ILLINOIS JOURNAL OF MATHEMATICS
Volume 41, Number 2, Summer 1997

ON AN INEQUALITY DUE TO BOURGAIN

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1. The inequality

The focus of this article is a key estimate behind J. Bourgain's pointwise ergodic theorems for arithmetic subsequences of the integers ([B]). It is an interesting variant on the Hardy-Littlewood maximal function estimate on L^2, and it has tantalizing connections to some deep questions in harmonic analysis.

Some notation is necessary to state the inequality. Define the Fourier transform by $\mathcal{F}f(\xi) = \hat{f}(\xi) = \int e^{-2\pi i \xi x} f(x)\, dx$. Let φ be a smooth function satisfying, say,

$$|\varphi(x)| \leq C|x|^{-3}, \quad |\hat{\varphi}(\xi) - 1| \leq |\xi| \quad \text{and} \quad |\hat{\varphi}(\xi)| \leq C|\xi|^{-2}.$$

Let $\varphi_j(x) = 2^{-j}\varphi(2^{-j}x)$. For $\lambda \in \mathbb{R}$, let $e_\lambda(x) = e^{2\pi i \lambda x}$.

THEOREM 1.1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_L \in \mathbb{R}$ be distinct points with $|\lambda_\ell - \lambda_{\ell'}| \geq 2^{-j_0}$ for $\ell \neq \ell'$. Then*

$$\left\| \sup_{j \geq j_0} \left| \sum_{\ell=1}^{L} e_{\lambda_\ell}(x)\, \varphi_j * (e_{-\lambda_\ell} f)(x) \right| \right\|_2 \leq C\, (\log L)^3\, \|f\|_2.$$

We do not have anything to add to Bourgain's proof of this lemma. But in some applications, one actually knows a little more than just separation of the base points of the multipliers. The points λ_ℓ are in fact rational points, with the common denominator not terribly large. Taking advantage of this fact, one can give a remarkably simple proof of the estimate. Specifically:

THEOREM 1.2. *With the notation of the previous theorem, assume further that $\lambda_1, \ldots, \lambda_L \in 2^{-j_0}\Lambda^{-1}\mathbb{Z}$, for some $\Lambda > 1$. Then*

$$\left\| \sup_{j \geq j_0} \left| \sum_{\ell=1}^{L} e_{\lambda_\ell}(x)\, \varphi_j * (e_{-\lambda_\ell} f)(x) \right| \right\|_2 \leq C\, \log\log(L + \Lambda)\, \|f\|_2.$$

The proof, under the restriction that the base points of the multipliers be in a lattice, will not employ the clever ideas of Bourgain. The tools will be standard. The

Received November 29, 1995
1991 Mathematics Subject Classification. Primary 42B.

proof offered here does extend to irrational λ_ℓ that admit a favorable simultaneous Diophantine approximation—but goes no further than that.

The theorem above is strong enough for the requirements of the polynomial ergodic theorems [B]. For them, one would apply the inequality above with the λ_ℓ given by

$$\mathbb{Q}_s = \{\lambda = a/q \mid 1 \le a < q\,;\ 2^s \le q < 2^{s+1}\,;\ \text{l.c.d. of } a \text{ and } q \text{ is } 1\}.$$

Notice that there are $O(2^s)$ such rational points; they are separated by $\delta = O(2^{-2s})$; and they have a common denominator $\Lambda = O(2^{s^2})$. Hence,

$$\left\| \sup_{j \ge 2s} \left| \sum_{\lambda \in \mathbb{Q}_s} \mathrm{e}_\lambda(x)\, \varphi_j * (\mathrm{e}_{-\lambda} f)(x) \right| \right\|_2 \le C(\log s) \|f\|_2.$$

The logarithmic estimate in s is sufficent to prove the polynomial ergodic theorems.

Our proof easily treats the case where the φ_j are replaced by an appropriate truncations of a singular integral. This is relevant to the investigations of [SW].

2. Proof of Theorem 1.2

For the proof of the second theorem, the important case to observe is this.

LEMMA 2.1. *Let $\lambda_1, \lambda_2, \ldots, \lambda_L$ be distinct points with $|\lambda_\ell - \lambda_{\ell'}| \ge 1$, and $\lambda_\ell \in \Lambda^{-1}\mathbb{Z}$. Assume $L \le \Lambda$. Then*

$$\left\| \sup_{j \ge 2 \log \Lambda} \left| \sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(x)\, \varphi_j * (\mathrm{e}_{-\lambda_\ell} f)(x) \right| \right\|_2 \le C \|f\|_2.$$

Here, the supremum is over $j \ge 2 \log \Lambda$, and the constant is independent of L and Λ.

Proof. The idea is that in the further restriction in the supremum, there is an extra degree of smoothness which can be used to introduce some orthogonality.

We begin with a decomposition of f. Let $\zeta(x)$ be a smooth function with $\hat{\zeta}(0) = 1$. Set $f_l(x) = \zeta * (\mathrm{e}_{-\lambda_\ell} f)(x)$. Then

$$\begin{aligned} \|\varphi_j * (\mathrm{e}_{-\lambda_\ell} f) - \varphi_j * f_\ell\|_2 &\le \|\mathcal{F}^{-1} \widehat{\varphi_j}(\xi)\,(1 - \hat{\zeta}(\xi)) \mathcal{F} f\|_2 \\ &= \|\mathcal{F}^{-1} \widehat{\varphi}(2^j \xi)\,(1 - \hat{\zeta}(\xi)) \mathcal{F} f\|_2 \\ &\le C 2^{-j} \|f\|_2. \end{aligned}$$

Summing this estimate over $1 \le \ell \le L$ and $j \ge 2 \log \Lambda$, we see that it suffices to estimate the L^2 norm of

$$\sup_{j \ge 2 \log \Lambda} \left| \sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(x)\, \varphi_j * f_\ell(x) \right|.$$

Fix a choice of $j \geq 2\log\Lambda$ and x. We exploit smoothness in the φ_j. For any $|u| \leq \Lambda$,

$$\begin{aligned}
|\varphi_j * f_\ell(x) - \varphi_j * f_\ell(x-u)| &\leq \int |\varphi_j(x-y-u) - \varphi_j(x-y)| \cdot |f_\ell(y)|\, dy \\
&= \int 2^{-j} \left|\varphi\left(\frac{x-y-u}{2^j}\right) - \varphi\left(\frac{x-y}{2^j}\right)\right| \cdot |f_\ell(y)|\, dy \\
&\leq C \int 2^{-j}\{\Lambda 2^{-j} \wedge (1+2^{-j}|x-y|)^{-3}\} \cdot |f_\ell(y)|\, dy \\
&\leq \int_{|x-y|\leq 2^{4j/3}\Lambda^{-1/3}} + \int_{|x-y|\geq 2^{4j/3}\Lambda^{-1/3}} \cdots\, dy \\
&\leq C(\Lambda 2^{-j})^{2/3} Mf(x) \\
&\leq C\Lambda^{-1} Mf(x).
\end{aligned}$$

This implies that for any $0 \leq u \leq \Lambda$,

$$\left|\sum_{\ell=1}^{L} \{\varphi_j * f_\ell(x) - \varphi_j * f_\ell(x-u)\}\right| \leq CMf(x).$$

But then, as these estimates are uniform in j, it suffices to estimate the L^2 norm below.

To make the argument clearer, we set $f_{j,\ell} = \varphi_j * f_\ell$. And we estimate the L^2 norm of

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{1}{\Lambda} \int_0^\Lambda \sup_{j\geq 2\log\Lambda} \left|\sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(x)\, f_{j,\ell}(x-u)\right|^2 du\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\Lambda} \int_0^\Lambda \sup_{j\geq 2\log\Lambda} \left|\sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(u)\, f_{j,\ell}(x)\right|^2 du\, dx
\end{aligned}$$

Notice that this line uses the periodicity of the exponentials. But $f_{j,\ell}$ is the convolution $\varphi_j * f_\ell(x)$, so that $\mathrm{e}_{\lambda_\ell}(u)\, f_{j,\ell}(x) = \varphi_j * (\mathrm{e}_{\lambda_\ell}(u)\, f_\ell(\cdot))(x)$, treating u as a constant. Hence

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{1}{\Lambda} \int_0^\Lambda \sup_{j\geq 2\log\Lambda} \left|\varphi_j * \left(\sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(u) f_j(\cdot)\right)(x)\right|^2 du\, dx \\
&\leq C^2 \int_{-\infty}^{\infty} \frac{1}{\Lambda} \int_0^\Lambda \left|\sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(u) f_\ell(x)\right|^2 du\, dx
\end{aligned}$$

This line follows by the ordinary maximal function estimate applied in the x variable.

Continuing the line of inequalities, we conclude the proof.

$$\begin{aligned} I &\le C^2 \int_{-\infty}^{\infty} \left| \sum_{\ell=1}^{L} f_\ell(x) \right|^2 dx \\ &\le C^2 \|f\|_2^2 \sup_{\xi} \sum_{\ell=1}^{L} |\zeta(\xi - \lambda_\ell)|^2 \\ &\le C^2 \|f\|_2^2. \end{aligned}$$

□

To conclude the proof of the theorem, we need to control the supremum over $1 \le j \le 2 \log \Lambda$, which can be done with the aid of this lemma.

LEMMA 2.2. *Let $R_1 \subset R_2 \subset \cdots \subset R_K$ be sets in $\widehat{\mathbb{R}}$. Then*

$$\left\| \sup_{1 \le l \le K} |\mathcal{F}^{-1} 1_{R_k} \mathcal{F} f| \right\|_2 \le C(\log K) \|f\|_2.$$

This is really just the Rademacher-Menschov Theorem, and we could deduce it directly from that theorem. Bourgain has however, an attractive proof of the lemma, reproduced below, which can be regarded as a dualization of the standard dyadic decomposition approach to this theorem.

Proof. Let $K = 2^s$ for an integer s. Let $(S_k f)\hat{} = 1_{R_k} \hat{f}$, and let B denote the best constant in the inequality dual to the one to be proved. Namely,

$$\left\| \sum_{k=1}^{2^s} S_k f_k \right\|_2 \le B \left\| \sum_{k=1}^{2^s} |f_k| \right\|_2 .$$

The best constant B is clearly finite. An upper bound on B will be provided.

The square of the left hand side can be expanded by taking advantage of the equalities $S_k^* = S_k$, and $S_k S_{k'} = S_{k \wedge k'}$. To get the logarithm into the picture, associate to each $1 \le k \le 2^s$ the terms $(\varepsilon_1(k), \varepsilon_2(k), \ldots, \varepsilon_s(k))$ in its dyadic expansion. Namely, $k = \sum_{t=1}^{s} \varepsilon_t(k) 2^{t-1}$, where $\varepsilon_t(k) \in \{0, 1\}$. Then for an initial string of 0's and 1's, $v = (\varepsilon_1, \ldots, \varepsilon_t)$, let $\mathcal{P}(v)$ be those integers whose first t terms in its dyadic expansion agree with v. Further, denote by $v0$ the string obtained by appending 0 to the end of v, and do likewise for $v1$. Let $|v|$ be the length of the string v. The point here is that for all $k \in \mathcal{P}(v0)$ and $k' \in \mathcal{P}(v1)$, we have $k < k'$. Taking advantage of all of these observations, we can write

$$\left\| \sum_{k=1}^{2^s} S_k f_k \right\|_2^2 \le \sum_{k=1}^{2^s} \|S_k f_k\|_2^2 + 2 \sum_{0 \le |v| < s} \left| \left\langle \sum_{k \in \mathcal{P}(v0)} S_k f_k \, , \sum_{k' \in \mathcal{P}(v1)} S_{k'} f_{k'} \right\rangle \right|$$

$$\leq \sum_{k=1}^{2^s} \|f_k\|_2^2 + 2\sum_{|v|<s}\left|\left\langle \sum_{k\in\mathcal{P}(v0)} S_k f_k\,,\ \sum_{k'\in\mathcal{P}(v1)} f_{k'}\right\rangle\right|$$

$$= \mathcal{D} + \mathcal{O}.$$

The first term is trivially less than $\| \sum_k |f_k| \|_2^2$. As for the second, use the assumed bound with best constant.

$$\begin{aligned}\mathcal{O} &\leq 2\sum_{0\leq|v|<s}\left\|\sum_{k\in\mathcal{P}(v0)} S_k f_k\right\|_2 \left\|\sum_{k'\in\mathcal{P}(v1)} f_{k'}\right\|_2\\ &\leq 2B\sum_{|v|<s}\left\|\sum_{k\in\mathcal{P}(v0)} f_k\right\|_2 \left\|\sum_{k'\in\mathcal{P}(v1)} f_{k'}\right\|_2\\ &\leq 2sB\left\|\sum_{k=1}^{2^s}|f_k|\right\|_2^2.\end{aligned}$$

As each integer k is in exactly s sets $\mathcal{P}(v)$, the last line follows.

Pulling the estimates together, we see that $B^2 \leq 1 + 2sB$, from which the estimate $B \leq 2s$ follows. □

Proof of Theorem 1.2. By using a dilation, we may assume that the λ_ℓ are all separated by 1; that is, it is enough to consider the case $j_0 = 0$. But then, by Lemma 1, we need only control the supremum over $1 \leq j \leq 2\log(\Lambda + L)$. To do this, let $R_j = \{\xi \mid \min_{1\leq\ell\leq L}|\xi - \lambda_\ell| \leq 2^{-j}\}$. Then, from Lemma 2,

$$\left\|\sup_{1\leq j\leq 2\log\Lambda+L}|\mathcal{F}^{-1}1_{R_j}\mathcal{F}|\right\|_2 \leq C\,\log\log(L+\Lambda)\,\|f\|_2.$$

Use a square function argument to directly compare these Fourier projections to the multipliers we wish to control.

$$\begin{aligned}&\sum_{j=1}^{2\log L+\Lambda}\left\|\sum_{\ell=1}^{L} \mathrm{e}_{\lambda_\ell}(x)\,\varphi_j * (\mathrm{e}_{-\lambda_\ell} f)(x) - \mathcal{F}^{-1}1_{R_j}\mathcal{F}f(x)\right\|_2^2\\ &\quad\leq \|f\|_2^2\,\sup_\xi \sum_{j=1}^{\infty}\left|1_{R_j}(\xi) - \sum_{\ell=1}^{L}\widehat{\varphi}_j(\xi-\lambda_\ell)\right|^2\\ &\quad\leq C\|f\|_2^2\sum_{j=1}^{\infty}2^{-j}\\ &\quad\leq C\|f\|_2^2.\end{aligned}$$

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