincide for an element $\tau \in \mathbf{m}\widehat{R}$; that is, τ is idealwise independent $\iff$ τ is residually algebraically independent $\iff$ τ is primarily independent.

We use the commutative square of (6.4) and obtain the following result for Henselizations.

7.1. PROPOSITION. *Let* ϕ: $(A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ *be an injective local morphism of normal local Noetherian domains, and let* ϕ^h: $A^h \longrightarrow B^h$ *denote the induced morphism of the Henselizations. Then:*

(1) ϕ *satisfies* $LF_k \iff \phi^h$ *satisfies* LF_k, *for each k with* $1 \le k \le \dim(B)$. *Thus, in particular,* ϕ *satisfies PDE* $\iff$ ϕ^h *satisfies PDE.*
(2) *(Descent) If* ϕ^h *is height-one preserving (respectively weakly flat), then* ϕ *is height-one preserving (respectively weakly flat).*

Using shorthand and diagrams, we show (7.1) schematically:

$$\boxed{\phi \text{ is } LF_k} \longleftrightarrow \boxed{\phi^h \text{ is } LF_k} \;;\; \boxed{\phi \text{ is PDE}} \longleftrightarrow \boxed{\phi^h \text{ is PDE}}$$

$$\boxed{\phi \text{ ht-1 pres}} \longleftarrow \boxed{\phi^h \text{ ht-1 pres}} \;;\; \boxed{\phi \text{ w.f.}} \longleftarrow \boxed{\phi^h \text{ w.f.}}.$$

Proof of (7.1). Consider the commutative diagram

$$\begin{array}{ccc} A^h & \xrightarrow{\phi^h} & B^h \\ {\scriptstyle\mu}\uparrow & & {\scriptstyle\nu}\uparrow \\ A & \xrightarrow{\phi} & B \end{array}$$

where μ and ν are the faithfully flat canonical injections [N, (43.8), page 182]. Since ϕ is injective and A is normal, ϕ^h is injective. By (6.4.2), (5.4) and (6.4.3) we need only show ($\Rightarrow$) in (1).

Let $Q' \in \operatorname{Spec}(B^h)$ with $\operatorname{ht}(Q') \leq k$. Put $Q = Q' \cap B$, $P' = Q' \cap A^h$, and $P = P' \cap A$. We consider the localized diagram

$$\begin{array}{ccc} A^h_{P'} & \xrightarrow{\phi^h_{Q'}} & B^h_{Q'} \\ {\scriptstyle \mu_{P'}}\uparrow & & {\scriptstyle \nu_{Q'}}\uparrow \\ A_P & \xrightarrow{\phi_Q} & B_Q. \end{array}$$

The faithful flatness of ν implies $\operatorname{ht}(Q) \leq k$.

In order to show that $\phi^h_{Q'}\colon A^h_{P'} \longrightarrow B^h_{Q'}$ is faithfully flat, we apply (5.5.2). First note that P' is a minimal prime divisor of PA^h and that $(A^h/PA^h)_{P'} = (A^h/P')_{P'}$ is a field [N, (43.20)]. Thus

$$\overline{\phi^h_{Q'}}\colon (A^h/PA^h)_{P'} \longrightarrow (B^h/PB^h)_{Q'}$$

is faithfully flat and it remains to show that

$$PA^h_{P'} \otimes_{A^h_{P'}} B^h_{Q'} \cong PB^h_{Q'}.$$

This can be seen as follows:

$$\begin{aligned} PA^h_{P'} \otimes_{A^h_{P'}} B^h_{Q'} &\cong (P \otimes_{A_P} A^h_{P'}) \otimes_{A^h_{P'}} B^h_{Q'} && \text{by flatness of } \mu \\ &\cong P \otimes_{A_P} B^h_{Q'} \\ &\cong (P \otimes_{A_P} B_Q) \otimes_{B_Q} B^h_{Q'} \\ &\cong PB_Q \otimes_{B_Q} B^h_{Q'} && \text{by flatness of } \phi_Q \\ &\cong PB^h_{Q'} && \text{by flatness of } \nu. \end{aligned}$$ □

7.2. COROLLARY. *Let* $(R, \mathbf{m})$ *and* $\{\tau_i\}_{i=1}^m$ *be as in the setting of* (2.1), *where* m *is either a positive integer or* $m = \infty$ *and* $\dim(R) = d \geq 2$. *Then*:

(1) $\{\tau_i\}_{i=1}^m$ *is primarily independent over* $R \iff \{\tau_i\}_{i=1}^m$ *is primarily independent over* R^h.
(2) $\{\tau_i\}_{i=1}^m$ *is residually algebraically independent over* $R \iff \{\tau_i\}_{i=1}^m$ *is residually algebraically independent over* R^h.
(3) *(Descent) If* $\{\tau_i\}_{i=1}^m$ *is idealwise independent over* R^h *then* $\{\tau_i\}_{i=1}^m$ *is idealwise independent over* R.

Proof. For (1) and (2) it suffices to show the equivalence for every positive integer $n \leq m$. Note that the local rings $R_n = R[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ and $\tilde{R}_n =$

$R^h[\tau_1, \ldots, \tau_n]_{(\mathbf{m}, \tau_1, \ldots, \tau_n)}$ have the same Henselization which we denote R_n^h. Also $R_n \subseteq \tilde{R}_n$. By (5.6) and (7.1):

$\tau_1, \ldots, \tau_n$ are primarily (respectively residually algebraically) independent over R

$$\Leftrightarrow R_n \longrightarrow \widehat{R} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1)$$
$$\Leftrightarrow R_n^h \longrightarrow \widehat{R} = \widehat{R^h} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1)$$
$$\Leftrightarrow \tilde{R}_n \longrightarrow \widehat{R} \text{ satisfies } LF_{d-1} \text{ (respectively } LF_1).$$

The third statement on idealwise independence follows from (6.4.3) by considering

$$\begin{array}{ccc} \tilde{R}_n & \xrightarrow{\phi'} & \widehat{R} \\ {\scriptstyle\mu}\uparrow & & \| \\ R_n & \xrightarrow{\phi} & \widehat{R}. \end{array}$$

□

7.3. *Remark.* The examples given in (4.7) and (4.9) show the converse to part (3) of (7.2) fails; weak flatness need not lift to the Henselization. With the notation of (7.1), if ϕ is weakly flat, then for every $P \in \operatorname{Spec}(A)$ of height one with $PB \neq B$ there exists by (2.10), $Q \in \operatorname{Spec}(B)$ of height one such that $P = Q \cap A$. In the Henselization A^h of A, the ideal PA^h is a finite intersection of height-one prime ideals P_i' of A^h [N, (43.20)]. Only one of the P_i' is contained in Q. Thus as in (4.7) and (4.9), one of the minimal prime divisors P_i' may fail the condition for weak flatness.

Let R be an excellent normal local domain and let K, respectively K^h, denote the fraction fields of R, respectively R^h. Let L be an intermediate field with $K \subseteq L \subseteq K^h$. It is shown in [R4] that the intersection ring $T = L \cap \widehat{R}$ is an excellent local normal domain with Henselization $T^h = R^h$. Excellent, Henselian, local, normal domains are algebraically closed in their completion and we obtain the next result.

7.4. COROLLARY. *Let $(R, \mathbf{m})$ and $\{\tau_i\}_{i=1}^m$ be as in the setting of* (2.1), *where m denotes a positive integer or $m = \infty$. Suppose that T is a local Noetherian domain dominating and algebraic over R and dominated by $\widehat{R}$ with $\widehat{R} = \widehat{T}$. Then*:

(1) *$\{\tau_i\}_{i=1}^m$ is primarily independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is primarily independent over T.*
(2) *$\{\tau_i\}_{i=1}^m$ is residually algebraically independent over R $\iff$ $\{\tau_i\}_{i=1}^m$ is residually algebraically independent over T.*
(3) *If $\{\tau_i\}_{i=1}^m$ is idealwise independent over T, then $\{\tau_i\}_{i=1}^m$ is idealwise independent over R.*

Proof. As mentioned above, R and T have a common Henselization and the statement follows by (7.2). □

We have seen in (4.4) that if $\tau \in \mathbf{m}\widehat{R}$ is residually algebraically independent over R, then τ is idealwise independent over R. In Proposition 7.5 we show that if R is Henselian or, more generally, if height-one prime ideals of R do not split in the completion of R, then idealwise independence and residual algebraic independence are equivalent for a single element τ in $\widehat{R}$. There is an example in [AHW] of a normal local domain R which is not Henselian but for each prime ideal P of R of height-one, the domain R/P is Henselian.

7.5. PROPOSITION. *Let $(R, \mathbf{m})$ and $\tau \in \widehat{m}$ be as in the setting of (2.1). Suppose R has the property that for each $P \in \operatorname{Spec}(R)$ with $\operatorname{ht}(P) = 1$, the ideal $P\widehat{R}$ is prime. Then τ is residually algebraically independent over $R \iff \tau$ is idealwise independent over R.*

In particular, if R is Henselian or if R/P is Henselian for each height-one prime P of R, then τ is residually algebraically independent over $R \iff \tau$ is idealwise independent over R.

Proof. By (4.4) it is enough to show τ idealwise independent $\Longrightarrow \tau$ is residually algebraically independent. Let $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ such that $\operatorname{ht}(\widehat{P}) = 1$ and $\widehat{P} \cap R \neq 0$. Then $\operatorname{ht}(\widehat{P} \cap R) = 1$ and $(\widehat{P} \cap R)R_1$ is a prime ideal of $R_1 = R[\tau]$ of height 1. Idealwise independence of τ implies that $(\widehat{P} \cap R)R_1 = (\widehat{P} \cap R)R_1\widehat{R} \cap R_1$. Since $(\widehat{P} \cap R)\widehat{R}$ is nonzero and prime, we have $\widehat{P} = (\widehat{P} \cap R)\widehat{R}$ and $\widehat{P} \cap R_1 = (\widehat{P} \cap R)R_1$. Therefore $\operatorname{ht}(\widehat{P} \cap R_1) = 1$ and Theorem 4.3.2 implies that τ is residually algebraically independent over R.

For the last statement, suppose that P is a height-one prime of R such that R/P is Henselian. Then the integral closure of the domain R/P in its fraction field is again local, in fact an excellent normal local domain and so analytically normal. But this implies that the extended ideal $P\widehat{R}$ is prime, because of the behavior of completions of finite integral extensions [N, (17.7), (17.8)]. □

Apparently (7.5) cannot be extended to more than one algebraically independent element $\tau \in \mathbf{m}\widehat{R}$, because even when R is Henselian, the localized polynomial ring $R[\tau]_{(\mathbf{m},\tau)}$ fails to be Henselian.

7.6. COROLLARY. *If R is an excellent Henselian normal local domain of dimension 2, then τ is idealwise independent over $R \iff \tau$ is residually algebraically independent over $R \iff \tau$ is primarily independent over R.*

Proof. This follows from (7.5) and (4.4.3). □

8. Summary diagram for the independence concepts

With the notation of (2.1) for R, $\mathbf{m}$, R_n, $\tau_1 \ldots, \tau_n$, let $d = \dim(R)$, L the quotient field of R_n, $\mathbf{p} \in \operatorname{Spec}(R_n)$ such that $\dim(R_n/\mathbf{p}) \leq d - 1$, $P \in \operatorname{Spec}(R_n)$ with

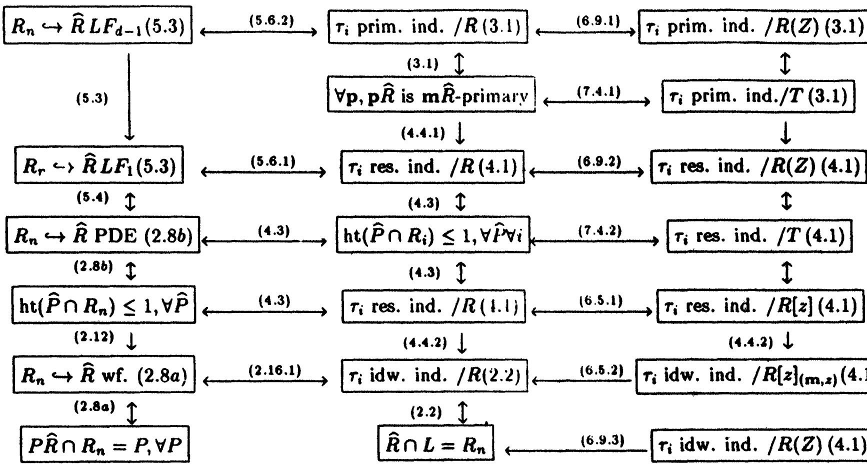


Figure 3. Independence concepts and results

ht(P) = 1, $\widehat{P} \in \operatorname{Spec}(\widehat{R})$ with ht($\widehat{P}$) = 1, R^h the Henselization of R in $\widehat{R}$, T a local Noetherian domain dominating and algebraic over R and dominated by $\widehat{R}$ with $\widehat{R} = \widehat{T}$, z an indeterminate over the quotient field of $\widehat{R}$ and Z a possibly uncountable set of set of indeterminates over the quotient field of $\widehat{R}$. Then we have the implications in Fig. 3. We use the abbreviations "prim. ind.", "res. ind." and "idw. ind" for "primarily independent", "residually independent" and "idealwise independent".

Note. $R_n \hookrightarrow \widehat{R}$ is always height-one preserving by (2.7).

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