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SOME REMARKS ON THE WHITEHEAD ASPHERICITY CONJECTURE

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ABSTRACT. The Whitehead asphericity conjecture claims that if $\langle \mathcal{A} \| \mathcal{R} \rangle$ is an aspherical group presentation, then for every $\mathcal{S} \subset \mathcal{R}$ the subpresentation $\langle \mathcal{A} \| \mathcal{S} \rangle$ is also aspherical. It is proven that if the Whitehead conjecture is false then there is an aspherical presentation $E = \langle \mathcal{A} \| \mathcal{R} \cup \{z\} \rangle$ of the trivial group E, where the alphabet $\mathcal{A}$ is finite or countably infinite and $z \in \mathcal{A}$, such that its subpresentation $\langle \mathcal{A} \| \mathcal{R} \rangle$ is not aspherical. It is also proven that if the Whitehead conjecture fails for finite presentations (i.e., with finite $\mathcal{A}$ and $\mathcal{R}$) then there is a finite aspherical presentation $\langle \mathcal{A} \| \mathcal{R} \rangle$, $\mathcal{R} = \{R_1, R_2, \dots, R_n\}$, such that for every $\mathcal{S} \subseteq \mathcal{R}$ the subpresentation $\langle \mathcal{A} \| \mathcal{S} \rangle$ is aspherical and the subpresentation $\langle \mathcal{A} \| R_1 R_2, R_3, \dots, R_n \rangle$ of aspherical $\langle \mathcal{A} \| R_1 R_2, R_2, R_3, \dots, R_n \rangle$ is not aspherical.

Introduction

The Whitehead asphericity conjecture (originally stated as a question in [W]) claims that every subcomplex L of an aspherical 2-complex K (meaning $\pi_2(K) = 0$) is also aspherical. Due to Howie [H1] it is known that if the Whitehead asphericity conjecture is false then there exists a counterexample (L, K) of one of the following two types:

1. K is a finite aspherical contractible 2-complex and L is non-aspherical subcomplex of K obtained from K by removing one 2-cell.
2. K is an aspherical contractible 2-complex, $K = \cup_{i=1}^{\infty} L_i$, $L_i \subset L_{i+1}$, the inclusion $L_i \to L_{i+1}$ is nullhomotopic, each L_i is finite and is not aspherical.

Recently Luft [L] reproved Howie's result and showed that the existence of a counterexample of type 1 implies the existence of a counterexample of type 2.

Recall that a group presentation $G = \langle \mathcal{A} \| \mathcal{R} \rangle$, where $\mathcal{A}$ is a group alphabet, $\mathcal{R}$ is a set of defining relators (which are reduced words in the free group $F(\mathcal{A})$ over $\mathcal{A}$), is called *aspherical* if the standard 2-complex K_G associated with the presentation $G = \langle \mathcal{A} \| \mathcal{R} \rangle$ (K_G has a single vertex and $\pi_1(K_G) = G$) is aspherical. The asphericity of $G = \langle \mathcal{A} \| \mathcal{R} \rangle$ can be rephrased in group-theoretic terms as follows (see [GR]): The relation module $\mathcal{M}(G)$ of $G = \langle \mathcal{A} \| \mathcal{R} \rangle$ is freely generated by images of the relators $R \in \mathcal{R}$. Accordingly, the Whitehead asphericity conjecture then states that every

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subpresentation $\langle \mathcal{A} \| \mathcal{R}' \rangle$ of an aspherical presentation $\langle \mathcal{A} \| \mathcal{R} \rangle$, where $\mathcal{R}' \subseteq \mathcal{R}$, is also aspherical. In this note, we will show that it is possible to assume in the Whitehead asphericity conjecture that the removed part $\mathcal{R} \setminus \mathcal{R}'$ of $\mathcal{R}$ is just a letter of $\mathcal{A}$. Recall that a presentation is called *finite* if both $\mathcal{A}$ and $\mathcal{R}$ are finite.

THEOREM 1. *If the Whitehead asphericity conjecture is false then there is an aspherical presentation $E = \langle \mathcal{A} \| \mathcal{R} \cup \{z\} \rangle$ of the trivial group E, where the alphabet $\mathcal{A}$ is finite or countably infinite and $z \in \mathcal{A}$, such that its subpresentation $\langle \mathcal{A} \| \mathcal{R} \rangle$ is not aspherical. In addition, if there is a finite presentation giving a counterexample to the Whitehead asphericity conjecture, then there is a finite presentation $\langle \mathcal{A} \| \mathcal{R} \cup \{z\} \rangle$ such that its subpresentation $\langle \mathcal{A} \| \mathcal{R} \rangle$ is not aspherical.*

Recall that elementary Andrews–Curtis *transformations* over a group presentation $\langle \mathcal{A} \| \mathcal{R} \rangle$ of types (T1)–(T3) are defined as follows:

(T1) Add a new letter $b \notin \mathcal{A}$ to both $\mathcal{A}$ and $\mathcal{R}$.
(T2) If $a \in \mathcal{A}$, $a \in \mathcal{R}$, and a, a^{-1} do not occur in relators $R \in \mathcal{R} \setminus \{a\}$ then delete a in both $\mathcal{A}$ and $\mathcal{R}$.
(T3) Replace $R \in \mathcal{R}$ by $R^{\varepsilon} C S^{\delta} C^{-1}$, where $\varepsilon, \delta \in \{\pm 1\}$, $C \in F(\mathcal{A})$, and $S \in \mathcal{R} \setminus \{R\}$.

Two finite presentations are called Andrews–Curtis *equivalent* if one of them can be obtained from the other by a finite sequence of elementary Andrews–Curtis transformations. (Recall that another major problem of low dimensional topology, the so-called Andrews–Curtis conjecture [AC], asks whether a finite aspherical presentation of the trivial group is Andrews–Curtis equivalent to $\langle \mathcal{A} \| \mathcal{A} \rangle$.)

Clearly, transformations (T1)–(T3) preserve the asphericity of a presentation $\langle \mathcal{A} \| \mathcal{R} \rangle$. Moreover, (T1)–(T2) evidently preserve the asphericity of subpresentations. Whether (T3) preserves the asphericity of subpresentations is unclear and turns out to be equivalent to the Whitehead asphericity conjecture for finite presentations following from the next result.

THEOREM 2. *Suppose $\langle \mathcal{A} \| \mathcal{R} \rangle$ is a finite aspherical presentation. Then $\langle \mathcal{A} \| \mathcal{R} \rangle$ is Andrews–Curtis equivalent (with a single (T1), no (T2) and several (T3)'s) to a finite aspherical presentation $\langle \mathcal{B} \| \mathcal{S} \rangle$ such that for every $\mathcal{S}' \subseteq \mathcal{S}$ the subpresentation $\langle \mathcal{B} \| \mathcal{S}' \rangle$ is aspherical.*

For 2-complexes (see [S] or [H2] for definitions), Theorem 2 implies:

COROLLARY. *Every finite aspherical 2-complex can be 3-deformed to a finite 2-complex all of whose subcomplexes are aspherical.*

Technical details in proving Theorem 2 enable us to sharpen the equivalence between the Whitehead asphericity conjecture for finite presentations and preservation of asphericity of subpresentations under (T3) as follows.

THEOREM 3. *If the Whitehead asphericity conjecture is false for finite presentations then there is a finite aspherical presentation* $\langle \mathcal{A} \| \mathcal{R} \rangle$, $\mathcal{R} = \{R_1, R_2, \dots, R_n\}$, *such that for every* $\mathcal{R}' \subseteq \mathcal{R}$ *the subpresentation* $\langle \mathcal{A} \| \mathcal{R}' \rangle$ *is aspherical and the subpresentation* $\langle \mathcal{A} \| R_1 R_2, R_3, \dots, R_n \rangle$ *of aspherical* $\langle \mathcal{A} \| R_1 R_2, R_2, R_3, \dots, R_n \rangle$ *is not aspherical.*

The proofs of Theorems 1–3 will be based on ideas of small cancellation theory (see [LS], [O]) and given in Section 2.

1. Preliminaries

Let a group G be given by a presentation

$$G = \langle \mathcal{A} \| \mathcal{R} \rangle,$$

where $\mathcal{A}$ is an alphabet, $\mathcal{A} = \{a_i \mid i \in I\}$, and $\mathcal{R} = \{R_j \mid j \in J\}$ is the set of defining relators of G such that each R_j is a nonempty cyclically reduced word over $\mathcal{A}^{\pm 1} = \mathcal{A} \cup \mathcal{A}^{-1}$.

By a *disk map* M we mean, as in [LS] and [O], a finite, planar, connected and simply connected simplicial 2-complex. Similarly, a *spherical map* is a finite 2-complex which is located on a 2-sphere and has no boundary. 0-, 1-, 2-cells of M are called *vertices*, *edges*, *cells* of M, respectively.

A disk (resp. spherical) diagram Δ over G given by (1) is a disk (resp. spherical) map that is equipped with a labeling function ϕ from the set of oriented edges of Δ to the alphabet $\mathcal{A}^{\pm 1}$ such that:

(L1) If $\phi(e) = a$, then $\phi(e^{-1}) = a^{-1}$.
(L2) If Π is a cell in Δ and $\partial \Pi = e_1 \dots e_k$ is the boundary cycle of Π, where $e_1, \dots, e_k$ are oriented edges, then $\phi(\partial \Pi) = \phi(e_1) \dots \phi(e_k)$ is a cyclic permutation of R^{ε}, where $\varepsilon = \pm 1$, $R \in \mathcal{R}$.

It is convenient to fix the positive (counterclockwise) orientation for the boundary $\partial \Pi$ of a cell Π in Δ and the negative (clockwise) orientation for the boundary $\partial \Delta$ (if any) of a diagram Δ.

If e is an oriented edge in a diagram Δ then e_- and e_+ will denote the initial and terminal vertices of e, respectively.

Let e be an oriented edge, Π_1, Π_2 cells in a diagram Δ, and $e \in \partial \Pi_1$, $e^{-1} \in \partial \Pi_2$. The cells Π_1, Π_2 in Δ are said to be a *reducible pair* provided the label $\phi(\partial \Pi_1 |_{e_-})$ of the (oriented) boundary $\partial \Pi_1 |_{e_-}$ starting at e_- is graphically (i.e., letter-by-letter) equal to $\phi(\partial \Pi_2 |_{e_-})^{-1}$. A diagram Δ over $G = \langle \mathcal{A} \| \mathcal{R} \rangle$ is termed *reduced* provided Δ contains no reducible pairs of cells.

If X, Y are words over $\mathcal{A}^{\pm 1}$ then $X = Y$ will denote the equality in the free group $F(\mathcal{A})$. The graphical (letter-by-letter) equality of X and Y is denoted by $X \equiv Y$.

The following lemma due to van Kampen is straightforward (see [LS], [O]); recall that a cyclic word is a word written on a circle).

LEMMA 1. *A nonempty cyclic word W equals 1 in the group $G = \langle \mathcal{A} \| \mathcal{R} \rangle$ if and only if there is a reduced disk diagram Δ over G such that $\phi(\partial \Delta) \equiv W$.*

2. Proofs of Theorems 1–3

Let $\langle \mathcal{A} \| \mathcal{R} \rangle$ be an aspherical group presentation, where $\mathcal{A}$ is finite or countably infinite, and its subpresentation $\langle \mathcal{A} \| \mathcal{S} \rangle$, where $\mathcal{S} \subset \mathcal{R}$ is finite, be not aspherical.

Let $y \in \mathcal{A}$ be a letter that occurs in some relator $R \in \mathcal{S}$ and $y \neq 1$ in the group given by $\langle \mathcal{A} \| \mathcal{S} \rangle$ (if there were no such letter the subpresentation $\langle \mathcal{A} \| \mathcal{S} \rangle$ would be obviously aspherical contrary to the assumption).

Let $z \notin \mathcal{A}$ be a new letter, $K = 10^5$, $L = 10^2$ (we will only require that L be "large enough" and $4L + 4 < 0.01K$), and the words $V_t(y, z)$, $t = 1, 2, \dots$, be defined as follows:

$$V_t(y,z) \equiv z^{Kt} y z^{Kt+1} y z^{Kt+2} y^{-1} z^{Kt+3} y^{-1} z^{Kt+4} y z^{Kt+5} y z^{Kt+6} y^{-1} z^{Kt+7} y^{-1} \cdots z^{Kt+4L} y z^{Kt+4L+1} y z^{Kt+4L+2} y^{-1} z^{Kt+4L+3} y^{-1} z^{Kt+4L+4}. \tag{1}$$

Now put $\mathcal{A}_z = \mathcal{A} \cup \{z\}$, $\mathcal{R}_0 = \mathcal{R} \setminus \mathcal{S}$, and $\mathcal{R}_0 = \{R_1, R_2, \dots\}$. Consider the presentations

$$\langle \mathcal{A}_z \| \mathcal{S} \cup \{R_1 V_1, R_2 V_2, \dots\} \cup \{z\} \rangle, \tag{2}$$

$$G = \langle \mathcal{A}_z \| \mathcal{S} \cup \{R_1 V_1, R_2 V_2, \dots\} \rangle, \tag{3}$$

$$H = \langle \mathcal{A} \| \mathcal{S} \rangle. \tag{4}$$

Since every V_t belongs to the normal closure of z in $F(\mathcal{A}_z)$, presentation (2) is aspherical (like $\langle \mathcal{A}_z \| \mathcal{R} \cup \{z\} \rangle$).

LEMMA 2. *Suppose $W = W(\mathcal{A})$ is a word in $\mathcal{A}^{\pm 1}$ and $W = 1$ in the group G given by* (3). *Then $W = 1$ in the group H given by* (4).

Proof. Let Δ be a diagram over G and e an edge of Δ with $\phi(e) = z^{\pm 1}$. We will call such an e a *z-edge* of Δ. A cell Π in Δ will be referred to as a *z-cell* if $\partial \Pi$ contains z-edges. Clearly, Π is a z-cell if and only if $\phi(\partial \Pi) = (R_t V_t)^{\pm 1}$ for some t. If T is a word in $\mathcal{A}_z^{\pm 1}$, then $|T|_z$ will denote the total number of occurrences of $z^{\pm 1}$ in T. Let $|p|_z$ denote the number of z-edges of a path p in Δ. Clearly, $|p|_z = |\phi(p)|_z$.

Assume that Lemma 2 is false and pick a disk diagram Δ over G with $\phi(\partial \Delta) \equiv W$ such that W is a word in $\mathcal{A}^{\pm 1}$, $W = 1$ in G, $W \neq 1$ in H, and Δ is minimal relative to the number N_z of z-cells in Δ and, if N_z is fixed, relative to the number of all cells in Δ.

It is clear that Δ contains z-cells and there are no z-edges on the boundary $\partial \Delta$ of Δ. Consequently, if Π is a z-cell and $e \in \partial \Pi$ is a z-edge then $e^{-1} \in \partial \Pi'$, where Π' is another z-cell in Δ.

By a *z-contiguity subdiagram* between two z-cells Π_1, Π_2 in Δ we mean a subdiagram Γ of Δ with $\partial\Gamma = pq$, where p, q are subpaths of $\partial\Pi_1, \partial\Pi_2$, respectively, such that $p = e_1 p_1 \dots e_{k-1} p_{k-1} e_k$, $q = e_k^{-1} q_{k-1} e_{k-1}^{-1} \dots q_1 e_1^{-1}$, where $e_1, \dots, e_k$ are all consecutive z-edges of p and $e_k^{-1}, \dots, e_1^{-1}$ are all consecutive z-edges of q. A z-contiguity subdiagram Γ between z-cells Π_1, Π_2 is referred to as *maximal* if Γ is not contained in a bigger z-contiguity subdiagram Γ' (relative to $|\partial\Gamma'|_z$).

It follows from the choice of Δ that if Γ is a z-contiguity subdiagram in Δ, then Γ has no z-cells (for otherwise, one of subdiagrams Δ_j in Γ with $\partial\Delta_j = p_j q_j$ would contain z-cells, contrary to the minimality of Δ). Therefore, it follows from definition (1) of words $V_t(y, z)$ that if Γ is a z-contiguity subdiagram between z-cells Π_1, Π_2 with $\partial\Gamma = pq$, where p, q are subpaths of $\partial\Pi_1, \partial\Pi_2$, respectively, then there is a subpath r of $\partial\Pi_1$ such that $\phi(r)$ is a subword of $V_{t_1}^{\pm 1}$ (provided $\phi(\partial\Pi_1) = (R_{t_1} V_{t_1})^{\pm 1}$), r^{-1} is a subpath of $\partial\Pi_2$ such that $\phi(r^{-1})$ is a subword of $V_{t_2}^{\pm 1}$ (provided $\phi(\partial\Pi_2) = (R_{t_2} V_{t_2})^{\pm 1}$), and $|r|_z \geq |p|_z/3$.

Since Δ is reduced (following from its choice) and so Π_1, Π_2 may not form a reducible pair, it follows from definition (1) again that

$$|r|_z < 2\,(\min(t_1, t_2)K + 4L + 4)\,.$$

Therefore, it follows from the inequality $4L + 4 < 0.01K$ that

$$\frac{|r|_z}{|\partial\Pi_1|_z}, \frac{|r|_z}{|\partial\Pi_2|_z} \leq \frac{2 \cdot 1.01K}{4LK} < \frac{1}{18},$$

whence

$$\frac{|p|_z}{|\partial\Pi_1|_z}, \frac{|p|_z}{|\partial\Pi_2|_z} < 3 \cdot \frac{2 \cdot 1.01K}{4LK} < \frac{1}{6}. \tag{5}$$

Let us construct an auxiliary graph Φ in Δ as follows: Pick a vertex o_Π inside each z-cell Π of Δ and connect vertices o_{Π_1} and o_{Π_2} by an edge e_Γ provided Γ is a maximal z-contiguity subdiagram between Π_1 and Π_2 so that e_Γ goes through a z-edge of Γ. It follows from definitions and inequality (5) that every vertex $o \in \Phi$ is incident to at least 6 edges of Φ and if $e_{\Gamma_1} e_{\Gamma_2} \dots e_{\Gamma_k}$ is a cycle in Φ then $k \geq 3$. Since Φ is planar and connected (following from choice of Δ), by Euler's formula we have $V - E + F = 1$, where V, E, F are the numbers of vertices, edges, faces of Φ, respectively. By the above observations, $3F \leq 2E$ and $6V \leq 2E$, whence $E/3 - E + 2E/3 = 0 \geq 1$. This contradiction shows no such Δ exists and Lemma 2 is proven. $\square$

Now suppose that both $\mathcal{A}$ and $\mathcal{R}$ are finite, $\mathcal{R} = \{R_1, \dots, R_n\}$, and the presentation $\langle \mathcal{A} \| \mathcal{R} \rangle$ is aspherical. As above, let $z \notin \mathcal{A}$ be a new letter, let $\mathcal{A}_z = \mathcal{A} \cup \{z\}$, pick $y \in \mathcal{A}$, and consider two more presentations

$$\langle \mathcal{A}_z \| R_1 V_1, R_2 V_2, \dots, R_n V_n, z \rangle, \tag{6}$$

$$G = \langle \mathcal{A}_z \| R_1 V_1, R_2 V_2, \dots, \ R_n V_n, z R_1 V_1 R_2 V_2 \dots R_n V_n \rangle, \tag{7}$$

where the words $V_t = V_t(y, z)$ are defined by (1).

LEMMA 3. *Suppose*

$$\mathcal{S} \subseteq \{R_1V_1, R_2V_2, \dots, R_nV_n, zR_1V_1R_2V_2\dots R_nV_n\}.$$

Then the subpresentation

$$\langle \mathcal{A}_z \| \mathcal{S} \rangle \tag{8}$$

of (7) *is aspherical.*

Proof. If $\mathcal{S} = \{R_1V_1, R_2V_2, \dots, R_nV_n, zR_1V_1R_2V_2\dots R_nV_n\}$, then the asphericity of presentation (8) follows from the asphericity of $\langle \mathcal{A}_z \| \mathcal{R} \cup \{z\}\rangle$ (recall that all V_t belong to the normal closure of z).

So we let $\mathcal{S} \neq \{R_1V_1, R_2V_2, \dots, R_nV_n, zR_1V_1R_2V_2\dots R_nV_n\}$ and Δ be a spherical reduced diagram over (8) such that Δ contains cells and is minimal relative to the number of cells. Keeping the notation introduced in the proof of Lemma 2, first assume that there are two cells π and Π in Δ and a z-contiguity subdiagram Γ with $\partial\Gamma = pq$, where p is a subpath of $\partial\Pi$, q is a subpath of $\partial\pi$, between Π and π such that $\phi(\partial\pi)^{\pm1} = R_iV_i$ and $|q|_z \ge \frac{1}{6}|\partial\pi|_z$. Then, by the same argument as in the proof of Lemma 2 (involving a reference to the analog of Lemma 2 for presentation (8) in which $H = F(\mathcal{A})$), one has $\phi(\partial\Pi)^{\pm1} = zR_1V_1R_2V_2\dots R_nV_n$ and $\partial\Gamma = pq$, where p is a subpath of the arc u of Π with $\phi(u) = V_i^{\pm1}$, q is a subpath of the arc v of π with $\phi(v) = V_i^{\mp1}$ and hence one can deform Δ so that $u = v^{-1}$, that is, $\partial\Gamma = uv^{-1}$. By this observation, we may assume that if Γ is a z-contiguity subdiagram with $\partial\Gamma = pq$ and $|q|_z \ge \frac{1}{6}|\partial\pi|_z$ (we will call such a Γ *exceptional*), then

$$\phi(\partial\pi)^{\pm1} = R_iV_i, \quad \phi(\partial\Pi)^{\mp1} = zR_1V_1R_2V_2\dots R_nV_n, \quad p = q^{-1},$$

and $|q|_z = |\partial\pi|_z$.

As in the proof of Lemma 2, we consider an auxiliary graph Φ in Δ constructed as follows: Pick a vertex o_π inside each cell π of Δ except when π is a cell with $\phi(\partial\pi)^{\pm1} = R_iV_i$ and there is an exceptional z-contiguity subdiagram between π and a cell π' with $\phi(\partial\pi')^{\mp1} = zR_1V_1R_2V_2\dots R_nV_n$. Connect vertices o_{π_1} and o_{π_2} by an edge e_Γ provided Γ is a maximal z-contiguity subdiagram between π_1 and π_2 (clearly, Γ is not exceptional) so that e_Γ goes through a z-edge of Γ. Since the set $\mathcal{S}$ omits at least one relator of the set $\{R_1V_1, R_2V_2, \dots, R_nV_n, zR_1V_1R_2V_2\dots R_nV_n\}$, it follows that for every cell Π with $\phi(\partial\Pi)^{\pm1} = zR_1V_1R_2V_2\dots R_nV_n$, the vertex o_Π is incident to some edges of Φ. Hence, it now follows from definitions and the analog of inequality (5) that every vertex $o \in \Phi$ is incident to more than 6 edges of Φ and if $e_{\Gamma_1}e_{\Gamma_2}\dots e_{\Gamma_k}$ is a cycle in Φ then $k \ge 3$. Let Φ_0 be a connected component of Φ. Arguing as in the proof of Lemma 2, we get $V_0 - E_0 + F_0 = 2$ (for Φ_0 is on 2-sphere) and, on the other hand, $3F_0 \le 2E_0$, $6V_0 \le 2E_0$ which, as in the proof of Lemma 2, contradicts the existence of Δ. $\square$

Proof of Theorem 1. If the Whitehead asphericity conjecture is false then, according to Howie [H1] (see also [L], [I]), there are an aspherical presentation $E =$

$\langle \mathcal{A} \| \mathcal{R} \rangle$, where $\mathcal{A}$ is finite or countably infinite, of the trivial group E and a finite subset $\mathcal{S} \subset \mathcal{R}$ such that $\langle \mathcal{A} \| \mathcal{S} \rangle$ is not aspherical. Applying the construction of presentation (2), it follows from Lemma 2 that we can naturally embed the group $H = \langle \mathcal{A} \| \mathcal{S} \rangle$ in the group G given by (3) and, therefore, presentation (3) is not aspherical. Since (3) is obtained from (2) by deleting the relator z, the first claim of Theorem 1 is proven. Next, if $\langle \mathcal{A} \| \mathcal{R} \rangle$ is finite and aspherical and $\langle \mathcal{A} \| \mathcal{S} \rangle$, $\mathcal{S} \subset \mathcal{R}$, is not aspherical, then, as above, presentation (3) is nonaspherical and obtained from finite aspherical presentation (2) by deleting relator z. □

Proof of Theorem 2. Applying a single transformation (T1) and several transformations (T3) to a finite aspherical presentation $\langle \mathcal{A} \| \mathcal{R} \rangle$ we will get presentation (7) all of whose subpresentation are aspherical by Lemma 3. □

Proof of Theorem 3. Suppose the Whitehead asphericity conjecture fails for finite presentations and $\langle \mathcal{A} \| \mathcal{R} \rangle$ is a finite aspherical presentation, $\mathcal{R} = \{R_1, \dots, R_n\}$, $\mathcal{S} = \{R_1, \dots, R_m\}$, $m < n$, and $\langle \mathcal{A} \| \mathcal{S} \rangle$ is nonaspherical. Consider presentations (6) and (7). Note that presentation (6) is also aspherical and its subpresentation $\langle \mathcal{A}_z \| \{R_1 V_1, \dots, R_m V_m, z\} \rangle$ is nonaspherical. Also observe that presentation (7) (all of whose subpresentations are aspherical by Lemma 3) can be obtained from (6) by n transformations (T3) that are multiplications of the last relator by all preceding ones. Applying the inverse transformations to get (6) back from (7), we see that the property of having all subpresentations aspherical must fail at one of these n reverse steps. This proves Theorem 3. □

References

[AC] J. J. Andrews and M. L. Curtis, *Free groups and handlebodies*, Proc. Amer. Math. Soc. **16**(1965), 192–195.

[GR] M. Gutierrez and J. G. Ratcliffe, *On the second homotopy group*, Quart. J. Math. Oxford (2) **32**(1981), 45–55.

[H1] J. Howie, *Some remarks on a problem of J. H. C. Whitehead*, Topology **22**(1983), 475–485.

[H2] ———, *On the asphericity of ribbon disk complements*, Trans. Amer. Math. Soc. **289**(1985), 281–302.

[I] S. V. Ivanov, *The Whitehead asphericity conjecture and periodic groups*, submitted.

[L] E. Luft, *On 2-dimensional aspherical complexes and a problem of J. H. C. Whitehead*, Math. Proc. Cambridge Phil. Soc. **119**(1996), 493–495.

[LS] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

[O] A. Yu. Ol'shanskii, *Geometry of defining relations in groups*, Nauka, Moscow, 1989; English translation in *Math. and Its Applications* (Soviet series), vol. 70, Kluwer Acad. Publishers, 1991.

[S] A. J. Sieradski, *Combinatorial isomorphisms and combinatorial homotopy equivalences*, J. Pure Appl. Algebra **7**(1976), 59–65.

[W] J. H. C. Whitehead, *On adding relations to homotopy groups*, Ann. of Math. **42**(1941), 409–428.

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