ILLINOIS JOURNAL OF MATHEMATICS
Volume 41, Number 2, Summer 1997

MEAN LIPSCHITZ SPACES AND BOUNDED MEAN OSCILLATION

DANIEL GIRELA

1. Introduction and statement of results

Let Δ denote the unit disc $\{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbf{T}$ the unit circle $\{\xi \in \mathbb{C}\colon |\xi| = 1\}$. For $0 < r < 1$ and g analytic in Δ we set

$$M_p(r, g) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|g(re^{i\theta})\right|^p d\theta\right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, g) = \max_{|z|=r} |g(z)|.$$

For $0 < p \leq \infty$ the Hardy space H^p consists of those functions g, analytic in Δ, for which

$$\|g\|_{H^p} = \sup_{0<r<1} M_p(r, g) < \infty.$$

The space $BMOA$ consists of those functions $f \in H^1$ whose boundary values have bounded mean oscillation on $\mathbf{T}$. We cite [2] and [9] as references for the main properties of BMOA-functions.

If f is a function which is analytic in Δ and has a non-tangential limit $f(e^{i\theta})$ at almost every $e^{i\theta} \in \mathbf{T}$, we define

$$\omega_p(\delta, f) = \sup_{0<|t|\leq\delta} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(e^{i(\theta+t)}) - f(e^{i\theta})\right|^p d\theta\right)^{1/p}, \delta > 0, 1 \leq p < \infty,$$

$$\omega_\infty(\delta, f) = \sup_{0<|t|\leq\delta} \left(\operatorname*{ess.sup}_{\theta\in[-\pi,\pi]} |f(e^{i(\theta+t)}) - f(e^{i\theta})|\right), \quad \delta > 0.$$

Then $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f.

Received November 17, 1995
1991 Mathematics Subject Classification. Primary 30D55, 30D45.
This research ha been supported in part by a D.G.I.C.Y.T. grant and by a grant from La Junta de Andalucia.

Given $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ^p_α consists of those functions f analytic in Δ having a non-tangential limit almost everywhere for which $\omega_p(\delta, f) = O(\delta^\alpha)$, as $\delta \to 0$. If $p = \infty$ we write Λ_α instead of Λ^∞_α. This is the usual Lipschitz space of order α. More precisely, a function f analytic in Δ belongs to Λ_α if and only if it has a continuous extension to the closed unit disc $\overline{\Delta}$ and its boundary values satisfy a Lipschitz condition of order α.

A classical result of Hardy and Littlewood [11] (see also Chapter 5 of [8]) asserts that for $1 \leq p \leq \infty$ and $0 < \alpha \leq 1$, we have $\Lambda^p_\alpha \subset H^p$ and

$$\Lambda^p_\alpha = \left\{ f \text{ analytic in } \Delta\colon\ M_p(r, f') = O\left(\tfrac{1}{(1-r)^{1-\alpha}}\right), \quad \text{as } r \to 1 \right\}. \tag{1.1}$$

Now, a well known result of Privalov [8, Th. 3.11] asserts that a function f analytic in Δ has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on $\mathbf{T}$ if and only if $f' \in H^1$. Consequently, we can state the following.

THEOREM A. *Let f be a function which is analytic in Δ. Then, the three following conditions are equivalent*:

(*i*) $f \in \Lambda^1_1$.
(*ii*) $f' \in H^1$.
(*iii*) *f has a continuous extension to the closed unit disc $\overline{\Delta}$ whose boundary values are absolutely continuous on* $\mathbf{T}$.

The author has recently proved in [10] that Theorem A is sharp in a very strong sense showing that no restriction on the growth of $M_1(r, f')$ other than its boundedness is enough to conclude that f is a normal function in the sense of Lehto and Virtanen [12]. We recall that a function f which is meromorphic in Δ is a normal function if and only if

$$\sup_{z\in\Delta}(1 - |z|^2)\frac{|f'(z)|}{1 + |f(z)|^2} < \infty. \tag{1.2}$$

We refer to [1] and [18] for the theory of normal functions. More precisely, we have proved the following result.

THEOREM B. *Let ϕ be any positive continuous function defined in* $[0, 1)$ *with $\phi(r) \to \infty$, as $r \to 1$. Then, there exists a function f analytic in Δ which is not a normal function and having the property that*

$$M_1(r, f') \leq \phi(r) \quad \textit{for all } r \textit{ sufficiently close to } 1.$$

Cima and Petersen proved in [7] that $\Lambda^2_{1/2} \subset BMOA$. This result was extended by Bourdon, Shapiro and Sledd who proved the following result in [6].

THEOREM C. For $1 \leq p < \infty$, $\Lambda^p_{1/p} \subset BMOA$.

In this paper we shall show that Theorem C is sharp in a very strong sense. In order to do so, let us introduce the generalized mean Lipschitz spaces.

Let ω: $[0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$. Then, for $1 \leq p \leq \infty$, the mean Lipschitz space $\Lambda(p, \omega)$ consists of those functions f analytic in Δ which have a non-tangential limit almost everywhere and satisfy

$$\omega_p(\delta, f) = O(\omega(\delta)), \quad \text{as } \delta \to 0.$$

With this notation we have $\Lambda^p_\alpha = \Lambda(p, \delta^\alpha)$.

The question of finding conditions on ω so that it is possible to obtain results on the spaces $\Lambda(p, \omega)$ analogous to those proved by Hardy and Littlewood for the spaces Λ^p_α has been studied by several authors (e.g., [4], [5] and [13]). We shall say that ω satisfies the Dini condition or that ω is a Dini-weight if there exists a positive constant C such that

$$\int_0^\delta \frac{\omega(t)}{t}\, dt \leq C\omega(\delta), \quad 0 < \delta < 1. \tag{1.3}$$

Given $0 < q < \infty$, we shall say that ω satisfies the condition b_q or that $\omega \in b_q$ if there exists a positive constant C such that

$$\int_\delta^\pi \frac{\omega(t)}{t^{q+1}}\, dt \leq C\frac{\omega(\delta)}{\delta^q}, \quad 0 < \delta < 1. \tag{1.4}$$

In [5, Th. 2.1], Blasco and de Souza proved the following extension of (1.1).

THEOREM D. *Let ω: $[0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$. If ω is a Dini-weight and satisfies the condition b_1 then,*

$$\Lambda(p, \omega) = \left\{f \text{ analytic in } \Delta\colon M_p(r, f') = O\left(\tfrac{\omega(1-r)}{1-r}\right), \text{ as } r \to 1\right\}. \tag{1.5}$$

The main results in this paper are contained in the following two theorems.

THEOREM 1. *Let $1 < p < \infty$ and let ω: $[0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$. Suppose that*

$$\omega \text{ is a Dini-weight}, \tag{1.6}$$

$$\omega \in b_1, \tag{1.7}$$

and

$$\frac{\omega(\delta)}{\delta^{1/p}} \to \infty \quad \text{as } \delta \to 0. \tag{1.8}$$

Then there exists a function $f \in \Lambda(p, \omega)$ which is not a Bloch function.

THEOREM 2. *Let $1 < p < \infty$ and let ρ: $[0, \pi] \to [0, \infty)$ be a be a continuous and increasing function with $\rho(0) = 0$. Suppose that* (1.6), (1.7) *and* (1.8) *are true with ρ in the place of ω and that*

$$\rho^{1/2} \in b_{\frac{1}{2p}}. \tag{1.9}$$

Then there exists a function $f \in \Lambda(p, \rho)$ which is not a normal function.

We recall that a function f analytic in Δ is a Bloch function if

$$\sup_{z \in \Delta}(1 - |z|^2)|f'(z)| < \infty.$$

Since it is well known that any $BMOA$-function is a Bloch function and a Bloch function is normal, Theorem 1 and Theorem 2 together with Theorem C (see also [10, Th. 2]), prove the sharpness of Theorem B. In particular, if ω is as in Theorem 1 then $\Lambda(p, \omega) \not\subset BMOA$. It is natural to expect the conclusion of Theorem 2 to remain true without assuming condition (1.9). However our methods do not yield this.

We remark that if n is a positive integer and $\beta > 0$ then the function ω of Theorem 1 can be taken to be

$$\omega(t) = t^{1/p}\left(\underbrace{\log \dots \log}_{n \text{ times}} \frac{1}{t}\right)^{\beta},$$

for t sufficiently close to 0.

2. Proof of Results

The arguments that we are going to use in the proof of our results are related to those used in [10]. In particular, we shall make use of certain sequences defined by K. I. Oskolkov. We start by stating certain properties of the weights ω and the Oskolkov's sequences associated with them, but first let us remark that from now on we shall be using the convention that C will denote a positive constant (which may depend on ω, λ, ρ and p but not on s, t, δ or n) and which may be different at each occurrence.

LEMMA 1. *Let ω: $[0, \pi] \to [0, \infty)$ be a continuous and increasing function with $\omega(0) = 0$. If ω satisfies the condition b_1 then there exists a positive constant C such that*

$$\frac{\omega(t)}{t} \le C\frac{\omega(s)}{s}, \quad 0 < s \le t \le 1. \tag{2.1}$$

Proof of Lemma 1. We have

$$\int_t^{\pi} \frac{dx}{x^2} = \frac{1}{t} - \frac{1}{\pi} \geq \frac{C}{t}, \quad 0 < t \leq 1,$$

and then, since ω is increasing and satisfies (1.4) with $q = 1$, we easily see that if $0 < s \leq t$ then

$$\frac{\omega(t)}{t} \leq C\omega(t) \int_t^{\pi} \frac{dx}{x^2} \leq C \int_t^{\pi} \frac{\omega(x)}{x^2}\, dx \leq C \int_s^{\pi} \frac{\omega(x)}{x^2}\, dx \leq C \frac{\omega(s)}{s}.$$

Definition 1. Let ω: $[0, \pi] \to [0, \infty)$ be a continuous function with $\omega(0) = 0$ and

$$\frac{\omega(\delta)}{\delta} \to \infty \quad \text{as } \delta \to 0. \tag{2.2}$$

Take a fixed number λ with $0 < \lambda < 1$ and let us consider the sequence of numbers $\{\delta_n\}_{n=0}^{\infty}$, defined inductively as follows:

$$\begin{cases} \delta_0 = 1, \\ \delta_{n+1} = \min\left\{\delta \in [0, 1) \colon \max\left[\frac{\omega(\delta)}{\omega(\delta_n)}, \frac{\omega(\delta_n)\delta}{\delta_n \omega(\delta)}\right] = \lambda\right\}, \quad n \geq 0. \end{cases} \tag{2.3}$$

Then $\{\delta_n\}_{n=0}^{\infty}$ will be called the λ-Oskolkov sequence associated with ω.

The sequence $\{\delta_n\}$ was defined by K. I. Oskolkov in [14], [15], [16] and [17] under the hypothesis that ω is a modulus of continuity, that is, increasing and subadditive (see Proposition 2.1 of [3]). However, it is clear that the definition of $\{\delta_n\}$ makes sense in our setting. In the following lemmas we shall list the main properties of the sequence $\{\delta_n\}$ which will be used in the sequel.

LEMMA 2. *Let ω: $[0, \pi] \to [0, \infty)$ be a continuous function with $\omega(0) = 0$ which satisfies* (2.2). *Let $0 < \lambda < 1$ and let $\{\delta_n\}_{n=0}^{\infty}$ be the λ-Oskolkov sequence associated with ω. Then $\{\delta_n\}$ is a decreasing sequence of positive numbers with $\delta_n \to 0$, as $n \to \infty$. Moreover, for all $n \geq 0$, we have*

$$\omega(\delta_{n+1}) \leq \lambda \omega(\delta_n), \tag{2.4}$$

$$\delta_{n+1} \leq \lambda^2 \delta_n, \tag{2.5}$$

$$\omega(\delta_{n+1})\delta_{n+1} \leq \lambda^3 \omega(\delta_n)\delta_n, \tag{2.6}$$

$$\frac{\omega(\delta_k)}{\delta_k} \leq \lambda^{n-k} \frac{\omega(\delta_n)}{\delta_n}, \quad 0 \leq k \leq n, \tag{2.7}$$

$$\omega(\delta_k) \leq \lambda^{k-n} \omega(\delta_n), \quad k \geq n. \tag{2.8}$$

Proof of Lemma 2. First let us notice that (2.4) and (2.5) are direct consequences from the definition of the sequence $\{\delta_n\}$ and then (2.6) and the fact that δ_n tends monotonically to zero follow trivially.

The definition of $\{\delta_n\}$ implies that

$$\frac{\omega(\delta_k)}{\delta_k} \leq \lambda \frac{\omega(\delta_{k+1})}{\delta_{k+1}}, \quad k \geq 0,$$

and then (2.7) follows. On the other hand, (2.8) follows from (2.4).

LEMMA 3. *Let ω, λ and $\{\delta_n\}_{n=0}^{\infty}$ be as in Lemma* 2. *Suppose also that ω is increasing and satisfies* (1.6) *and* (1.7). *Then there exists a positive constant C such that*

$$\frac{\delta_n}{\delta_{n+1}} \leq C, \quad n \geq 0. \tag{2.9}$$

Proof of Lemma 3. If ω were a modulus of continuity then (2.9) would follow from Remark 2, pp. 145–146 of [17]. However, it is not difficult to see that (2.9) remains true in our setting. Indeed, take $n \in \mathbb{N}$. Notice that at least one of the two relations

$$\text{(R1)}\quad \omega(\delta_{n+1}) = \lambda\omega(\delta_n), \qquad \text{(R2)}\quad \frac{\omega(\delta_n)}{\delta_n} = \lambda\frac{\omega(\delta_{n+1})}{\delta_{n+1}} \tag{2.10}$$

holds. If (R1) holds, then, using (1.6) and the fact that ω is increasing, we obtain

$$C\omega(\delta_n) \geq \int_{\delta_{n+1}}^{\delta_n} \frac{\omega(t)}{t}\,dt \geq \omega(\delta_{n+1})\log\frac{\delta_n}{\delta_{n+1}} = \lambda\omega(\delta_n)\log\frac{\delta_n}{\delta_{n+1}},$$

and hence

$$\log\frac{\delta_n}{\delta_{n+1}} \leq \frac{C}{\lambda}. \tag{2.11}$$

On the other hand, if (R2) holds, then (1.7) and Lemma 1 show that

$$C\omega(\delta_{n+1}) \geq \delta_{n+1}\int_{\delta_{n+1}}^{\delta_n} \frac{\omega(t)}{t^2}\,dt \geq C\delta_{n+1}\frac{\omega(\delta_n)}{\delta_n}\log\frac{\delta_n}{\delta_{n+1}} = \lambda\omega(\delta_{n+1})\log\frac{\delta_n}{\delta_{n+1}}$$

which, again, gives (2.11). Consequently, we have proved (2.9). This finishes the proof of Lemma 3.

Proof of Theorem 1. Let p and ω be as in Theorem 1. Notice that, since $p > 1$, (1.8) implies (2.2) and so we see that ω satisfies the conditions of Lemma 3. Take a fixed $\lambda \in (0, 1)$ and let $\{\delta_n\}_{n=0}^{\infty}$ be defined by (2.3). We set

$$f(z) = \sum_{j=0}^{\infty} \frac{\omega(\delta_j)}{(1 - z + \delta_j)^{1/p}}, \quad z \in \Delta. \tag{2.12}$$

Clearly, this series converges uniformly on each compact subset of Δ and therefore it defines a function which is analytic in Δ. We have

$$f'(z) = \frac{1}{p}\sum_{j=0}^{\infty} \frac{\omega(\delta_j)}{(1-z+\delta_j)^{1+\frac{1}{p}}}, \quad z \in \Delta, \tag{2.13}$$

and, hence, for $0 < r < 1$, we have

$$\begin{aligned} M_p(r, f') &\leq C\sum_{j=0}^{\infty}\omega(\delta_j)\left(\int_{-\pi}^{\pi} \frac{d\theta}{|1+\delta_j - re^{i\theta}|^{p+1}}\right)^{1/p} \\ &\leq C\sum_{j=0}^{\infty}\omega(\delta_j)\left(\int_{-\pi}^{\pi} \frac{d\theta}{|1 - \frac{r}{1+\delta_j}e^{i\theta}|^{p+1}}\right)^{1/p} \\ &\leq C\sum_{j=0}^{\infty}\omega(\delta_j)\frac{1}{|1-\frac{r}{1+\delta_j}|} \\ &\leq C\sum_{j=0}^{\infty}\frac{\omega(\delta_j)}{1+\delta_j - r}. \end{aligned} \tag{2.14}$$

Set

$$r_n = 1 - \delta_n, \quad n \geq 1. \tag{2.15}$$

Then (2.14) implies

$$M_p(r_{n+1}, f') \leq C\sum_{j=0}^{\infty}\frac{\omega(\delta_j)}{\delta_j + \delta_{n+1}}, \quad n \geq 0. \tag{2.16}$$

Using (2.7), we obtain

$$\begin{aligned} \sum_{j=0}^{n}\frac{\omega(\delta_j)}{\delta_j+\delta_{n+1}} &\leq \frac{\omega(\delta_n)}{\delta_n}\sum_{j=0}^{n}\lambda^{n-j}\frac{\delta_j}{\delta_j+\delta_{n+1}} \\ &\leq \frac{\omega(\delta_n)}{\delta_n}\sum_{j=0}^{n}\lambda^{n-j} \leq \frac{\omega(\delta_n)}{\delta_n}\sum_{k=0}^{\infty}\lambda^k \\ &= C\frac{\omega(\delta_n)}{\delta_n}. \end{aligned} \tag{2.17}$$

On the other hand, (2.8) and (2.9) imply

$$\begin{aligned}\sum_{j=n+1}^{\infty}\frac{\omega(\delta_j)}{\delta_j+\delta_{n+1}} &\le \omega(\delta_n)\sum_{j=n+1}^{\infty}\lambda^{j-n}\frac{1}{\delta_j+\delta_{n+1}}\\ &\le C\frac{\omega(\delta_n)}{\delta_n}\sum_{j=n+1}^{\infty}\lambda^{j-n}\frac{\delta_n}{\delta_{n+1}} \le C\frac{\omega(\delta_n)}{\delta_n}\sum_{j=n+1}^{\infty}\lambda^{j-n}\\ &\le C\frac{\omega(\delta_n)}{\delta_n}\sum_{k=0}^{\infty}\lambda^{k} = C\frac{\omega(\delta_n)}{\delta_n},\end{aligned}$$

which, with (2.17) and (2.16), implies

$$M_p(r_{n+1}, f') \le C\frac{\omega(\delta_n)}{\delta_n}, \quad n \ge 0. \tag{2.18}$$

Now, since $M_p(r, f')$ is an increasing function of r, using Lemma 1 we deduce that

$$M_p(r, f') \le M_p(r_{n+1}, f') \le C\frac{\omega(\delta_n)}{\delta_n} \le C\frac{\omega(1-r)}{1-r}, \quad r_n \le r \le r_{n+1}, \quad n \ge 1.$$

Hence

$$M_p(r, f') \le C\frac{\omega(1-r)}{1-r}, \quad 0 < r < 1, \tag{2.19}$$

which, together with (1.5), shows that $f \in \Lambda(p, \omega)$.

Now, using (2.13) we easily see that

$$|f'(r_n)| \ge \frac{1}{p}\frac{\omega(\delta_n)}{(2\delta_n)^{1+\frac{1}{p}}} = C\frac{\omega(\delta_n)}{\delta_n^{1+\frac{1}{p}}}, \quad n \ge 1,$$

which, using (2.15), implies

$$(1-r_n)|f'(r_n)| \ge C\frac{\omega(\delta_n)}{\delta_n^{1/p}}, \quad n \ge 1.$$

Then (1.8) shows that $(1-r_n)|f'(r_n)| \to \infty$, as $n \to \infty$. Consequently, f is not a Bloch function. Since $f \in \Lambda(p, \omega)$, this finishes the proof of Theorem 1.

Proof of Theorem 2. Let p and ρ be as in Theorem 2. We may assume without loss of generality that

$$\rho(\delta) \ge \delta^{1/p}, \quad 0 < \delta \le 1. \tag{2.20}$$

Define

$$\omega(t) = t^{1-\frac{1}{2p}}\rho(t)^{1/2}, \quad 0 \le t \le \pi. \tag{2.21}$$

Clearly, ω is a positive, increasing and continuous function in $[0,\pi]$ with $\omega(0)=0$ which satisfies

$$\lim_{t\to 0}\frac{\omega(t)}{t}=\infty \text{ and } \omega(t)\geq t \quad \text{for all } t\in[0,1]. \tag{2.22}$$

Furthermore, since ρ is a Dini-weight, using Hölder's inequality we obtain

$$\begin{aligned}\int_0^\delta \frac{\omega(t)}{t}\,dt &= \int_0^\delta \frac{\rho(t)^{1/2}}{t^{\frac{1}{2p}}}\,dt = \int_0^\delta \left(\frac{\rho(t)}{t}\right)^{1/2}\frac{dt}{t^{\frac{1}{2}(\frac{1}{p}-1)}}\\ &\leq \left(\int_0^\delta \frac{\rho(t)}{t}\,dt\right)^{1/2}\left(\int_0^\delta \frac{dt}{t^{\frac{1}{p}-1}}\right)^{1/2} \leq C\delta^{1-\frac{1}{2p}}\rho(\delta)\\ &= C\omega(\delta), \quad 0<\delta<1.\end{aligned}$$

Hence,

$$\omega \text{ is a Dini-weight.} \tag{2.23}$$

On the other hand, (1.9) implies

$$\int_\delta^\pi \frac{\omega(t)}{t}\,dt = \int_\delta^\pi \frac{\rho(t)^{1/2}}{t^{1+\frac{1}{2p}}}\,dt \leq C\frac{\rho(\delta)^{1/2}}{\delta^{\frac{1}{2p}}} = C\frac{\omega(\delta)}{\delta}, \quad 0<\delta<1,$$

so that

$$\omega\in b_1. \tag{2.24}$$

Consequently, we have shown that ω satisfies the conditions of Lemma 3. Take $0<\lambda<1$ and let $\{\delta_n\}_{n=0}^\infty$ be the λ-Oskolkov sequence associated with ω.

The function f that we are going to construct to prove Theorem 1 will be of the form $f(z)=B(z)F(z)$ where B will be a Blaschke product while the function F will be given by a series of analytic functions in Δ which converges uniformly on every compact subset of Δ. We start with the construction of the function F. We set

$$F(z)=\sum_{j=1}^\infty \frac{\omega(\delta_j)\delta_j}{1-z+\omega(\delta_j)\delta_j}, \quad z\in\Delta. \tag{2.25}$$

Clearly, this series converges uniformly on each compact subset of Δ and therefore it defines a function which is analytic in Δ. We have

$$F'(z)=\sum_{j=1}^\infty \frac{\omega(\delta_j)\delta_j}{\left(1-z+\omega(\delta_j)\delta_j\right)^2}, \quad z\in\Delta,$$

and therefore we conclude that, for $0 < r < 1$,

$$\text{(2.26)} \quad \begin{aligned} M_p(r, F') &\le C \sum_{j=1}^{\infty} \omega(\delta_j)\delta_j \left(\int_{-\pi}^{\pi} \frac{d\theta}{|1 + \omega(\delta_j)\delta_j - re^{i\theta}|^{2p}} \right)^{1/p} \\ &= C \sum_{j=1}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1 + \omega(\delta_j)\delta_j)^2} \left(\int_{-\pi}^{\pi} \frac{d\theta}{\left|1 - \frac{re^{i\theta}}{1+\omega(\delta_j)\delta_j}\right|^{2p}} \right)^{1/p} \\ &\le C \sum_{j=1}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1 + \omega(\delta_j)\delta_j)^2} \frac{1}{\left(1 - \frac{r}{1+\omega(\delta_j)\delta_j}\right)^{2-\frac{1}{p}}} \\ &\le C \sum_{j=1}^{\infty} \frac{\omega(\delta_j)\delta_j}{(1 - r + \omega(\delta_j)\delta_j)^{2-\frac{1}{p}}}. \end{aligned}$$

Set $r_n = 1 - \delta_n$ $(n \ge 1)$. Then (2.26) implies

$$\text{(2.27)} \qquad M_p(r_{n+1}, F') \le C \sum_{j=1}^{\infty} \frac{\omega(\delta_j)\delta_j}{(\delta_{n+1} + \omega(\delta_j)\delta_j)^{2-\frac{1}{p}}}, \qquad n \ge 1.$$

Now, recalling that ω is in the conditions of Lemma 3 and using (2.22), (2.7) and (2.9), we obtain

(2.28)

$$\begin{aligned} \sum_{j=1}^{n} \frac{\omega(\delta_j)\delta_j}{(\delta_{n+1} + \omega(\delta_j)\delta_j)^{2-\frac{1}{p}}} &\le C \sum_{j=1}^{n} \frac{\omega(\delta_j)^2}{(\delta_{n+1} + \omega(\delta_j)\delta_j)^{2-\frac{1}{p}}} \\ &\le C \left(\frac{\omega(\delta_n)}{\delta_n} \right)^2 \sum_{j=1}^{n} \lambda^{2(n-j)} \frac{\delta_j^2}{(\delta_{n+1} + \omega(\delta_j)\delta_j)^{2-\frac{1}{p}}} \\ &\le C \left(\frac{\omega(\delta_n)}{\delta_n} \right)^2 \sum_{j=1}^{n} \lambda^{2(n-j)} \frac{\omega(\delta_j)\delta_j}{(\delta_{n+1} + \omega(\delta_j)\delta_j)^{2-\frac{1}{p}}} \\ &\le C \left(\frac{\omega(\delta_n)}{\delta_n} \right)^2 \sum_{j=1}^{n} \lambda^{2(n-j)} \frac{1}{\delta_{n+1}^{1-\frac{1}{p}}} \\ &\le C \frac{\omega(\delta_n)^2}{\delta_n^{3-\frac{1}{p}}} \sum_{k=0}^{\infty} \lambda^{2k} \\ &= C \frac{\omega(\delta_n)^2}{\delta_n^{3-\frac{1}{p}}}, \quad n \ge 1. \end{aligned}$$

Also, using (2.22), (2.8) and (2.9), we see that

$$\begin{aligned}
\sum_{j=n+1}^{\infty} \frac{\omega(\delta_j)\delta_j}{(\delta_{n+1}+\omega(\delta_j)\delta_j)^{2-\frac{1}{p}}} &\leq \sum_{j=n+1}^{\infty} \frac{\omega(\delta_j)^2}{(\delta_{n+1}+\omega(\delta_j)\delta_j)^{2-\frac{1}{p}}} \\
&\leq \frac{\omega(\delta_n)^2}{\delta_n^{2-\frac{1}{p}}} \sum_{j=n+1}^{\infty} \lambda^{2(j-n)} \\
&\leq \frac{\omega(\delta_n)^2}{\delta_n^{2-\frac{1}{p}}} \sum_{k=0}^{\infty} \lambda^{2k} \\
&= C\frac{\omega(\delta_n)^2}{\delta_n^{2-\frac{1}{p}}} \\
&\leq C\frac{\omega(\delta_n)^2}{\delta_n^{3-\frac{1}{p}}}, \quad n \geq 1,
\end{aligned}$$

which, with (2.28), (2.27) and (2.21), shows that

$$M_p(r_{n+1}, F') \leq C\frac{\rho(\delta_n)}{\delta_n}, \quad n \geq 1,$$

which, since $\rho \in b_1$, using Lemma 1 and arguing as in the proof of (2.19) implies

$$M_p(r, F') \leq C\frac{\rho(1-r)}{1-r}, \quad r_1 \leq r < 1. \tag{2.29}$$

Using (2.25) and the definition of r_{n+1}, we obtain

$$M_\infty(r_{n+1}, F) \leq \sum_{j=N}^{\infty} \frac{\omega(\delta_j)\delta_j}{\delta_{n+1}+\omega(\delta_j)\delta_j}, \quad n \geq 1. \tag{2.30}$$

Now, (2.22) and (2.7) imply

$$\begin{aligned}
\sum_{j=1}^{n} \frac{\omega(\delta_j)\delta_j}{\delta_{n+1}+\omega(\delta_j)\delta_j} &\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{j=1}^{n} \lambda^{j-n} \frac{\delta_j^2}{\delta_{n+1}+\omega(\delta_j)\delta_j} \\
&\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{j=1}^{n} \lambda^{j-n} \frac{\omega(\delta_j)\delta_j}{\delta_{n+1}+\omega(\delta_j)\delta_j} \\
&\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{k=0}^{\infty} \lambda^{k} = C\frac{\omega(\delta_n)}{\delta_n}, \quad n \geq 1.
\end{aligned} \tag{2.31}$$

Also, (2.8) and (2.9) imply

$$\sum_{j=n+1}^{\infty} \frac{\omega(\delta_j)\delta_j}{\delta_{n+1}+\omega(\delta_j)\delta_j} \le \frac{\omega(\delta_n)}{\delta_n} \sum_{j=n+1}^{\infty} \lambda^{j-n} \frac{\delta_n\delta_j}{\delta_{n+1}+\omega(\delta_j)\delta_j} \le C\frac{\omega(\delta_n)}{\delta_n}, \quad n \ge 1. \tag{2.32}$$

Then (2.30), (2.31) and (2.32) imply

$$M_\infty(r_{n+1}, F) \le C\frac{\omega(\delta_n)}{\delta_n}, \quad n \ge 1,$$

which, using (2.24), Lemma 1, arguing as in the proof of (2.19) and having in mind (2.21) gives

$$M_\infty(r, F) \le C\frac{\omega(1-r)}{1-r} = C\frac{\rho(1-r)^{1/2}}{(1-r)^{\frac{1}{2p}}}, \quad r_1 \le r < 1. \tag{2.33}$$

Notice that for every j,

$$\frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j}$$

is a positive increasing function of r in $(0, 1)$ and hence, using the Lebesgue's Monotone Convergence Theorem, we deduce that

$$\lim_{r\to 1^-} F(r) = \sum_{j=1}^{\infty} \lim_{r\to 1^-} \frac{\omega(\delta_j)\delta_j}{1-r+\omega(\delta_j)\delta_j} = \infty. \tag{2.34}$$

To finish the proof of Theorem 2 we shall use the following result which may be of independent interest.

THEOREM 3. *Let ϕ be any positive and continuous function defined in $[0, 1)$ with $\phi(r) \to \infty$ as $r \to 1$. Then there exists an interpolating Blaschke product B with positive zeros having the property that*

$$M_1(r, B') = O(\phi(r)) \quad \text{as } r \to 1. \tag{2.35}$$

We recall (see Chapter 9 of [8]) that a Blaschke product B is said to be an interpolating Blaschke product if its sequence of zeros $\{a_k\}$ is a universal interpolation sequence or, equivalently, uniformly separated, that is, if there exists a number $\gamma > 0$ such that

$$\prod_{\substack{j=1\\ j\ne k}}^{\infty} \left|\frac{a_k-a_j}{1-\overline{a}_j a_k}\right| \ge \gamma \quad \text{for all } k.$$

Once Theorem 3 has been stated, we continue the proof of Theorem 2. We take

$$\phi(r) = \frac{\rho(1-r)^{p/2}}{(1-r)^{1/2}}, \quad 0 \le r < 1. \tag{2.36}$$

Since (1.8) is true with ρ in the place of ω, we see that ϕ is as in the conditions of Theorem 3. Hence, there exists a Blaschke sequence $\{a_k\}_{k=1}^{\infty} \subset (0, 1)$ which is uniformly separated such that the Blaschke product

$$B(z) = \prod_{k=1}^{\infty} \frac{a_k - z}{1 - a_k z}, \quad z \in \Delta, \tag{2.37}$$

satisfies (2.35).

Now,

$$M_p(r, B') \le M_1(r, B')^{1/p} M_\infty(r, B')^{1-\frac{1}{p}}, \quad 0 < r < 1. \tag{2.38}$$

Since B is a Blaschke product, $\|B\|_{H^\infty} = 1$ and then $|B'(z)| \le (1 - |z|^2)^{-1}$ for all $z \in \Delta$(see e.g., Lemma 1.2 of [9]). Consequently, we have

$$M_\infty(r, B') \le \frac{1}{1-r} \quad \text{for all } r \in [0, 1),$$

which, with (2.35), (2.36) and (2.38), shows that

$$M_p(r, B') \le C \frac{\rho(1-r)^{1/2}}{(1-r)^{\frac{1}{2p}}} \frac{1}{(1-r)^{1-\frac{1}{p}}} = C \frac{\rho(1-r)^{1/2}}{(1-r)^{1-\frac{1}{2p}}} \tag{2.39}$$

for all r sufficiently close to 1.

Now we define

$$f(z) = F(z)B(z), \quad z \in \Delta. \tag{2.40}$$

Then $f' = F'B + FB'$ and hence

$$M_p(r, f') \le M_p(r, F')M_\infty(r, B) + M_\infty(r, F)M_p(r, B')$$

which, using (2.29), the fact that $M_\infty(r, B) \le 1$ for all r, (2.33) and (2.39) gives

$$M_p(r, f') = O\left(\frac{\rho(1-r)}{1-r}\right) \quad \text{as } r \to 1,$$

or, equivalently,

$$f \in \Lambda(p, \rho). \tag{2.41}$$

Since the sequence $\{a_k\}$ is uniformly separated, there exists $\gamma > 0$ such that

$$(1 - |a_k|^2)|B'(a_k)| = \prod_{\substack{j=1 \\ j \ne k}}^{\infty} \left| \frac{a_j - a_k}{1 - a_j a_k} \right| \ge \gamma \quad \text{for all } k. \tag{2.42}$$

Since $B(a_k) = 0$, computing the spherical derivative of f at a_k yields

$$\begin{aligned}(1-|a_k|^2)\frac{|f'(a_k)|}{1+|f(a_k)|^2} &= (1-|a_k|^2)\frac{|F'(a_k)B(a_k)+F(a_k)B'(a_k)|}{1+|B(a_k)F(a_k)|^2} \\ &= (1-|a_k|^2)|B'(a_k)||F(a_k)|\end{aligned}$$

which, using (2.42) and (2.34), implies

$$(1-|a_k|^2)\frac{|f'(a_k)|}{1+|f(a_k)|^2} \to \infty \quad \text{as } k \to \infty$$

and, hence, we see that f is not a normal function which, with (2.41), finishes the proof of Theorem 2.

Now it only remains to prove Theorem 3. We remark that even though it is not explicitely stated there, Theorem 3 is essentially proved in the proof of Theorem 1 of [10]. However, we shall include here the proof for the sake of completeness but before embarking into it, let us remark that the function ω and the sequences $\{\delta_n\}$ and $\{r_n\}$ that we are going to use to construct the Blaschke product are not the same that those used before. We believe that this will not cause any confusion.

Proof of Theorem 3. Clearly, we may assume without loss of generality that

$$\phi(r) \geq 1 \quad \text{for all } r \in [0,1).$$

Let us define

$$\phi_1(r) = \min\left(\phi(r), \frac{2}{(1-r)^{1/2}}\right), \quad 0 \leq r < 1,$$

and let ϕ_2 denote the highest increasing minorant of ϕ_1; that is,

$$\phi_2(r) = \inf_{r \leq s < 1} \phi_1(s), \quad 0 \leq r < 1.$$

Then, it is clear that ϕ_2 is a positive, continuous and increasing function in $[0,1)$ which satisfies

$$1 \leq \phi_2(r) \leq \phi(r), \quad 0 \leq r < 1, \tag{2.43}$$

$$\phi_2(r) \to \infty \quad \text{as } r \to 1, \tag{2.44}$$

and

$$(1-r)\phi_2(r) \to 0 \quad \text{as } r \to 1. \tag{2.45}$$

Let ω: $[0,1] \to \mathbb{R}$ be defined as follows

$$\begin{cases} \omega(0) = 0, \\ \omega(\delta) = \delta\phi_2(1-\delta), \quad 0 < \delta \leq 1. \end{cases} \tag{2.46}$$

Hence,

$$\phi_2(r) = \frac{\omega(1-r)}{1-r}, \quad 0 < r < 1. \tag{2.47}$$

Using (2.45), it is easy to see that ω is positive and continuous in [0, 1]. Moreover,

$$\frac{\omega(\delta)}{\delta} \to \infty \quad \text{as } \delta \to 0, \tag{2.48}$$

and

$$\omega(\delta) \geq \delta \quad \text{for all } \delta \in [0, 1]. \tag{2.49}$$

Hence we see that ω is in the conditions of Definition 1 and Lemma 2. Take a fixed number λ with $0 < \lambda < 1$ and let $\{\delta_n\}_{n=0}^{\infty}$ be the λ-Oskolkov sequence associated with ω. Since $\omega(\delta_n) \to 0$, as $n \to \infty$, there exists a positive integer N such that $\omega(\delta_n) < 1$, if $n \geq N$. Define

$$a_n = 1 - \delta_n \omega(\delta_n), \quad n \geq N. \tag{2.50}$$

Then $a_n \in (0, 1)$ for all n and (2.6) implies that the sequence $\{a_n\}_{n=N}^{\infty}$ satisfies the Blaschke condition; that is, $\sum_{n=N}^{\infty}(1 - |a_n|) < \infty$. Even more, (2.6) implies that the sequence $\{a_n\}$ is uniformly separated (see Chapter 9 of [8]). Let B denote the Blaschke product whose zeros are $\{a_n\}_{n=N}^{\infty}$; that is,

$$B(z) = \prod_{n=N}^{\infty} \frac{a_n - z}{1 - a_n z}, \quad z \in \Delta. \tag{2.51}$$

Then B is an interpolating Blaschke product. We set

$$r_n = 1 - \delta_n, \quad n \geq N. \tag{2.52}$$

Protas proved in [19, p. 394] that

$$\int_{-\pi}^{\pi} |B'(re^{i\theta})| \, d\theta \leq 8\pi \sum_k \frac{1 - |a_k|}{1 - r + 1 - |a_k|}, \quad 0 < r < 1.$$

Using this inequality and the definitions of r_n and a_n, for every $n \geq N$ we have

$$M_1(r_{n+1}, B') \leq C \sum_{k=N}^{\infty} \frac{\omega(\delta_k)\delta_k}{\delta_{n+1} + \delta_k \omega(\delta_k)}. \tag{2.53}$$

Now, (2.7) and (2.49) show that, for $n \geq N$, we have

$$\begin{aligned} \sum_{k=N}^{n} \frac{\omega(\delta_k)\delta_k}{\delta_{n+1} + \delta_k \omega(\delta_k)} &\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{k=N}^{n} \lambda^{n-k} \frac{\delta_k^2}{\delta_k \omega(\delta_k)} \\ &\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{k=0}^{\infty} \lambda^k = C \frac{\omega(\delta_n)}{\delta_n}. \end{aligned} \tag{2.54}$$

On the other hand, using (2.8) and the fact that δ_n is decreasing, we see that, for $n \geq N$,

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{\omega(\delta_k)\delta_k}{\delta_{n+1} + \delta_k\omega(\delta_k)} &\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{k=n+1}^{\infty} \lambda^{k-n} \frac{\delta_n\delta_k}{\delta_{n+1} + \delta_k\omega(\delta_k)} \\ &\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{k=n+1}^{\infty} \lambda^{k-n} \frac{\delta_k}{\delta_{n+1}} \\ &\leq \frac{\omega(\delta_n)}{\delta_n} \sum_{k=0}^{\infty} \lambda^k \leq C\frac{\omega(\delta_n)}{\delta_n},\end{aligned}$$

which, with (2.54), (2.53), (2.52) and (2.47), implies

$$M_1(r_{n+1}, B') \leq C\phi_2(r_n), \quad n \geq N. \tag{2.55}$$

Since $M_1(r, B')$ and $\phi_2(r)$ are increasing functions of r, arguing as in the proof of (2.19), we see that (2.55) implies that $M_1(r, B') \leq C\phi_2(r)$ for all r sufficiently close to 1. In view of (2.43), this implies (2.35) and finishes the proof of Theorem 3.

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Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
Girela@anamat.cie.uma.es