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ABEL SUMMABILITY OF JACOBI TYPE SERIES

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1. Introduction

A brief description of the Jacobi type series is as follows: Let $\mathbf{P}_n^{(\alpha,\beta)}(y)$ be the n-th normalized Jacobi polynomial of parameters α, β; namely

$$\int_{-1}^{1} \mathbf{P}_n^{(\alpha,\beta)}(y)\mathbf{P}_\ell^{(\alpha,\beta)}(y)(1-y)^\alpha(1+y)^\beta dy = \delta_{n,\ell},$$

for $\alpha, \beta > -1$. For their definitions and estimates see [S]. The m-dimensional Jacobi polynomial of order $n = (n_1, \dots, n_m)$ is given by

$$\mathbf{P}_n^{(\alpha,\beta)}(X) = \prod_{j=1}^{m} \mathbf{P}_{n_j}^{(\alpha_j,\beta_j)}(x_j),$$

where $X = (x_1, \dots, x_m) \in \mathbb{R}^m$, $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m)$, $\alpha_j > -1$, $\beta_j > -1$, $j = 1, \dots, m$. We will let $J^{(\alpha,\beta)}$ denote the measure defined on the cube $Q = [-1, 1] \times \cdots \times [-1, 1] = [-1, 1]^m$ by

$$dJ^{(\alpha,\beta)} = \prod_{j=1}^{m} \left(1-y_j\right)^{\alpha_j} \left(1+y_j\right)^{\beta_j} dy_1 \cdots dy_m.$$

Clearly,

$$\int_Q \mathbf{P}_n^{(\alpha,\beta)}(X)\mathbf{P}_\ell^{(\alpha,\beta)}(X)\, dJ^{(\alpha,\beta)} = \delta_{n,\ell} = \delta_{(n_1,\dots,n_m),(\ell_1,\dots,\ell_m)}.$$

Likewise, we introduce the Jacobi functions, namely,

$$\mathbf{F}_n^{(\alpha,\beta)}(y) = \mathbf{P}_n^{(\alpha,\beta)}(y)(1-y)^{\alpha/2}(1+y)^{\beta/2},$$

for $\alpha, \beta \geq 0$, $y \in \mathbb{R}$. It is immediate that these functions are orthonormal with respect to the Lebesgue measure on the interval $[-1, 1]$; i.e.,

$$\int_{-1}^{1} \mathbf{F}_n^{(\alpha,\beta)}(y)\mathbf{F}_\ell^{(\alpha,\beta)}(y)dy = \delta_{n,\ell}.$$

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The m-dimensional Jacobi functions are

$$\mathbf{F}_n^{(\alpha,\beta)}(X) = \prod_{j=1}^{m} \mathbf{F}_{n_j}^{(\alpha_j,\beta_j)}(x_j),$$

where $X = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $j = 1, \ldots, m$ and $n = (n_1, \ldots, n_m)$.

Remark 1. Throughout this paper, all the single and m-dimensional parameters α and β will be non negative.

We shall be concerned with multiple Jacobi functions series of the type

$$\sum C_{n_1,\ldots,n_m} \mathbf{F}_{n_1}^{(\alpha_1,\beta_1)}(x_1) \cdots \mathbf{F}_{n_m}^{(\alpha_m,\beta_m)}(x_m),$$

where $\alpha_j \geq 0$, $\beta_j \geq 0$, $j = 1, 2, \ldots, m$, and

$$\tilde{C}_{n_1,\ldots,n_m}(f) = \int_Q f \, \mathbf{F}_n^{(\alpha,\beta)} dY,$$

for $f \in L^1(Q)$. For short we write

$$\sum_{n=(n_1,\ldots,n_m)} \tilde{C}_n \, \mathbf{F}_n^{(\alpha,\beta)}(X) \, .$$

The Abel summability of the multiple Jacobi functions series is given by

$$\lim_{(r_1,\ldots,r_m)\to(1^-,\ldots,1^-)} \sum r_1^{n_1} \cdots r_m^{n_m} \tilde{C}_{n_1,\ldots,n_m} \mathbf{F}_{n_1}^{(\alpha_1,\beta_1)}(x_1) \cdots \mathbf{F}_{n_m}^{(\alpha_m,\beta_m)}(x_m),$$

whenever this limit exists. The Abel approximation will be denoted by

$$\begin{aligned} \tilde{f}(r, X) &= \tilde{f}(r_1, \ldots, r_m, x_1, \ldots, x_m) \\ &= \sum_{n_1,\ldots,n_m} r_1^{n_1} \cdots r_m^{n_m} \tilde{C}_{n_1,\ldots,n_m} \mathbf{F}_{n_1}^{(\alpha_1,\beta_1)}(x_1) \cdots \mathbf{F}_{n_m}^{(\alpha_m,\beta_m)}(x_m). \end{aligned}$$

Likewise, we define the coefficients with respect to the normalized Jacobi polynomials:

$$C_{n_1,\ldots,n_m}(f) = \int_Q f \, \mathbf{P}_{n_1,\ldots,n_m}^{(\alpha,\beta)} \, dJ^{(\alpha,\beta)}.$$

The Abel approximation for the Jacobi series is given by

$$\begin{aligned} f(r, X) &= f(r_1, \ldots, r_m, x_1, \ldots, x_m) \\ &= \sum_{n_1,\ldots,n_m} r_1^{n_1} \cdots r_m^{n_m} C_{n_1,\ldots,n_m} \mathbf{P}_{n_1}^{(\alpha_1,\beta_1)}(x_1) \cdots \mathbf{P}_{n_m}^{(\alpha_m,\beta_m)}(x_m), \end{aligned}$$

and its restricted maximal operator is

$$f^{***}(x_1, \dots, x_m) = \sup_{(r_1,\dots,r_m)} |f(r_1, \dots, r_m, x_1, \dots, x_m)| : 0 < r_j < 1,$$

$$\tfrac{1}{M} \le \tfrac{1-r_i}{1-r_j} \le M, i, j = 1, \dots, m.$$

The well-known estimates on the Jacobi polynomials allow us to write the Abel approximation as the following integral (see L. A. Caffarelli-C. P. Calderón, [CC2] p. 278):

$$f(r, X) = f(r_1, \dots, r_m, x_1, \dots, x_m) = \int_Q \mathbf{K}^{(\alpha,\beta)}(r, X, Y)\ f(Y)\, dJ^{(\alpha,\beta)},$$

for $0 < r_j < 1$, $j = 1, \dots, m$. Here

$$\mathbf{K}^{(\alpha,\beta)}(r, X, Y) = \prod_{j=1}^{m} \mathbf{K}^{(\alpha_j,\beta_j)}(r_j, x_j, y_j)$$

is the multiple Watson kernel for the Jacobi polynomials. For an expression of the one-dimensional Watson kernel see Bateman [B], p. 272, and §4 below. We shall alternatively use the Watson kernel for the Jacobi functions and the Watson kernel for the Jacobi polynomials. The first kernel can be obtained from the second one by multiplication by the factor $(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}$. Thus, the Abel approximation for the Jacobi functions series can be expressed as

$$\tilde{f}(r, X) = \tilde{f}(r_1, \dots, r_m, x_1, \dots, x_m) = \int_Q \tilde{\mathbf{K}}^{(\alpha,\beta)}(r, X, Y)\ f(Y)\, dY, \qquad (1.1)$$

where $\tilde{\mathbf{K}}^{(\alpha,\beta)}(r, x, y) = \mathbf{K}^{(\alpha,\beta)}(r, x, y)(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}$, is the modified Watson kernel for Jacobi functions.

We consider the maximal operator

$$f^{**}(x_1, \dots, x_m) = \sup_{(r_1,\dots,r_m)} \left|\tilde{f}(r_1, \dots, r_m, x_1, \dots, x_m)\right|,$$

$$0 < r_j < 1, \frac{1}{M} \le \frac{1-r_i}{1-r_j} \le M, i, j = 1, \dots, m,$$

for a given constant $M > 1$. The unrestricted maximal operator is defined as

$$f^{*}(x_1, \dots, x_m) = \sup_{(r_1,\dots,r_m)} \left|\tilde{f}(r_1, \dots, r_m, x_1, \dots, x_m)\right|, 0 < r_j < 1, i, j = 1, \dots, m.$$

The properties of these maximal operators are the key to understanding and proving convergence a.e. of the Abel approximation. The first result in this direction was published in 1974 by L.A. Caffarelli and C.P. Calderón, [CC2], where they developed a method for handling the maximal operator for the restricted Abel sums of arbitrary $L^1(J^{(\alpha,\beta)})$ functions. Their result is the L^1-weak type estimate for the restricted maximal operator, for details we refer the reader to [CC2] and also §6 below. One of the results in this paper is the weak type estimate for the maximal operator of the restricted Abel sums of multiple Jacobi functions series of arbitrary $L^1\,(Q)$ functions. In order to achieve this result we use two different methods. The first one, which is argued in §5 below, uses an approach based on a domination of the modified Watson kernel by an infinite superposition of Poisson type kernels, namely,

$$\begin{aligned}\tilde{K}^{(\alpha,\beta)}(r,x,y) \;\leq\; & C(\alpha,\beta) + C(\alpha,\beta)\sum_{j=0}^{\infty} b_j \frac{2^j\varphi(r)}{(x-y)^2 + 2^{2j}\varphi^2(r)} \\ & + C(\alpha,\beta)\sum_{j=0}^{\infty} b_j \frac{2^j\varphi(r)\left(2^j\varphi(r) + h(x)\right)}{\left((x-y)^2 + 2^j\varphi(r)\left(2^j\varphi(r)+h(x)\right)\right)^{3/2}},\end{aligned}$$

where $b_j \sim 2^{-j/2}$, $\varphi(r) = k-1$ and $h(x) = 1-x$. This method deals basically with the Lebesgue measure case.

The second method is an adaptation of the corresponding one in [CC2] and is discussed in §6 below. In this context, it gives a generalization of the main theorem in [CC2] to a family of measures given by the weights

$$dJ^{(\bar{\alpha},\bar{\beta})} = \prod_{j=1}^{m}(1-y_j)^{\bar{\alpha}_j}(1+y_j)^{\bar{\beta}_j}\,dy_1\cdots dy_m,$$

where $\bar{\alpha}_j \geq \alpha_j \geq 0$, $\bar{\beta}_j \geq \beta_j \geq 0$, $j = 1,\ldots,m$. The main tool in this approach is the following domination of the Watson kernel:

$$K^{\alpha,\beta}(r,X,Y) \leq C \sum_{n=(n_1,\ldots,n_m)} \frac{1}{2^{n_1/2}\cdots 2^{n_m/2}} \frac{1}{\mu\{I_n(X,r)\}} \chi_{I_n(X,r)}(Y),$$

for $n_j \in \mathbf{Z}$, $n_j \geq -1$; where

$$I_n(X,r) = I_{n_1}(x_1,r_1)\times\cdots\times I_{n_m}(x_m,r_m), \quad I_{-1}(x_i,r_i) = [-1,1],$$

$\chi_{I_n(X,r)}$ is the characteristic function of $I_n(X,r)$, the $I_{n_i}(x_i,r_i)$'s are suitable intervals and μ is either the Lebesgue measure restricted to Q or the m-dimensional Jacobi measure $J^{(\bar{\alpha},\bar{\beta})}$, $\bar{\alpha}_j \geq \alpha_j$, $\bar{\beta}_j \geq \beta_j$, with only a possible change in the constant $C = C(\alpha,\beta,\bar{\alpha},\bar{\beta})$. $K^{\alpha,\beta}(r,X,Y)$ denotes the modified Watson kernel $\tilde{K}$ in the case of the Lebesgue measure.

We have included in §4 and §6 below and in the Appendix some results that were stated in [CC2] with proofs merely outlined or omitted in the simplest cases. It is not only in the spirit of further clarification that these proofs have been included here in detail; in fact, the density and complexity of the exposition in [CC2] make some repetition unavoidable.

As indicated above the method of §6 below gives both results, Theorem 2.1 and Theorem 2.2. In fact, the estimates for the Watson kernel are obtained simultaneously for both scenarios in the various auxiliary lemmata in §6, for $\alpha \geq 0$ and $\beta \geq 0$. In §5, as pointed out above, a different approach to estimate the Watson kernel is found and discussed throughout. This is the first of a series of papers on these and related topics.

The paper is organized as follows: In §2 the main results are stated; §3 includes some auxiliary lemmas; §4 provides some estimates for the single Watson kernel. In §5 a theory for the Lebesgue measure case is discussed, while in §6 a unified theory for both measures is given. Finally, we have added the Appendix in §7.

2. Main results

THEOREM 2.1. *If $f \in L^1(Q)$ and*

$$f^{**}(x_1, \dots, x_m) = \sup_{(r_1,\dots,r_m)} \left| \tilde{f}(r_1, \dots, r_m, x_1, \dots, x_m) \right|,$$

$$0 < r_j < 1, \frac{1}{M} \leq \frac{1 - r_i}{1 - r_j} \leq M, i, j = 1, \dots, m,$$

where

$$\tilde{f}(r_1, \dots, r_m, x_1, \dots, x_m) = \sum_{n_1,\dots,n_m} r_1^{n_1} \cdots r_m^{n_m} \tilde{C}_{n_1,\dots,n_m} \mathbf{F}_{n_1}^{(\alpha_1,\beta_1)}(x_1) \cdots \mathbf{F}_{n_m}^{(\alpha_m,\beta_m)}(x_m),$$

and

$$\tilde{C}_{n_1,\dots,n_m}(f) = \int_Q f\, \mathbf{F}_{n_1,\dots,n_m}^{(\alpha,\beta)}\, dY,$$

for $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m)$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $j = 1, \dots, m$, then

(*i*) $|\{f^{**} > \lambda\} \cap Q| < \frac{C}{\lambda} \|f\|_1, \lambda > 0.$

Here C is an independent constant. Furthermore, $\tilde{f}(r_1, \dots, r_m, x_1, \dots, x_m)$ converges a.e. to $f(x_1, \dots, x_m)$ as $(r_1, \dots, r_m) \to (1^-, \dots, 1^-)$, restrictedly; that is, when $\frac{1}{M} \leq \frac{1-r_i}{1-r_j} \leq M$, $i, j = 1, \dots, m$, for some fixed constant $M > 1$. If $f \in L^1(\log^+ L)^{m-1}$, then the condition of restricted convergence can be relaxed to convergence. Moreover, an estimate in the spirit of Jessen-Marcinkiewicz-Zygmund inequality is valid:

(*ii*) $\int_Q (f^*)^\gamma\, dx \leq C_1 + C_2 \int_Q |f| \left(\log^+ |f|\right)^{m-1} dx,$

where $0 < \gamma < 1$, C_1 and C_2 depend on γ only.

If $p > 1$, *then*

(*iii*) $\int\limits_Q (f^*)^p \, dx \leq C_p \int\limits_Q |f|^p \, dx,$

whenever f *belongs to* $L^p(Q)$.

As indicated in the introduction, the second method used to prove the above theorem also gives the following result:

THEOREM 2.2. *Let* $f \in L^1(J^{(\alpha,\beta)})$, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m)$, $\alpha_j \geq 0$, $\beta_j \geq 0$, $j = 1, \ldots, m$,

$$C_{n_1,\ldots,n_m}(f) = \int_Q f \, \mathbf{P}^{(\alpha,\beta)}_{n_1,\ldots,n_m}(Y) \, dJ^{(\alpha,\beta)}$$

and let

$$f(r_1, \ldots, r_m, x_1, \ldots, x_m) = \sum_{n_1,\ldots,n_m} r_1^{n_1} \cdots r_m^{n_m} C_{n_1,\ldots,n_m} \mathbf{P}^{(\alpha_1,\beta_1)}_{n_1}(x_1) \cdots \mathbf{P}^{(\alpha_m,\beta_m)}_{n_m}(x_m).$$

Then, its restricted maximal operator,

$$f^{***}(x_1, \ldots, x_m) = \sup_{(r_1,\ldots,r_m)} |f(r_1, \ldots, r_m, x_1, \ldots, x_m)| : 0 < r_j < 1,$$

$$\frac{1}{M} \leq \frac{1 - r_i}{1 - r_j} \leq M, i, j = 1, \ldots, m$$

satisfies

$$J^{(\bar{\alpha},\bar{\beta})}\left\{f^{***} > \lambda\right\} \leq \frac{C}{\lambda} \int_Q |f| \, dJ^{(\bar{\alpha},\bar{\beta})},$$

for any $\lambda > 0, \bar{\alpha}_j \geq \alpha_j \geq 0,$ $\bar{\beta}_j \geq \beta_j \geq 0,$ $j = 1, \ldots, m$, *for some independent positive constant* $C = C(\alpha, \beta, \bar{\alpha}, \bar{\beta})$.

Remark 2. Here $dJ^{(\bar{\alpha},\bar{\beta})}$ stands for a whole family of measures with parameters $\bar{\alpha}$ and $\bar{\beta}$, $\bar{\alpha}_j \geq \alpha_j \geq 0,$ $\bar{\beta}_j \geq \beta_j \geq 0,$ $j = 1, \ldots, m$, where α_j and β_j are the original parameters of the Jacobi series in question. As a particular case we get the known result already proved by L. A. Caffarelli and C. P. Calderón [CC2] in 1974.

3. Auxiliary lemmas

LEMMA 3.1. *Let* S *be a bounded set in* $\mathbb{R}^m$. *Suppose that for each* $X \in S$ *there is associated a non-degenerate rectangle* $R(X)$ *with edges parallel to the coordinate* *axes and center X, such that the edge parallel to the j-th axis has length given by $h_j(t)$, $j = 1, \dots, m$. The functions $h_j(t)$ are assumed to be continuous, non-negative and satisfying the monotonicity type condition*

$$h_j(t_1) \leq k_j h_j(t_2) \text{ whenever } t_1 \leq t_2,\ 0 \leq t_1, t_2 \in \mathbb{R},$$

for some positive constants k_j depending only on j; $h_j(t) > 0$ for $t > 0$, $h_j(0) = 0$, $h_j(t) \to \infty$ as $t \to \infty$, for $j = 1, \dots, m$.

Then there exists a denumerable subfamily $\{R(X_n)\}$ of rectangles that covers S such that each $X \in \mathbb{R}^m$ belongs to at most

$$2^m m! \prod_{j=1}^{m} \left(2 + \log_2 \left(1 + k_j\right)\right)$$

rectangles. Here $\log_2$ stands for the logarithm to the basis 2.

The proof of this lemma can be found in L.A. Caffarelli-C.P. Calderón [CC1], pp. 222–223. A consequence of this lemma is the following one, whose proof is in L. A. Caffarelli-C. P. Calderón, [CC2], p. 279.

LEMMA 3.2. *Let S be a bounded set in $\mathbb{R}^m$ such that for each X belonging to S there is a non-degenerate rectangle $R(X)$ associated with it, with edges parallels to the coordinate axes and center X, such that the edge parallel to the j-th axis has length given by*

$$K_j \varphi_j^{1/2}(t) \left[h_j(x_j) + \varphi_j(t)\right]^{1/2},$$

where $t = t(X)$ is a parameter and h_j is a function that depends on x_j only, satisfying the Lipschitz condition

$$\left|h_j(s_1) - h_j(s_2)\right| < C_j \left|s_1 - s_2\right|, \qquad C_j > 0, j = 1, \dots, m.$$

The $\varphi_j(t)$ are increasing functions of the parameter $t \geq 0$, continuous at $t = 0$, $\varphi_j(0) = 0$, $j = 1, \dots, m$.

Then there exists a denumerable subfamily $\{R(X_n)\}$ of rectangles that covers S and such that each $X \in \mathbb{R}^m$ belongs to at most

$$C(m) \prod_{j=1}^{m} \left(1 + \log_2 \left(1 + C_j K_j\right)\right)$$

of such rectangles.

LEMMA 3.3. *Let $\mu_i \geq 0$ be a finite measure on the interval $[-1,1]$, $i = 1, \dots, m$, and let μ be the product measure $\mu_1 \times \cdots \times \mu_m$ on $Q = [-1,1] \times \cdots \times [-1,1]$. Let $Q = Q_1 \times Q_2$, where $Q_1 = [-1,1]^j$ and $Q_2 = [-1,1]^{m-j}$, and let $\nu_1 = \mu_1 \times \cdots \times \mu_j$, $\nu_2 = \mu_{j+1} \times \cdots \times \mu_m$, $1 \leq j \leq m$. Let f be a function belonging to $L^1(Q)$ and consider the maximal operator*

$$f_n^*(X) = \sup_{t>0} \frac{1}{\nu_1\left(I_{n_1}(x_1,t) \times \cdots \times I_{n_j}(x_j,t)\right)} \times \int\limits_{I_{n_1}(x_1,t)\times\cdots\times I_{n_j}(x_j,t)} \left[\frac{1}{\nu_2(Q_2)} \int\limits_{Q_2} |f| \, d\nu_2\right] d\nu_1.$$

where $I_{n_i}(x_i,t)$ is the interval

$$\left[x_i - K_i\varphi_i^{1/2}(t)\left(h_i(x_i)+\varphi_i(t)\right)^{1/2}, x_i + K_i\varphi_i^{1/2}(t)\left(h_i(x_i)+\varphi_i(t)\right)^{1/2}\right] \cap [-1,1],$$

$i = 1, \dots, j$. Then

$$\mu\left\{f_n^*(X) > \lambda\right\} \leq \frac{C(m)\prod_{i=1}^{j}\left(1+\log_2(1+C_iK_i)\right)}{\lambda} \int\limits_Q |f| \, d\mu,$$

Here $h_i(x_i)$, $\varphi_i(t)$, C_i and K_i are the functions and constants already defined in Lemma 3.2.

Proof. It follows from the standard procedure applied to

$$g\left(x_1, \dots, x_j\right) = \frac{1}{\nu_2(Q_2)} \int\limits_{Q_2} |f| \, d\nu_2,$$

by using Lemma 3.2. The passage from Q_1 to the whole cube Q is immediate after taking $\bar{g}(x_1, \dots, x_j, x_{j+1}, \dots, x_m) = g(x_1, \dots, x_j)$. $\square$

4. Auxiliary estimates

From [B], p. 272, the single Watson kernel for the Jacobi polynomials can be written as

$$K^{\alpha,\beta}(r,x,y) = r^{(1-\alpha-\beta)/2}\frac{d}{dr}\left(k^{1+\alpha+\beta}\int_0^{\pi/2} \frac{\sec^{2+\alpha+\beta}\omega\,\cos(\alpha-\beta)\,\omega}{Z_1^{\alpha}Z_2^{\beta}Y}\,d\omega\right),$$

where $k = \frac{1}{2}(r^{\frac{1}{2}} + r^{-\frac{1}{2}})$, $s = k \sec \omega$,

$$Y = \left(\left(\frac{x-y}{2} \right)^2 + \left(s^2 - 1\right)\left(s^2 - xy\right) \right)^{\frac{1}{2}},$$

$$Z_1 = s^2 - \frac{1}{2}(x+y) + Y, \text{ and}$$

$$Z_2 = s^2 + \frac{1}{2}(x+y) + Y.$$

The Watson kernel for the Jacobi functions is the modified Watson kernel obtained from the previous one by multiplication by a convenient factor, namely,

$$\tilde{K}^{\alpha,\beta}(r,x,y) = K^{\alpha,\beta}(r,x,y)\,(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}.$$

ASSUMPTION. $0 \le x, x_i \le 1$, $-1 \le y, y_i \le 1$ *for* $i = 1, \ldots, m$ *and* $1/2 < r < 1$.

LEMMA 4.1. *The following estimate for the single Watson kernel is valid*:

$$K^{\alpha,\beta}(r,x,y) \le C(\alpha,\beta)(1+L),$$

where $C(\alpha,\beta)$ *is a positive constant,* L *is the integral*

$$L = (1-r)\int_k^2 \frac{(s - \min(x,y))^{1-\alpha}}{\left((x-y)^2 + (s-1)(s-\min(x,y))\right)^{3/2}} \frac{ds}{(s-k)^{1/2}}$$

and $k = \frac{1}{2}(r^{\frac{1}{2}} + r^{-\frac{1}{2}})$, $0 \le x \le 1$.

Before beginning the proof of this lemma, let us state some estimates that will be needed. (Their proofs can be found in the Appendix.)

Let $1 \le s \le 2$, $0 \le x \le 1$, $|y| \le 1$. Then:

(i) $s^2 - \min(x,y) \le 4(s - \min(x,y))$;
(ii) $s - \min(x,y) \le 2(s - xy) \le 4(s - \min(x,y))$;
(iii) $C_1((x-y)^2 + (s-1)(s-\min(x,y))) \le Y^2$
$\le C_2((x-y)^2 + (s-1)(s-\min(x,y)))$;
(iv) $s^2 - \min(x,y) \le Z_1 \le C(s^2 - \min(x,y))$;
(v) $1 \le s^2 + \max(x,y) \le Z_2 \le C$;
(vi) if $\varphi(x,r) = (k-1)^{\frac{1}{2}}(k-x)^{\frac{1}{2}}$, then $k - 1 \le \varphi(x,r) \le k - x$, for $k > 1$;
(vii) $C_1(1-r)^2 \le k - 1 \le C_2(1-r)^2$, if $0 < r_o < r < 1$.

Here C, C_1 and C_2 denote positive constants.

Proof of Lemma 4.1. The proof is carried out through some estimates given in four claims. By taking derivatives, the Watson kernel can be decomposed into the sum of the following four kernels A, B, C and D:

$$A = r^{(1-\alpha-\beta)/2}\frac{d}{dr}\left(k^{1+\alpha+\beta}\right)\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\ \cos(\alpha-\beta)\omega}{Z_1^{\alpha}Z_2^{\beta}Y}\,d\omega,$$

$$B = r^{(1-\alpha-\beta)/2}k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{d}{dr}\left(Y^{-1}\right)\frac{\sec^{2+\alpha+\beta}\omega\ \cos(\alpha-\beta)\omega}{Z_1^{\alpha}Z_2^{\beta}}\,d\omega,$$

$$C = r^{(1-\alpha-\beta)/2}k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{d}{dr}\left(Z_1^{-\alpha}\right)\frac{\sec^{2+\alpha+\beta}\omega\ \cos(\alpha-\beta)\omega}{Z_2^{\beta}Y}\,d\omega,$$

$$D = r^{(1-\alpha-\beta)/2}k^{1+\alpha+\beta}\int_0^{\pi/2}\frac{d}{dr}\left(Z_2^{-\beta}\right)\frac{\sec^{2+\alpha+\beta}\omega\ \cos(\alpha-\beta)\omega}{Z_1^{\alpha}Y}\,d\omega.$$

CLAIM 1.

$$\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\ \cos(\alpha-\beta)\omega}{Z_1^{\alpha}Z_2^{\beta}Y}\,d\omega \le C(\alpha,\beta)\left(1+k^{-(1+\alpha+\beta)}\int_k^2\frac{s^{1+\alpha+\beta}}{Z_1^{\alpha}Z_2^{\beta}Y}\frac{ds}{(s-k)^{1/2}}\right)$$

Proof. By changing variable and letting $s = k\sec\omega$ we get

$$\int_0^{\pi/2}\frac{\sec^{2+\alpha+\beta}\omega\ \cos(\alpha-\beta)\omega}{Z_1^{\alpha}Z_2^{\beta}Y}\,d\omega \le k^{-(2+\alpha+\beta)}\int_k^{\infty}\frac{s^{2+\alpha+\beta}}{Z_1^{\alpha}Z_2^{\beta}Y}\frac{k\,ds}{s\left(s^2-k^2\right)^{1/2}}.$$

Since we assume that $1/2 < r < 1$, we have $1 < k < 3/2 < 2$. Now, for $2 < s < \infty$, the last integral is dominated by a constant depending on α and β only; in fact,

$$\frac{s^{1+\alpha+\beta}}{Z_1^{\alpha}Z_2^{\beta}Y\left(s^2-k^2\right)^{1/2}} \le \frac{s^{1+\alpha+\beta}}{\left(s^2\right)^{\alpha}\left(s^2\right)^{\beta}s^2 s} = \frac{C}{s^{2+\alpha+\beta}},$$

because $Y, Z_1, Z_2 \ge s^2$, and therefore it follows that

$$\int_2^{\infty}\frac{s^{1+\alpha+\beta}\,ds}{Z_1^{\alpha}Z_2^{\beta}Y\left(s^2-k^2\right)^{1/2}} \le C(\alpha,\beta).$$

Consequently,

$$\int_0^{\pi/2} \frac{\sec^{2+\alpha+\beta}\omega\,\cos(\alpha-\beta)\,\omega}{Z_1^\alpha Z_2^\beta Y}\,d\omega \le C(\alpha,\beta)+Ck^{-(1+\alpha+\beta)}\int_k^2 \frac{s^{1+\alpha+\beta}}{Z_1^\alpha Z_2^\beta Y}\,\frac{ds}{(s-k)^{1/2}}. \tag{4.1}$$

CLAIM 2. $|A| \le C(\alpha,\beta)(1+L)$.

Proof. The change of variable $s = k\sec\omega$ and (4.1) show that $|A|$ is bounded above by

$$r^{(1-\alpha-\beta)/2}(1+\alpha+\beta)\,k^{\alpha+\beta}\frac{(1-r)}{4r^{3/2}}\left[C(\alpha,\beta)+Ck^{-(1+\alpha+\beta)}\int_k^2 \frac{s^{1+\alpha+\beta}}{Z_1^\alpha Z_2^\beta Y}\,\frac{ds}{(s-k)^{1/2}}\right],$$

which in turn is dominated by

$$C(\alpha,\beta)+C(\alpha,\beta)(1-r)\int_k^2 \frac{Y^2}{Z_1^\alpha Z_2^\beta Y^3}\,\frac{ds}{(s-k)^{1/2}}. \tag{4.2}$$

The estimate (ii) above for $s = 1$, applied to $(x-y)^2 = x^2+y^2-2xy \le 2(1-xy)$ implies that

$$(x-y)^2 \le 4(1-\min(x,y)) \le 4(s-\min(x,y)). \tag{4.3}$$

From (4.3) and the estimate (iii) above, it follows that

$$Y^2 \le C\left((x-y)^2+(s-1)(s-\min(x,y))\right) \le C(s-\min(x,y)). \tag{4.4}$$

Thus, (4.2), (4.4), and the estimates (iii)–(v) above give

$$|A| \le C(\alpha,\beta)(1+L).$$

CLAIM 3. $|B| \le C(\alpha,\beta)(1+L)$. □

Proof. Similarly, by taking derivatives and changing variables, it follows that

$$|B| \le C(\alpha,\beta)+C(\alpha,\beta)(1-r)\int_k^2 \frac{2s^2-1-xy}{Z_1^\alpha Z_2^\beta Y^3}\,\frac{ds}{(s-k)^{1/2}}. \tag{4.5}$$

As a consequence of the estimates (i)–(v) above, we get

$$\frac{1}{Z_1^\alpha Z_2^\beta Y^3} \le C(\alpha,\beta)\frac{1}{(s-\min(x,y))^\alpha\left((x-y)^2+(s-1)(s-\min(x,y))\right)^{3/2}}.$$

Therefore, it only remains to be shown that $2s^2 - 1 - xy \le C\,(s - \min(x, y))$. Now,

$$2s^2 - 1 - xy \le 2\left(s^2 - xy\right).$$

If $y \le 0$, then

$$s^2 - xy \le 5 \le 5s \le C(s - \min(x, y)).$$

Suppose $y > 0$. Without loss of generality assume that $\min(x, y) = x$; $m(x, y) = y$ is similar. Then

$$s^2 - xy \le s^2 - x^2 = (s + x)(s - x) \le C\,(s - \min(x, y)). \tag{4.6}$$

Thus, in any situation,

$$2s^2 - 1 - xy \le 2\left(s^2 - xy\right) \le C(s - \min(x, y)). \tag{4.7}$$

Then, estimate (iii) above, (4.5) and (4.7) give $|B| \le C\,(\alpha, \beta)\,(1 + L)$.

CLAIM 4. $|C| \le C\,(\alpha, \beta)\,(1 + L)$ and $|D| \le C\,(\alpha, \beta)\,(1 + L)$.

Proof. Once again, by taking derivatives and changing variables, it follows that $|C|$ and $|D|$ are dominated by

$$C\,(\alpha, \beta) + C\,(\alpha, \beta)\,(1 - r) \int_k^2 \frac{1}{Z_1^{\alpha+1} Z_2^{\beta} Y} \left(1 + \frac{2s^2 - 1 - xy}{Y}\right) \frac{ds}{(s - k)^{1/2}}$$

and

$$C\,(\alpha, \beta) + C\,(\alpha, \beta)\,(1 - r) \int_k^2 \frac{1}{Z_1^{\alpha} Z_2^{\beta+1} Y} \left(1 + \frac{2s^2 - 1 - xy}{Y}\right) \frac{ds}{(s - k)^{1/2}},$$

respectively. Since $0 < Y \le Z_i$, $i = 1, 2$, and $Z_2 \sim C$, we get

$$\frac{1}{Z_1^{\alpha+1} Z_2^{\beta} Y} \le \frac{1}{Z_1^{\alpha} Z_2^{\beta} Y^2} \le \frac{C\,(\beta)}{Z_1^{\alpha} Y^2}$$

and

$$\frac{1}{Z_1^{\alpha} Z_2^{\beta+1} Y} \le \frac{1}{Z_1^{\alpha} Z_2^{\beta} Y^2} \le \frac{C\,(\beta)}{Z_1^{\alpha} Y^2}.$$

Because of (4.7) it suffices to prove that $Y \le C\,(s - \min(x, y))$. Now,

$$Y = \left(\left(\frac{x - y}{2}\right)^2 + \left(s^2 - 1\right)\left(s^2 - xy\right)\right)^{\frac{1}{2}},$$

$$(x-y)^2 \le (1-\min(x,y))^2 \le (s-\min(x,y))^2,$$

and by virtue of (4.6) we have

$$\left(s^2-1\right)\left(s^2-xy\right) \le \left(s^2-xy\right)^2 \le C\,(s-\min(x,y))^2.$$

Hence,

$$Y \le C\,(s-\min(x,y)),$$

and so the required inequalities: $|C| \le C(\alpha,\beta)(1+L)$ and $|D| \le C(\alpha,\beta)(1+L)$ hold.

Collecting results, Lemma 4.1 follows. □

Remark 3. In the case of the Lebesgue measure, the integral L and the Watson kernel have to be multiplied by the bounded factors $(1-x)^{\alpha/2}(1-y)^{\alpha/2}$ and $(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2}$, respectively. So, the above lemma is also valid for the modified Watson kernel and the corresponding modified integral L.

5. A theory for the Lebesgue measure case

5.1. *Estimates for the modified Watson kernel.* The modified Watson kernel $\tilde{K}^{(\alpha,\beta)}(r,x,y)$ is given by

$$\tilde{K}^{(\alpha,\beta)}(r,x,y) = K^{(\alpha,\beta)}(r,x,y)\,(1-x)^{\alpha/2}(1+x)^{\beta/2}(1-y)^{\alpha/2}(1+y)^{\beta/2}.$$

Hence,

$$\tilde{K}^{(\alpha,\beta)}(r,x,y) \le 2^{\beta} K^{(\alpha,\beta)}(r,x,y)\,(1-x)^{\alpha/2}(1-y)^{\alpha/2}.$$

By considering the estimate given in Lemma 4.1, the fact that

$$(1-x)^{\alpha/2}(1-y)^{\alpha/2} \le (s-\min(x,y))^{\alpha},$$

for $s \ge 1$, and the estimate (vii) in Section 4, it follows that $\tilde{K}^{(\alpha,\beta)}(r,x,y)$ is dominated by

$$C(\alpha,\beta) + C(\alpha,\beta)\,(k-1)^{1/2} \int_k^2 \frac{s-\min(x,y)}{\left((x-y)^2+(s-1)(s-x)\right)^{3/2}} \frac{ds}{(s-k)^{1/2}}.$$

We want to get an expression involving a superposition of Poisson type kernels. The elementary estimate

$$s-\min(x,y) \le s-x+|x-y|,$$

gives

$$\tilde{K}^{(\alpha,\beta)}(r,x,y) \le C(\alpha,\beta) + C(\alpha,\beta)(k-1)^{1/2} \times \int_k^2 \frac{s-x}{\left((x-y)^2+(s-1)(s-x)\right)^{3/2}} \frac{ds}{(s-k)^{1/2}} + C(\alpha,\beta)(k-1)^{1/2} \int_k^2 \frac{|x-y|}{\left((x-y)^2+(s-1)(s-x)\right)^{3/2}} \frac{ds}{(s-k)^{1/2}}.$$

Now,

$$\int_k^2 \frac{s-x}{\left((x-y)^2+(s-1)(s-x)\right)^{3/2}} \frac{ds}{(s-k)^{1/2}} = \int_k^2 \frac{1}{(s-k)^{1/2}} \frac{1}{(s-1)} \frac{\left(\frac{1}{(s-1)(s-x)}\right)^{1/2} ds}{\left(\left(\frac{x-y}{((s-1)(s-x))^{1/2}}\right)^2+1\right)^{3/2}}.$$

On the other hand,

$$\int_k^2 \frac{|x-y|}{\left((x-y)^2+(s-1)(s-x)\right)^{3/2}} \frac{ds}{(s-k)^{1/2}} = \int_k^2 \frac{1}{(s-k)^{1/2}} \frac{1}{(s-1)(s-x)} \frac{\frac{|x-y|}{((s-1)(s-x))^{1/2}} ds}{\left(\left(\frac{x-y}{((s-1)(s-x))^{1/2}}\right)^2+1\right)^{3/2}} \le \int_k^2 \frac{1}{(s-k)^{1/2}(s-1)} \frac{1}{((s-1)(s-x))^{1/2}} \frac{\frac{|x-y|}{((s-1)(s-x))^{1/2}} ds}{\left(\left(\frac{x-y}{((s-1)(s-x))^{1/2}}\right)^2+1\right)^{3/2}}.$$

Thus, the modified Watson kernel is dominated by an infinite superposition of Poisson type kernels. The crucial estimate is

$$(k-1)^{1/2} \int_k^2 \frac{1}{(s-k)^{1/2}} \frac{1}{(s-1)^{1/2}(s-x)^{1/2}} ds \le C, \tag{5.1}$$

which in turn follows from

$$(k-1)^{1/2} \int_k^2 \frac{1}{(s-k)^{1/2}} \frac{1}{(s-1)} ds \le C. \tag{5.2}$$

Remark 4. At this point, this estimate can be applied to the multiple Watson kernel to give the Jessen-Marcinkiewicz-Zygmund inequality. In this case the well known iteration method gives part (ii) of Theorem 2.1 because of (5.1) and (5.2) above.

In order to get the weak type (1,1) inequality, we have to replace the continuous estimate involving an integral by a series.

LEMMA 5.1. *The modified single Watson kernel admits a domination by a series of Poisson type kernels, namely,*

$$\begin{aligned}\tilde{K}^{(\alpha,\beta)}(r,x,y) \leq\ & C(\alpha,\beta) + C(\alpha,\beta)\sum_{j=0}^{\infty} b_j \frac{2^j\varphi(r)}{(x-y)^2 + 2^{2j}\varphi^2(r)} \\ & + C(\alpha,\beta)\sum_{j=0}^{\infty} b_j \frac{2^j\varphi(r)\left(2^j\varphi(r)+h(x)\right)}{\left((x-y)^2 + 2^j\varphi(r)\left(2^j\varphi(r)+h(x)\right)\right)^{3/2}},\end{aligned}$$

where $b_j \sim 2^{-j/2}$, $\varphi(r) = k-1$ *and* $h(x) = 1-x$.

Proof. Recall the basic inequality

$$\begin{aligned}\tilde{K}^{(\alpha,\beta)}(r,x,y) \leq\ & C(\alpha,\beta) + C(\alpha,\beta)(k-1)^{1/2} \\ & \times \int_k^2 \frac{s-x+|x-y|}{\left((x-y)^2+(s-1)(s-x)\right)^{3/2}} \frac{ds}{(s-k)^{1/2}},\end{aligned}$$

where, as before, we have used $s-\min(x,y) \leq s-x+|x-y|$. By setting $s-1 = u(k-1)$, we have the above integral dominated by

$$(k-1)^{1/2}\int_1^{\infty} \frac{u(k-1)+(1-x)+|x-y|}{\left((x-y)^2+u(k-1)(u(k-1)+(1-x))\right)^{3/2}} \frac{(k-1)\,du}{(u(k-1)-k-1)^{1/2}}$$

$$= (k-1)\int_1^{\infty} \frac{u(k-1)+(1-x)+|x-y|}{\left((x-y)^2+u(k-1)(u(k-1)+(1-x))\right)^{3/2}} \frac{du}{(u-1)^{1/2}}. \tag{5.3}$$

Set $\varphi = \varphi(r) = k-1$ and $h(x) = 1-x$. Taking into account that

$$\varphi\left(u\varphi + h(x) + |x-y|\right) \leq \varphi\left(u\varphi + h(x)\right) + \varphi\left((x-y)^2 + u^2\varphi^2\right)^{1/2},$$

we have (5.3) bounded above by the sum of the following two integrals

$$\int_1^{\infty} \frac{\varphi\left((x-y)^2+u^2\varphi^2\right)^{1/2}}{\left((x-y)^2+u^2\varphi^2\right)^{3/2}} \frac{du}{(u-1)^{1/2}}. \tag{5.4}$$

and

$$\int_1^\infty \frac{\varphi\,(u\varphi + h\,(x))}{\left((x-y)^2 + u\varphi\,(u\varphi + h\,(x))\right)^{3/2}} \frac{du}{(u-1)^{1/2}}. \tag{5.5}$$

The one in (5.4) is dominated by

$$\int_1^\infty \frac{u\varphi}{\left((x-y)^2 + u^2\varphi^2\right)} \frac{du}{u\,(u-1)^{1/2}}$$

$$\leq \sum_{j=0}^{\infty} \left(\int_{2^j}^{2^{j+1}} \frac{du}{u\,(u-1)^{1/2}} \right) \frac{2^{j+1}\varphi}{(x-y)^2 + 2^{2j}\varphi^2}.$$

On the other hand, (5.5) is less than or equal to

$$C \sum_{j=0}^{\infty} \left(\int_{2^j}^{2^{j+1}} \frac{du}{u\,(u-1)^{1/2}} \right) \frac{2^j\varphi\left(2^j\varphi + h(x)\right)}{\left((x-y)^2 + 2^j\varphi\left(2^j\varphi + h(x)\right)\right)^{3/2}}. \qquad \square$$

5.2. *Proof of Theorem* 2.1. In order to simplify the writing of the proof we may consider without loss of generality $m = 2$, a typical case. The proof of the theorem relies on a very simple fact, namely, that $T = \sum_{j=0}^{\infty} b_j T_j$ is weak type $(1, 1)$ if $\sum_{j=0}^{\infty} b_j^{1/2} < \infty$ and each T_j is sublinear, weak type $(1, 1)$ with uniformly bounded type constants;, i.e., $\mu\left(\{T_j\,(f) > \lambda\}\right) < \frac{C}{\lambda} \|f\|_1$, $j = 0, 1, 2, \ldots$. For a proof of this property see, for example, C. P. Calderón [C, p. 121]. Taking into account (1.1) and Lemma 5.1, it will be enough to show that each one of the following operators is of weak type $(1, 1)$ with uniformly bounded type constants:

$$G_1 = \sup_r \iint_Q \prod_{i=1}^{2} \frac{2^{j_i}\varphi\,(r)\left(2^{j_i}\varphi\,(r) + h_i(x_i)\right)}{\left((x_i - y_i)^2 + 2^{j_i}\varphi\,(r)\left(2^{j_i}\varphi\,(r) + h_i(x_i)\right)\right)^{3/2}} \,|f\,(y_1, y_2)|\, dy_1 dy_2,$$

$$G_2 = \sup_r \iint_Q \frac{2^{j_1}\varphi\,(r)\left(2^{j_1}\varphi\,(r) + h_1(x_1)\right)}{\left((x_1 - y_1)^2 + 2^{j_1}\varphi\,(r)\left(2^{j_1}\varphi\,(r) + h_1(x_1)\right)\right)^{3/2}}$$
$$\times \frac{2^{j_2}\varphi\,(r)}{(x_2 - y_2)^2 + 2^{2j_2}\varphi^2\,(r)} \,|f\,(y_1, y_2)|\, dy_1 dy_2,$$

$$G_3 = \sup_r \iint_Q \prod_{i=1}^{2} \frac{2^{j_i}\varphi\,(r)}{(x_i - y_i)^2 + 2^{2j_i}\varphi^2\,(r)} \,|f\,(y_1, y_2)|\, dy_1 dy_2,$$

$$G_4 = \sup_r \int_I \frac{2^j\varphi\,(r)\left(2^j\varphi\,(r) + h_1(x_1)\right)}{\left((x_1 - y_1)^2 + 2^j\varphi\,(r)\left(2^j\varphi\,(r) + h_1(x_1)\right)\right)^{3/2}} \left(\int_I |f\,(y_1, y_2)|\, dy_2 \right) dy_1$$

and

$$G_5 = \sup_r \int_I \frac{2^j \varphi(r)}{(x_1 - y_1)^2 + 2^{2j}\varphi^2(r)} \left(\int_I |f(y_1, y_2)|\, dy_2 \right) dy_1,$$

where $\varphi(r) = k - 1$ and $h_i(x_i) = 1 - x_i$. Let us write

$$\theta_i(x_i, r) = 2^{j_i}\varphi(r)\left(2^{j_i}\varphi(r) + h_i(x_i)\right).$$

Now, the integral in G_1 is dominated by

$$C \sum_{l_1, l_2 \geq 0} 2^{-2(l_1+l_2)} \frac{1}{2^{l_1+l_2}(\theta_1(x_1, r)\theta_2(x_2, r))^{1/2}} \iint_{|x_i - y_i| \leq 2^{l_i}(\theta_i(x_i, r))^{1/2}} |f(y_1, y_2)| dy_1 dy_2.$$

which follows from a standard procedure by considering both estimates

$$\frac{\theta_i(x_i, r)}{\left((x_i - y_i)^2 + \theta_i(x_i, r)\right)^{3/2}} \leq \frac{1}{2^{3l_i}(\theta_i(x_i, r))^{1/2}},$$

for $2^{l_i}(\theta_i(x_i, r))^{1/2} \leq |x_i - y_i| \leq 2^{l_i+1}(\theta_i(x_i, r))^{1/2}$, $l_i \geq 0$, and

$$\frac{\theta_i(x_i, r)}{\left((x_i - y_i)^2 + \theta_i(x_i, r)\right)^{3/2}} \leq \frac{1}{(\theta_i(x_i, r))^{1/2}}.$$

From Lemma 3.2 with $\varphi_i(t) = 2^{j_i}\varphi(r)$, $t = 1 - r$, $K_i = 2^{l_i+1}$ and $C_i = 1$, we have

$$\left|\left\{ \sup_r \frac{1}{2^{l_1+l_2}(\theta_1(x_1, r)\theta_2(x_2, r))^{1/2}} \iint_{|x_i - y_i| \leq 2^{l_i}(\theta_i(x_i, r))^{1/2}} |f(y_1, y_2)|\, dy_1 dy_2 > \lambda \right\}\right|$$

$$\leq \frac{C(m)}{\lambda} \prod_{i=1}^{2} (1 + l_i)\, \|f\|_1.$$

Therefore, G_1 is weak type (1,1) because

$$\sum_{l_1, l_2 \geq 0} \left((1 + l_1)(1 + l_2)\, 2^{-2(l_1+l_2)}\right)^{1/2} < \infty.$$

G_2 and G_3 are handled in a similar way by making an appropriate use of Lemma 3.2; G_4 and G_5 follow from Lemma 3.3. □

6. A unified theory for both cases

6.1. *Estimates for the Watson kernel.* This section provides an estimate for the multiple Watson kernel which is based on an estimate of the integral L that dominates the single Watson kernel in Lemma 4.1. These estimates are summarized in the following two lemmas:

LEMMA 6.1. *$L \leq C(\alpha, \beta) \sum_{n=0}^{\infty} \frac{1}{2^{n/2}} \frac{1}{\mu\{I_n(x,r)\}} \chi_{I_n(x,r)}(y)$, where $I_n(x,r)$ denotes the interval*

$$\left[x - 2^n \varphi(x,r); x + 2^n \varphi(x,r)\right] \cap [-1, 1],$$

and $\chi_{I_n(x,r)}$ is its characteristic function, $n = 0, 1, 2, \ldots$. μ is either the $J^{(\alpha,\beta)}$ measure or the Lebesgue measure restricted to $[-1, 1]$. In this latter case the integral L is modified by multiplication by the factor $(1-x)^{\alpha/2}(1-y)^{\alpha/2}$. $\varphi(x,r) = (k-1)^{1/2}(k-x)^{1/2}$.

LEMMA 6.2. *The multiple Watson kernel satisfies the inequality*

$$K^{\alpha,\beta}(r, X, Y) \leq C \sum_{n=(n_1,\ldots,n_m)} \frac{1}{2^{n_1/2} \cdots 2^{n_m/2}} \frac{1}{\mu\{I_n(X,r)\}} \chi_{I_n(X,r)}(Y),$$

for $n_j \in \mathbf{Z}$, $n_j \geq -1$, where

$$I_n(X,r) = I_{n_1}(x_1, r_1) \times \cdots \times I_{n_m}(x_m, r_m), \quad I_{-1}(x_i, r_i) = [-1, 1],$$

$\chi_{I_n(X,r)}$ is the characteristic function of $I_n(X,r)$ and μ is either the Lebesgue measure restricted to Q or the m-dimensional Jacobi measure $J^{(\bar{\alpha},\bar{\beta})}$, $\bar{\alpha}_j \geq \alpha_j$, $\bar{\beta}_j \geq \beta_j$, with only a possible change in the constant $C = C(\alpha, \beta, \bar{\alpha}, \bar{\beta})$. $K^{\alpha,\beta}(r, X, Y)$ denotes the modified Watson kernel $\tilde{K}$ in the case of the Lebesgue measure.

For the proof of Lemma 6.2, it will be convenient to consider the following estimates of the integral L and the $J^{(\alpha,\beta)}$ measure of the intervals $I_n(x,r)$.

LEMMA 6.3. $L \leq \dfrac{C(1-r)}{(k - \min(x,y))^{\alpha+1/2}} \dfrac{1}{k-1}, 0 \leq x \leq 1.$

Proof. It is clear that

$$\begin{aligned}
L &= (1-r) \int_k^2 \frac{(s - \min(x,y))^{1-\alpha}}{\left((x-y)^2 + (s-1)(s - \min(x,y))\right)^{3/2}} \frac{ds}{(s-k)^{1/2}} \\
&\leq (1-r) \int_k^2 \frac{(s - \min(x,y))}{(s - \min(x,y))^{\alpha} (s-1)^{3/2} (s - \min(x,y))^{3/2}} \frac{ds}{(s-k)^{1/2}} \\
&\leq \frac{(1-r)}{(k - \min(x,y))^{\alpha+1/2}} \int_k^2 \frac{1}{(s-1)^{3/2}} \frac{ds}{(s-k)^{1/2}}.
\end{aligned} \tag{6.1}$$

The fact that $1 < k$ and an integration by parts show that for any a, b with $b > a \geq k > 1$,

$$\begin{aligned}
\int_a^b \frac{1}{(s-1)^{3/2}} \frac{ds}{(s-k)^{1/2}} &= 2\left[\frac{(b-k)^{1/2}}{(b-1)^{3/2}} - \frac{(a-k)^{1/2}}{(a-1)^{3/2}}\right] + 3\int_a^b \frac{(s-k)^{1/2}}{(s-1)^{5/2}} ds \\
&\leq 2\left[\frac{1}{(b-1)} - \frac{(a-k)^{1/2}}{(a-1)^{3/2}}\right] + 3\int_a^b \frac{1}{(s-1)^2} ds \\
&= 2\left[\frac{1}{(b-1)} - \frac{(a-k)^{1/2}}{(a-1)^{3/2}}\right] \\
&\quad - \frac{3}{b-1} + \frac{3}{a-1} < \frac{3}{a-1}
\end{aligned} \tag{6.2}$$

Therefore,

$$L \leq \frac{C(1-r)}{(k-\min(x,y))^{\alpha+1/2}} \frac{1}{k-1}. \qquad \square$$

LEMMA 6.4. *For any Borel set E in $\mathbb{R}^m$, $J^{(\bar{\alpha},\bar{\beta})}\{E\} \leq C(\alpha,\beta,\bar{\alpha},\bar{\beta}) J^{(\alpha,\beta)}\{E\}$, whenever $\bar{\alpha}_j \geq \alpha_j$, $\bar{\beta}_j \geq \beta_j$, $j = 1, \ldots, m$.*

Proof. Observe that for $\alpha, \beta, y \in \mathbb{R}$, $|1-y| < 2$ and $|1+y| < 2$ imply that

$$\begin{aligned}
(1-y)^{\bar{\alpha}}(1+y)^{\bar{\beta}} &= 2^{\bar{\alpha}+\bar{\beta}} \left(\frac{1-y}{2}\right)^{\bar{\alpha}-\alpha+\alpha} \left(\frac{1+y}{2}\right)^{\bar{\beta}-\beta+\beta} \\
&\leq C\left(\alpha,\beta,\bar{\alpha},\bar{\beta}\right)(1-y)^{\alpha}(1+y)^{\beta},
\end{aligned}$$

for any $\bar{\alpha} \geq \alpha$, $\bar{\beta} \geq \beta$. Thus, the lemma follows immediately.

LEMMA 6.5. *$J^{(\bar{\alpha},\bar{\beta})}\{I_0(x,r)\} \leq C\left(\alpha,\beta,\bar{\alpha},\bar{\beta}\right)\varphi(x,r)(k-x)^{\alpha}$ for $\bar{\alpha} \geq \alpha$, $\bar{\beta} \geq \beta$.*

Proof. By Lemma 6.4, it is enough to show the case $J^{(\alpha,\beta)}$. The $J^{(\alpha,\beta)}$-measure of the interval $I_0(x,r)$ satisfies

$$J^{(\alpha,\beta)}\{I_0(x,r)\} = \int_{I_0} (1-y)^{\alpha}(1+y)^{\beta} dy \leq 2^{\beta} \int_{I_0} (1-y)^{\alpha} dy.$$

Suppose that $x + \varphi(x,r) \leq 1$. For short, we will sometimes write I_0 and φ. Then

$$J^{(\alpha,\beta)}\{I_0\} \leq C(\beta) \int_{x-\varphi}^{x+\varphi} (1-y)^{\alpha} dy = C(\alpha,\beta)((1-x+\varphi)^{\alpha+1} - (1-x-\varphi)^{\alpha+1}). \tag{6.3}$$

The Mean Value Theorem applied to $f(u) = (1 - u + \varphi(x,r))^{\alpha+1}$ gives

$$\begin{aligned}(1 - x + \varphi)^{\alpha+1} - (1 - x - \varphi)^{\alpha+1} &\leq C(\alpha)\, 2\varphi(x,r)\,(1-x)^{\alpha} \\ &\leq C(\alpha)\,\varphi(x,r)\,(k-x)^{\alpha}.\end{aligned}$$

If $x + \varphi(x,r) > 1$, then

$$J^{(\alpha,\beta)}\{I_0\} \leq C(\beta) \int_{x-\varphi}^{1} (1-y)^{\alpha}\, dy = C(\alpha,\beta)\,(1 - x + \varphi)^{\alpha+1}. \tag{6.4}$$

In this case, $(1-x) < \varphi(x,r)$, hence

$$(1 - x + \varphi)^{\alpha+1} < 2\varphi(x,r)\,(2\varphi(x,r))^{\alpha}.$$

But, $\varphi(x,r) = (k-1)^{\frac{1}{2}}(k-x)^{\frac{1}{2}} \leq (k-x)$, and so

$$(1 - x + \varphi)^{\alpha+1} \leq C(\alpha)\,\varphi(x,r)\,(k-x)^{\alpha}.$$

Therefore,

$$J^{(\alpha,\beta)}\{I_0(x,r)\} \leq C(\alpha,\beta)\,\varphi(x,r)\,(k-x)^{\alpha}. \tag{6.5}$$

□

LEMMA 6.6. *$L \leq C(\alpha,\beta)\frac{1-r}{|x-y|^{\alpha+3/2}}$ if $y \in I_n(x,r) - I_{n-1}(x,r)$ and $1 - x < 2^{n-1}\varphi(x,r)$.*

Proof. Notice that in this case, $y < x$ and $k - 1 < \varphi(x,r) < |x-y|$. Then $\min(x,y) = y$ and

$$\begin{aligned} L_1 &= (1-r) \int_{k}^{1+|x-y|} \frac{(s - \min(x,y))^{1-\alpha}}{\left((x-y)^2 + (s-1)(s-\min(x,y))\right)^{3/2}} \frac{ds}{(s-k)^{1/2}} \\ &\leq (1-r)\frac{(1+|x-y|-y)}{(k-y)^{\alpha}\,|x-y|^3} \int_{k}^{1+|x-y|} \frac{ds}{(s-k)^{1/2}}. \end{aligned}$$

Note that $1 + x - 2y = 1 - x + 2(x-y) < 2^{n-1}\varphi(x,r) + 2|x-y| < 3|x-y|$. Hence

$$L_1 \leq C(1-r)\frac{|x-y|}{(x-y)^{\alpha}\,|x-y|^3}(1 + |x-y| - k)^{1/2} \leq \frac{C(1-r)}{|x-y|^{\alpha+3/2}}.$$

Now, assume that $k \leq 1 + |x-y| < 2$ and consider the integral

$$L_2 = (1-r)\int_{1+|x-y|}^{2} \frac{(s-\min(x,y))^{1-\alpha}}{\left((x-y)^2 + (s-1)(s-\min(x,y))\right)^{3/2}} \frac{ds}{(s-k)^{1/2}}.$$

As in (6.1) and (6.2), we get

$$L_2 \leq \frac{C\,(1-r)}{(k-y)^{\alpha+1/2}} \frac{3}{(1+|x-y|-1)},$$

so, $y < x < k$ implies

$$L_2 \leq \frac{C\,(1-r)}{|x-y|^{\alpha+3/2}}.$$

Altogether, it follows that

$$L \leq \frac{C\,(1-r)}{|x-y|^{\alpha+3/2}}. \qquad \square$$

LEMMA 6.7. *If* $y \in I_n(x,r) - I_{n-1}(x,r)$ *and* $1-x < 2^{n-1}\varphi(x,r)$, *then* $J^{(\bar{\alpha},\bar{\beta})}\{I_n(x,r)\} \leq C(\alpha,\beta,\bar{\alpha},\bar{\beta})|x-y|^{\alpha+1}$, *for* $\bar{\alpha} \geq \alpha$, $\bar{\beta} \geq \beta$.

Proof. Because of Lemma 6.4, it is enough to show the case $J^{(\alpha,\beta)}$. Observe that $1 < x + 2^{n-1}\varphi(x,r) \leq x + 2^n\varphi(x,r)$. Thus

$$\begin{aligned}
J^{(\alpha,\beta)}\{I_n\} &= \int_{I_n} (1-y)^\alpha (1+y)^\beta \, dy \leq 2^\beta \int_{x-2^n\varphi}^{1} (1-y)^\alpha \, dy \\
&= C\,(\alpha,\beta)\left(1-x+2^n\varphi(x,r)\right)^{\alpha+1} \\
&\leq C\,(\alpha,\beta)\left(2^{n-1}\varphi(x,r)\right)^{\alpha+1} \\
&\leq C\,(\alpha,\beta)\,|x-y|^{\alpha+1}.
\end{aligned} \qquad \square$$

LEMMA 6.8. $L \leq C\,(\alpha,\beta)\,\frac{\varphi(x,r)}{(1-x)^\alpha|x-y|^2}$ *if* $y \in I_n\,(x,r) - I_{n-1}(x,r)$ *and* $1-x \geq 2^{n-1}\varphi(x,r)$.

Proof. First, suppose that $k > 1 + \frac{(x-y)^2}{1-x}$. From Lemma 6.3, it follows that

$$\begin{aligned}
L &\leq \frac{C\,(1-r)}{(k-\min(x,y))^{\alpha+1/2}} \frac{1}{k-1} \\
&\leq \frac{C\,(1-r)}{(1-x)^{\alpha+1/2}} \frac{1-x}{(x-y)^2} \\
&\leq C \frac{(k-1)^{1/2}\,(1-x)^{1/2}}{(1-x)^\alpha\,(x-y)^2} \\
&\leq C \frac{(k-1)^{1/2}\,(k-x)^{1/2}}{(1-x)^\alpha\,(x-y)^2} \\
&= C \frac{\varphi(x,r)}{(1-x)^\alpha\,(x-y)^2}.
\end{aligned}$$

Now, assume that $k \leq 1 + \frac{(x-y)^2}{1-x}$. Observe that $|x-y| \leq 2^n\varphi(x,r) \leq 2(1-x)$. Then, for $k \leq s \leq 1 + \frac{(x-y)^2}{1-x}$, we have

$$s - \min(x,y) \leq 1 + \frac{(x-y)^2}{1-x} - \min(x,y) \leq 1 - \min(x,y) + 4(1-x)$$

and

$$1 - \min(x,y) \leq 1 - x + |x - \min(x,y)| \leq 1 - x + 2(1-x) = 3(1-x),$$

so

$$1 - x \leq s - \min(x,y) \leq 7(1-x).$$

With this estimate and the ones of the integrals L_1 and L_2 in the proof of the previous lemma, with $1 + |x-y|$ replaced by $1 + \frac{(x-y)^2}{1-x}$, we get

$$L \leq C(\alpha,\beta)\frac{1-r}{(1-x)^{\alpha-1/2}|x-y|^2}.$$

Hence

$$L \leq C(\alpha,\beta)\frac{(k-1)^{1/2}(1-x)^{1/2}}{(1-x)^{\alpha}\;|x-y|^2} \leq C(\alpha,\beta)\frac{\varphi(x,r)}{(1-x)^{\alpha}\;|x-y|^2}. \qquad \square$$

LEMMA 6.9. $J^{(\bar\alpha,\bar\beta)}\{I_n(x,r)\} \leq C(\alpha,\beta,\bar\alpha,\bar\beta)(1-x)^{\alpha}2^n\varphi(x,r)$, *whenever* $1 - x \geq 2^{n-1}\varphi(x,r), \bar\alpha \geq \alpha, \bar\beta \geq \beta$.

Proof. This result follows similarly as in the I_0 case. Indeed, the inequalities $2^{n-1}\varphi(x,r) \leq 1-x \leq 2^n\varphi(x,r)$, give (6.3) and (6.4) with $\varphi(x,r)$ replaced by $2^{n-1}\varphi(x,r)$ and $2^n\varphi(x,r)$, respectively. Thus, the factor 2^n gets in (6.5) to yield the desired estimate. $\square$

Proof of Lemma 6.1. *Case* 1. $\mu = J^{(\bar\alpha,\bar\beta)}$.
Let $y \in I_0(x,r)$. Lemmas 6.3 and 6.5 give

$$L \leq \frac{C(1-r)}{(k-\min(x,y))^{\alpha+1/2}}\frac{1}{k-1}$$

and $J^{(\bar\alpha,\bar\beta)}\{I_0(x,r)\} \leq C(\alpha,\beta,\bar\alpha,\bar\beta)\varphi(x,r)(k-x)^{\alpha}$. Then

$$\begin{aligned} L &\leq \frac{C(1-r)}{(k-\min(x,y))^{\alpha+1/2}}\frac{1}{k-1}\frac{C(\alpha,\beta,\bar\alpha,\bar\beta)\varphi(x,r)(k-x)^{\alpha}}{J^{(\bar\alpha,\bar\beta)}\{I_0(x,r)\}}\chi_{I_0(x,r)}(y) \\ &\leq \frac{C(\alpha,\beta,\bar\alpha,\bar\beta)(1-r)}{(k-x)^{\alpha+1/2}}\frac{1}{k-1}\frac{(k-1)^{1/2}(k-x)^{1/2}(k-x)^{\alpha}}{J^{(\bar\alpha,\bar\beta)}\{I_0(x,r)\}}\chi_{I_0(x,r)}(y) \\ &\leq C(\alpha,\beta,\bar\alpha,\bar\beta)\frac{1}{J^{(\bar\alpha,\bar\beta)}\{I_0(x,r)\}}\chi_{I_0(x,r)}(y). \end{aligned}$$

Here we have used the fact that $(k-1) \sim (1-r)^2$, which follows from the estimate (vii) in Section 4 above.

Assume that $y \in I_n(x,r) - I_{n-1}(x,r)$. If $1-x < 2^{n-1}\varphi(x,r)$, then $L \leq C(\alpha,\beta)\frac{1-r}{|x-y|^{\alpha+3/2}}$ and $J^{(\bar\alpha,\bar\beta)}\{I_n(x,r)\} \leq C(\alpha,\beta,\bar\alpha,\bar\beta)\,|x-y|^{\alpha+1}$, from Lemmas 6.6 and 6.7 respectively. Hence

$$\begin{aligned} L &\leq C\left(\alpha,\beta,\bar\alpha,\bar\beta\right)\frac{1-r}{|x-y|^{\alpha+3/2}}\frac{|x-y|^{\alpha+1}}{J^{(\bar\alpha,\bar\beta)}\{I_n\}}\,\chi_{I_n(x,r)}(y) \\ &\leq C\left(\alpha,\beta,\bar\alpha,\bar\beta\right)\frac{1-r}{(2^n\varphi)^{1/2}}\frac{1}{J^{(\bar\alpha,\bar\beta)}\{I_n\}}\,\chi_{I_n(x,r)}(y) \\ &\leq \frac{C\left(\alpha,\beta,\bar\alpha,\bar\beta\right)}{2^{n/2}}\frac{1-r}{(k-1)^{1/2}}\frac{1}{J^{(\bar\alpha,\bar\beta)}\{I_n\}}\,\chi_{I_n(x,r)}(y) \\ &\leq \frac{C\left(\alpha,\beta,\bar\alpha,\bar\beta\right)}{2^{n/2}}\frac{1}{J^{(\bar\alpha,\bar\beta)}\{I_n\}}\,\chi_{I_n(x,r)}(y). \end{aligned} \tag{6.6}$$

When $1-x \geq 2^{n-1}\varphi(x,r)$, Lemmas 6.8 and 6.9 imply that $L \leq C(\alpha,\beta)\frac{\varphi(x,r)}{(1-x)^\alpha|x-y|^2}$ and $J^{(\bar\alpha,\bar\beta)}\{I_n(x,r)\} \leq C(\alpha,\beta,\bar\alpha,\bar\beta)\,(1-x)^\alpha\,2^n\varphi(x,r)$. Finally, one can obtain the required estimate for L by handling the case as in (6.6) above.

Case 2. μ is the Lebesgue measure.
The kernel to be considered is

$$\begin{aligned} \tilde K^{(\alpha,\beta)}(r,x,y) &= K^{(\alpha,\beta)}(r,x,y)\,(1-x)^{\alpha/2}(1-y)^{\alpha/2}(1+x)^{\beta/2}(1+y)^{\beta/2} \\ &\leq 2^\beta K^{(\alpha,\beta)}(r,x,y)\,(1-x)^{\alpha/2}(1-y)^{\alpha/2}, \end{aligned}$$

and the corresponding modified integral $\tilde L$ is

$$\tilde L = L\,(1-x)^{\alpha/2}(1-y)^{\alpha/2}.$$

Now, Lemma 6.3 gives

$$\tilde L \leq \frac{C\,(1-r)}{(k-\min(x,y))^{\alpha+1/2}}\frac{1}{k-1}\,(1-x)^{\alpha/2}(1-y)^{\alpha/2}.$$

Notice that

$$\frac{(1-x)^{\alpha/2}(1-y)^{\alpha/2}}{(k-\min(x,y))^{\alpha}} \leq 1.$$

Suppose that $y \in I_0(x,r)$, i.e., $|x-y| \le \varphi(x,r)$, $|y| \le 1$. Then

$$\begin{aligned}
\tilde{L} &\le C\frac{(1-r)}{(k-\min(x,y))^{1/2}}\frac{1}{k-1}\frac{2\varphi(x,r)}{|I_0|}\chi_{I_0}(y)\\
&= C\frac{(1-r)}{(k-\min(x,y))^{1/2}}\frac{1}{k-1}\frac{(k-1)^{\frac{1}{2}}(k-x)^{\frac{1}{2}}}{|I_0|}\chi_{I_0}(y)\\
&\le C\frac{(1-r)}{(k-1)^{1/2}}\frac{1}{|I_0|}\chi_{I_0}(y)\\
&\le C\frac{1}{|I_0|}\chi_{I_0}(y).
\end{aligned}$$

The last inequality follows because of estimate (vii): $(k-1) \sim (1-r)^2$.

Now, assume that $y \in I_n(x,r) - I_{n-1}(x,r)$, namely, $2^{n-1}\varphi(x,r) < |x-y| \le 2^n\varphi(x,r)$, $n = 1, 2, \ldots$. If $1-x < 2^{n-1}\varphi(x,r)$, Lemma 6.6 gives

$$L \le C(\alpha,\beta)\frac{1-r}{|x-y|^{\alpha+3/2}},$$

hence

$$\tilde{L} \le C(\alpha,\beta)\frac{1-r}{|x-y|^{\alpha+3/2}}(1-x)^{\alpha/2}(1-y)^{\alpha/2}.$$

Notice that

$$|1-y| \le |1-x| + |x-y| \le 2^{n-1}\varphi(x,r) + 2^n\varphi(x,r) \le 2^{n+1}\varphi(x,r)$$

and

$$|I_n| = 2^{n+1}\varphi(x,r).$$

Thus, $\tilde{L}$ is dominated by

$$\begin{aligned}
&C(\alpha,\beta)\frac{1-r}{|x-y|^{\alpha+3/2}}\left(2^{n-1}\varphi(x,r)\right)^{\alpha/2}\left(2^{n+1}\varphi(x,r)\right)^{\alpha/2}\frac{2^{n+1}\varphi(x,r)}{|I_n|}\chi_{I_n}(y)\\
&\quad\le C(\alpha,\beta)\frac{1-r}{|x-y|^{1/2}}\frac{1}{|I_n|}\chi_{I_n}(y)
\end{aligned}$$

because $|x-y| \sim 2^n\varphi(x,r)$. By using this fact once again,

$$\tilde{L} \le C(\alpha,\beta)\frac{1-r}{|2^n\varphi(x,r)|^{1/2}}\frac{1}{|I_n|}\chi_{I_n}(y).$$

But, $(\varphi(x,r))^{1/2} \ge (k-1)^{1/2}$ and $(k-1)^{1/2} \sim (1-r)$ imply

$$\tilde{L} \le \frac{C(\alpha,\beta)}{2^{n/2}}\frac{1}{|I_n|}\chi_{I_n}(y).$$

In the case that $1 - x \geq 2^{n-1}\varphi(x, r)$, Lemma 6.8 implies that

$$L \leq \frac{C(\alpha, \beta)\varphi(x, r)}{(1-x)^{\alpha}|x-y|^2};$$

consequently,

$$\begin{aligned}
\tilde{L} &\leq \frac{C(\alpha, \beta)\varphi(x, r)}{(1-x)^{\alpha}|x-y|^2}(1-x)^{\alpha/2}(1-y)^{\alpha/2} \\
&\leq \frac{C(\alpha, \beta)\varphi(x, r)}{(1-x)^{\alpha}|x-y|^2}(1-x)^{\alpha/2}(1-y)^{\alpha/2}\frac{2^{n+1}\varphi(x, r)}{|I_n|}\chi_{I_n}(y) \\
&\leq \frac{C(\alpha, \beta)}{2^n}\frac{(1-y)^{\alpha/2}}{(1-x)^{\alpha/2}}\frac{1}{|I_n|}\chi_{I_n}(y),
\end{aligned}$$

because $|x-y| \sim 2^n\varphi(x, r)$. Now,

$$|1-y|^{\alpha/2} \leq (|1-x| + |x-y|)^{\alpha/2} \leq 2^{\alpha/2}\left(|1-x|^{\alpha/2} + |x-y|^{\alpha/2}\right).$$

Then

$$\begin{aligned}
\frac{(1-y)^{\alpha/2}}{(1-x)^{\alpha/2}} &\leq C(\alpha)\left(1 + \frac{|x-y|^{\alpha/2}}{(1-x)^{\alpha/2}}\right) \\
&\leq C(\alpha)\left(1 + \frac{(2^n\varphi(x, r))^{\alpha/2}}{\left(2^{n-1}\varphi(x, r)\right)^{\alpha/2}}\right) \\
&= C(\alpha).
\end{aligned}$$

Therefore,

$$\tilde{L} \leq \frac{C(\alpha, \beta)}{2^n}\frac{1}{|I_n|}\chi_{I_n}(y) \leq \frac{C(\alpha, \beta)}{2^{n/2}}\frac{1}{|I_n|}\chi_{I_n}(y). \qquad \square$$

Proof of Lemma 6.2. It follows immediately from the one dimensional estimates in Lemmas 4.1 and 6.1. $\square$

6.2. *The joint proof of Theorems* 2.1 *and* 2.2. We consider the maximal functions

$$M_n(f)(X) = \sup_r \frac{1}{\mu\{I_n(X, r)\}}\int_{I_n(X,r)}|f(Y)|\,d\mu(Y),$$

for $n = (n_1, \ldots, n_m)$, where μ is either the Lebesgue measure restricted to Q or the m-dimensional Jacobi measure $J^{(\bar{\alpha},\bar{\beta})}$, $\bar{\alpha}_j \geq \alpha_j$, $\bar{\beta}_j \geq \beta_j$. By Lemma 3.2 applied to the family of rectangles $\{I_n(X, r)\}$ with

$$\varphi_j(r) = k(r) - 1, \quad h_j\left(x_j\right) = 1 - x_j, \quad C_i = 1, \quad K_j = 2^{n_j},$$

we get the weak type estimate

$$\mu\left\{M_n(f)(X) > \lambda\right\} \leq \frac{C\prod_{j=1}^{m} n_j}{\lambda} \|f\|_{L^1(\mu)}, \tag{6.7}$$

for any $\lambda > 0$. Taking into account the estimate for the multiple Watson kernel given in Lemma 6.2 and (6.7), Theorems 2.1 and 2.2 follow from the observation made at the beginning of the proof of Theorem 2.1 in Section 5 ([C], p. 121, Lemma 1.3). □

7. Appendix

Suppose that $1 \leq s \leq 2$, $0 \leq x \leq 1$, $|y| \leq 1$,

$$\begin{aligned} Y &= \left(\left(\frac{x-y}{2}\right)^2 + \left(s^2-1\right)\left(s^2-xy\right)\right)^{\frac{1}{2}}, \\ Z_1 &= s^2 - \frac{1}{2}(x+y) + Y, \\ Z_2 &= s^2 + \frac{1}{2}(x+y) + Y. \end{aligned}$$

Then it follows that

(i) $s^2 - \min(x, y) \leq 4\,(s - \min(x, y))$,
(ii) $s - \min(x, y) \leq 2(s - xy) \leq 4\,(s - \min(x, y))$,
(iii) $C_1((x-y)^2 + (s-1)(s - \min(x, y))) \leq Y^2$
$\leq C_2((x-y)^2 + (s-1)(s - \min(x, y)))$
(iv) $s^2 - \min(x, y) \leq Z_1 \leq C(s^2 - \min(x, y))$
(v) $1 \leq s^2 + \max(x, y) \leq Z_2 \leq C$.

Furthermore, if $k = \frac{1}{2}(r^{\frac{1}{2}} + r^{-\frac{1}{2}})$ and $\varphi(x, r) = (k-1)^{\frac{1}{2}}(k-x)^{\frac{1}{2}}$, then

(vi) $k - 1 \leq \varphi(x, r) \leq k - x$, for $k > 1$, and
(vii) $C_1\,(1-r)^2 \leq k - 1 \leq C_2\,(1-r)^2$ for $0 < r_0 < r < 1$.

Here C, C_1 and C_2 denote positive constants.

Proof. (i) For $1 \leq s \leq 2$ and $a \leq 1$, we have

$$s^2 - a - 4(s-a) = s^2 - 4s + 3a \leq s^2 - 4s + 3 = s(s-4) + 3 \leq 0.$$

Hence

$$s^2 - \min(x, y) \leq 4\,(s - \min(x, y))\,,$$

because $\min(x, y) \leq 1$.

(ii) To show that $s - \min(x, y) \leq 2(s - xy)$, it is equivalent to prove that

$$2xy - \min(x, y) \leq s.$$

For let $c = 2xy - \min(x, y)$. It can be readily seen that for $y \leq 0$,

$$c \leq -\min(x, y) = -y \leq 1 \leq s.$$

Suppose $\min(x, y) = y > 0$. In this case, we have

$$c = y(2x - 1) \leq y \leq 1 \leq s.$$

Finally, if $\min(x, y) = x$, then

$$c = x(2y - 1) \leq x \leq 1 \leq s.$$

Altogether, we get

$$s - \min(x, y) \leq 2(s - xy).$$

In order to prove that $s - xy \leq 2(s - \min(x, y))$, let $d = 2\min(x, y) - xy$, and handle this case as in the above paragraph.

Note that this last proof works for any $s \geq 1$.

(iii) By (ii) applied to $s = 1$, it follows that $1 - \min(x, y)$ is of the same order of magnitude as $1 - xy$. Moreover, $s^2 - 1 = (s + 1)(s - 1)$ is of the same order of magnitude as $s - 1$. Thus, $s^2 - xy = (s^2 - 1) + (1 - xy)$ is of the same order of magnitude as $(s - 1) + (1 - \min(x, y)) = s - \min(x, y)$. Hence, the statement follows immediately from the definition of Y.

(iv) Notice that $Y^2 \geq \left(\frac{x-y}{2}\right)^2$ implies $Y \geq \frac{|x-y|}{2}$. Then

$$Z_1 = s^2 - \frac{1}{2}\,(x + y) + Y \geq s^2 - \frac{1}{2}\,(x + y) + \frac{1}{2}\,(x - y) = s^2 - y,$$

and

$$Z_1 \geq s^2 - \frac{1}{2}\,(x + y) - \frac{1}{2}\,(x - y) = s^2 - x.$$

These inequalities imply that

$$Z_1 \geq s^2 - \min(x, y).$$

From the last remark, we may apply (ii) to s^2, in order to get

$$s^2 - xy \le 2\left(s^2 - \min(x, y)\right).$$

Then,

$$\begin{aligned} Y^2 &= \frac{1}{4}(x-y)^2 + (s^2-1)(s^2-xy) \\ &\le \frac{1}{4}(1-\min(x,y))^2 + 2\left(s^2-\min(x,y)\right)\left(s^2-xy\right) \\ &\le \frac{1}{4}\left(s^2-\min(x,y)\right)^2 + 4\left(s^2-\min(x,y)\right)^2 \\ &\le C\left(s^2-\min(x,y)\right)^2. \end{aligned}$$

Therefore, since $\min(x, y) \le \frac{1}{2}(x+y)$,

$$\begin{aligned} Z_1 &= s^2 - \frac{1}{2}(x+y) + Y \\ &\le s^2 - \min(x,y) + C\left(s^2-\min(x,y)\right)^2 \\ &= C\left(s^2-\min(x,y)\right)^2. \end{aligned}$$

(v) $Z_2 = s^2 + \frac{1}{2}(x+y) + Y \ge s^2 + \frac{1}{2}(x+y) + \frac{1}{2}(x-y) = s^2 + x$ and

$$Z_2 \ge s^2 + \frac{1}{2}(x+y) - \frac{1}{2}(x-y) = s^2 + y.$$

Therefore,

$$Z_2 \ge s^2 + \max(x, y) \ge 1 + x \ge 1.$$

On the other hand, Y, x, y and s are bounded, so Z_2 is bounded as well.

(vi) This follows immediately from $0 < k - 1 < k - x$.

(vii) $k - 1 = \frac{1}{2}(r^{\frac{1}{2}} + r^{-\frac{1}{2}} - 2) = \frac{1}{2}(r^{-\frac{1}{4}} - r^{\frac{1}{4}})^2 = \frac{r^{-\frac{1}{2}}}{2}(1 - r^{\frac{1}{2}})^2 = O(1-r)^2$.

Since $0 < r_0 < r < 1$, we have

$$C_1(1-r)^2 \le k - 1 \le C_2(1-r)^2.$$

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