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BILIPSCHITZ GROUP ACTIONS AND HOMOGENEOUS JORDAN CURVES

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ABSTRACT. We analyze bilipschitz group actions on Jordan curves and present a list of alternative descriptions for bilipschitz homogeneous bounded turning Jordan curves in doubling metric spaces.

1. Introduction

Recall that a map f between metric spaces is *bilipschitz* provided

$$K^{-1}|x-y| \leq |f(x)-f(y)| \leq K|x-y| \quad \text{for all } x, y;$$

we abbreviate this by the phrase 'f is K-BL'. We say that G is a bilipschitz group *acting* on a Jordan curve Γ if G is a uniformly bilipschitz group of orientation preserving self-homeomorphisms of Γ. In this situation there is no harm in assuming that G is closed with respect to the topology of local uniform convergence. Our first result asserts that such groups have a simple algebraic structure.

THEOREM A. *Let Γ be a Jordan curve in any metric space. Suppose G is a bilipschitz group acting on Γ. Then either G is cyclic (hence finite in the case of a compact curve) or G is a one-parameter group.*

A metric space X is *bilipschitz homogeneous* if there is a family of uniformly bilipschitz self-homeomorphisms of X which acts transitively on X; i.e., there is a constant K such that for each pair of points $x, y \in X$ there exists a K-bilipschitz $f\colon X \to X$ with $f(x) = y$. A natural class of mappings associated with such spaces are the *quasihomogeneous*, or QH, embeddings $h\colon X \to Y$ which satisfy

$$\frac{|hx-hy|}{|hu-hv|} \leq \eta\left(\frac{|x-y|}{|u-v|}\right)$$

for distinct points x, y, u, v; here $\eta\colon [0,+\infty) \to [0,+\infty)$ is a homeomorphism, and we abbreviate this by saying that h is η-QH. For example, if $h\colon X \to Y$ is η-quasihomogeneous and X is K-bilipschitz homogeneous, then $h(X)$ is $\eta(K)$-bilipschitz homogeneous.

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Our interest here lies in the case where $X = \Gamma$ is a Jordan curve in some ambient *doubling* metric space. Bilipschitz homogeneous curves, especially in $\mathbf{R}^n$, have been considered in [GH$_2$], [M], and recently by Chris Bishop [B]. Standard examples in the plane include all chordarc curves; however, in contrast with the quasiconformal case, (Erkama [E] established that the curves homogeneous with respect to global quasiconformal maps are precisely the quasicircles), Tukia [T] showed that there are fractal bilipschitz homogeneous curves, e.g., the von Koch snowflake. Other fractal examples are the so called quasi-self-similar circles like Julia sets of polynomials $p(z) = z^2 + c$ with c near 0 and limit sets of certain Kleinian groups. These examples all share a common property: to wit, each has a parametrization ψ (a homeomorphism from either $\mathbf{S}^1$ or $\mathbf{R}$ to the curve) which satisfies $|\psi(t) - \psi(s)|^\delta \approx |s-t|$ where δ is the Hausdorff dimension of the image curve.

Clearly, quasihomogeneous maps of $\mathbf{S}^1$ or $\mathbf{R}$ need not satisfy such a Hölder condition. On the other hand, it is easy to see, by conjugating either rotations of $\mathbf{S}^1$ or translations of $\mathbf{R}$, that any curve which admits a quasihomogeneous parametrization is bilipschitz homogeneous. In fact, if a Jordan curve has a VWQH parametrization, then it is bilipschitz homogeneous with respect to a one-parameter group (cf. [GH$_2$, 4.1,4.3]); Section 2 contains basic definitions, terminology and notation. Thus a fundamental question is whether or not every bilipschitz homogeneous curve admits a VWQH (or WQH or QH) parametrization. A partial answer is given by the following easy consequence of Theorem A.

THEOREM B. *A Jordan curve Γ in any metric space is bilipschitz homogeneous with respect to a group G if and only if Γ has a VWQH parametrization, which conjugates G to an isometry group (rotations of $\mathbf{S}^1$ when Γ is compact and translations of $\mathbf{R}$ otherwise).*

We pause here to raise this question: Does bilipschitz homogeneity imply bilipschitz homogeneity with respect to a group? Note that for a Jordan curve this is equivalent to asking whether or not bilipschitz homogeneity guarantees the existence of a VWQH parametrization. We do not know of an example where there is bilipschitz homogeneity but not with respect to a group; in fact for plane Jordan curves bilipschitz homogeneity always implies bilipschitz homogeneity with respect to a group.

According to Lemma 2.2, the parametrization provided by Theorem B is WQH when Γ is bounded turning (and, by Fact 2.3, quasihomogeneous when Γ is also doubling). It turns out that bilipschitz homogeneous plane Jordan curves are necessarily bounded turning, a significant result just established by Chris Bishop [B, Thm. 1.1]. (We learned of this as we were completing our manuscript; in subsection 4.D we sketch our proof of this for the special case where the curve is bilipschitz homogeneous with respect to a group.) This, in conjunction with Theorem E and other results of Bishop's [B, Cor. 1.2], gives a long list of equivalent conditions for certain Jordan plane curves.

Thus the class of plane Jordan curves which are homogeneous with respect to uniformly bilipschitz self-homeomorphisms is precisely the class of plane curves which are homogeneous with respect to a family (or group) of uniformly bilipschitz homeomorphisms acting on the whole plane. In fact, not only are these the same curves, but one can also extend any group action on the curve to a group action on the whole plane.

THEOREM C. *Let G be any L-bilipschitz group acting on a K-bilipschitz homogeneous Jordan curve in the plane $\mathbf{R}^2$. Then one can extend G to an M-bilipschitz group acting on the plane, with M depending only on K, L and the quasicircle constant.*

Our proof of the above is based on the next result, which has its own interest. Here we let Γ^1 denote either the real line $\mathbf{R}$ or the circle $\mathbf{S}^1$, and we call Γ a *line* or a *circle* if Γ has a parametrization $\psi\colon \Gamma^1 \to \Gamma$ which is an isometry.

THEOREM D. *Let Γ be a line or a circle in any metric space. Suppose G is an L-bilipschitz group acting on Γ. Then there is an L-bilipschitz self-map f of Γ such that $f\,G\,f^{-1}$ is a group of isometries of Γ. (Thus $\psi^{-1} f G f^{-1} \psi$ is a group of translations when $\Gamma^1 = \mathbf{R}$ and a group of rotations when $\Gamma^1 = \mathbf{S}^1$.)*

Since bilipschitz groups are convergence groups, we already know that there is a topological conjugacy [Ga]. The new information here is that we can choose a bilipschitz conjugacy. There are similar results for quasisymmetric groups, due to Hinkkanen [H], and for strongly quasisymmetric groups, due to Mayer and Zinsmeister [MZ]. Of course such a result is false for bilipschitz groups acting on the plane, since there are non-chordarc, even fractal, bilipschitz homogeneous curves.

We draw attention to the special cases in Theorems C,D where G is a non-discrete group. When such a G acts on a line or a circle, then the parametrization given by Theorem B—which conjugates G to an isometry group—is in fact bilipschitz. When G acts on a quasicircle in $\mathbf{R}^2$, we can extend its action to all of $\mathbf{R}^2$ simply by extending the (quasihomogeneous) parametrization obtained via Theorem B.

In contrast with the situation described above for plane curves, there are no such results for bilipschitz homogeneous curves in $\mathbf{R}^n$ when $n \geq 3$. For example, a helix in $\mathbf{R}^3$ is a 1-bilipschitz homogeneous curve whose bounded turning constant can be arbitrarily large. In fact there are bilipschitz homogeneous curves in $\mathbf{R}^3$ which are not bounded turning; see Example 5.6 and also [B, Example 4.1].

Now we turn to the general problem of understanding bilipschitz homogeneous Jordan curves in arbitrary metric spaces. We remind the reader of the basic problem as to whether or not such curves have VWQH parametrizations. Our final result provides some basic information about these curve (see [B] also), and demonstrates, in particular, that *quasihomogeneous circles* (curves which have a quasihomogeneous parametrization) are precisely the curves which are simultaneously bilipschitz homogeneous, bounded turning, and doubling.

THEOREM E. *For a Jordan curve* Γ, *in a doubling metric space, the following assertions are quantitatively equivalent.*

(a) Γ *is bilipschitz homogeneous and bounded turning.*
(b) Γ *admits a quasihomogeneous parametrization.*
(c) Γ *enjoys a bounded covering property.*
(d) Γ *supports a geometric measure.*
(e) Γ *satisfies a generalized chordarc condition.*

Again we observe that when the bilipschitz homogeneity in (a) is with respect to a group, then (b) follows from Theorem B in conjunction with Lemma 2.2 and Fact 2.3. We verify these characterizations for bilipschitz homogeneous curves in Section 3, where we also define and explain the above terminology. Note that conditions (b) through (e) all imply that Γ is bounded turning; as mentioned above, this hypothesis is essential in (a). Also, as Examples 5.4, 5.5 illustrate, the doubling hypothesis is necessary, and in fact an integral ingredient in our proofs. We point out that there is no similar result for higher dimensional bilipschitz homogeneous sets: There are bilipschitz homogeneous surfaces in $\mathbf{R}^3$ which even fail to admit quasisymmetric parametrizations. Rickman showed that $\Gamma \times \mathbf{R}$, with Γ a snowflake curve, cannot be the image of $\mathbf{R}^2$ under a quasisymmetric map, and Tukia [T] established the bilipschitz homogeneity of this surface.

Mayer proved the equivalence of conditions (b), (c), (d) for plane curves [M, Thm. 1.1]. Gamshari and Herron established the equivalence of (a) and (b) for curves in $\mathbf{R}^n$ which have positive finite Hausdorff measure [GH$_2$, Thms. C,4.6]. However, there are bilipschitz homogeneous curves which fail to satisfy this criterion; e.g., see [M. p. 160] or [GH$_2$, 5.3]. Thus we are forced to use generalized Hausdorff measures; it turns out that we can always associate a 'natural' generalized Hausdorff measure to a given bilipschitz homogeneous bounded turning Γ and then (e) holds. This result has two important applications. First, there is an easy way to calculate the Hausdorff dimension of such a curve: it coincides with the lower Minkowski or box dimension (see Corollary 3.8). Second, we can classify these curves modulo bilipschitz maps (see Corollary 3.9); this extends a corresponding result of Falconer and Marsh [FM] concerning quasi-self-similar circles.

This paper is organized as follows: We examine bilipschitz group actions in Section 4 and corroborate Theorems A,B,C,D in subsections 4.B, 4.C, 4.F, 4.E respectively. Section 3 is devoted to explaining and substantiating Theorem E. We conclude with Section 5 where we exhibit illustrative examples.

2. Preliminaries

Our notation is relatively standard. We let $B(x; r) = \{y\colon |x - y| < r\}$ and $S(x; r) = \partial B(x; r)$ denote the open ball and sphere of radius r centered at the point x. We write $c = c(a, \ldots)$ to indicate a constant c which depends only on $a, \ldots$;

typically c will depend on various parameters, and we try to make this as clear as possible often giving explicit values. We write $a \approx b$ to mean there exists a positive finite constant c, depending only on the given data, with $a/c \le b \le ac$.

A metric space X is said to be *doubling* if there is a constant ν_0 such that any ball B in X can be covered with at most ν_0 balls each having radius half the radius of B. In other words, $N(r; B(x; 2r)) \le \nu_0$ for all $r > 0$ and all $x \in X$, where

$$N(r; E) = \min\{n \in \mathbb{N}\colon\ E \subset \cup_{i=1}^n B(x_i; r)\}.$$

Examples of doubling spaces include $\mathbf{R}^n$, the Heisenberg groups, and Ahlfors regular spaces [DS]. Every complete doubling metric space carries a doubling measure [LS].

We utilize the fact that every subspace of a doubling space is itself doubling, and especially the trivial, but crucial, observation that in a doubling metric space there exists an increasing function $\nu\colon [1, \infty) \to [1, \infty)$ such that for every compact set E

$$N(\sigma r; E) \le N(r; E) \le \nu(\sigma) N(\sigma r; E) \quad \text{for all } r > 0 \text{ and } \sigma \ge 1.$$

In fact $\nu(\sigma) = \nu_0^{p+1}$ works where $p = \log_2 \sigma$. Henceforth ν always denotes this associated 'doubling function' in our ambient doubling space.

As indicated in the Introduction, we are primarily interested in Jordan curves which live in some ambient doubling space. For the record, by a Jordan curve Γ we mean a homeomorphic image of either the circle $\mathbf{S}^1$ (the compact case) or the real line $\mathbf{R}$ (the non-compact case). We let $\Gamma(x, y)$ denote the component of $\Gamma \setminus \{x, y\}$ with minimal diameter. We say that Γ satisfies a *bounded turning* condition provided there is a constant a such that for all $x, y \in \Gamma$,

$$\operatorname{diam}(\Gamma(x, y)) \le a|x - y|.$$

In the sequel we say that Γ is a-BT when the above holds. Thanks to Ahlfors, we know that the bounded turning plane Jordan curves are precisely the quasicircles (i.e., images of a circle or line under a quasiconformal self-homeomorphism of the plane).

There is an alternative way of estimating $N(r; \Gamma)$ for a bounded turning Jordan curve Γ in a doubling spaces. An *r-chain* of *length ℓ* along Γ from x to y is an ordered sequence of points $x = x_0, x_1, \ldots, x_\ell = y$ on Γ with the property that $|x_{i-1} - x_i| = r$ for $1 < i < \ell$ and $|x - x_1| \le r$, $|y - x_{\ell-1}| \le r$.

2.1. LEMMA. *Let $\gamma = \Gamma(x, y)$ be a subarc of an a-BT Jordan curve in a doubling metric space. Then any r-chain $x = x_0, x_1, \ldots, x_\ell = y$ on γ satisfies*

$$\frac{1}{\nu(a)} N(r; \gamma) \le \ell \le 2\nu(2)\nu(3a) N(r; \gamma).$$

Proof. Put $\gamma_i = \gamma(x_{i-1}, x_i)$. Since Γ is a-BT, $N(ar; \gamma_i) = 1$, from which we deduce the lower bound on ℓ. To verify the upper bound, we can assume that $\ell \ge 2$. Fix distinct points x_i, x_j different from x_0, x_1 and from $x_{\ell-1}, x_\ell$. Then there is an index k, $1 < k < \ell$, such that the subarc γ_k lies between x_i and x_j. Hence $|x_i - x_j| \geq r/a$, and we see that the balls $B(x_i; r/3a)$ $(i = 1, 2, \ldots, \ell - 1)$ are disjoint. Thus $N(r/6a; \gamma) \geq \ell - 1$ and so

$$\ell \leq 2(\ell - 1) \leq 2N(r/3a; \gamma) \leq 2\nu(2)\nu(3a)N(r; \gamma). \qquad \square$$

In addition to the quasihomogeneous maps defined in the Introduction, we also require several related classes of homeomorphisms. An embedding h: $X \to Y$ is called *weakly quasisymmetric*, or $H-$WQS, provided

$$|h(x) - h(y)| \leq H|h(x) - h(z)| \quad \text{when } |x - y| \leq |x - z|.$$

Next, h is *weakly quasihomogeneous*, or $K-$WQH, provided

$$|h(x) - h(y)| \leq K|h(u) - h(v)| \quad \text{when} \quad |x - y| \leq |u - v|.$$

Finally, we say that h is $K-$VWQH, or *very weakly quasihomogeneous*, if

$$|h(x) - h(y)| \leq K|h(u) - h(v)| \quad \text{when } |x - y| = |u - v|.$$

We remind the reader that a Jordan curve is bounded turning if and only if it has a WQS parametrization.

We mention that a K-VWQH map need not be K-WQH [GH_2, Example 4.2]. In fact a VWQH homeomorphism may not be WQH; see Example 4.2. The following result furnishes a useful criterion for determining when a VWQH map is WQH; we utilize it to verify that certain parametrizations are WQH.

2.2. LEMMA. *Any K-VWQH H-WQS embedding h of a connected space X is HK-WQH.*

Proof. Fix $x, y, u, v \in X$ with $|x - y| \leq |u - v|$. Choose $z \in X$ with $|x - y| = |u - z| \leq |u - v|$. Then $|h(x) - h(y)| \leq K|h(u) - h(z)| \leq HK|h(u) - h(v)|$. $\square$

Next we point out that there are WQH homeomorphisms which fail to be QH; see Examples 5.4,5.5. Since the inverse of a WQH map may not be WQH, we consider the more general class of QH maps. Fortunately, in many important cases a WQH homeomorphism is QH; the following result, in the quasisymmetric case, is due to Tukia and Väisälä [TV, 2.15],[V, 2.9].

2.3. FACT. *Every K-weakly quasihomogeneous homeomorphism between doubling metric spaces is η-quasihomogeneous, where η depends only on K and the doubling constant.*

On several occasions we require the following information [GH_2, 2.5].

2.4. FACT. *If C is a K-BLH Jordan curve, then there is a K^2-BL family of orientation preserving self-homeomorphisms acting transitively on C. When C is K-BLH with respect to a group, we can assume each group element is orientation preserving.*

3. Bilipschitz Homogeneous Curves

Here we prove Theorem E. It is easy to see that a curve with a QH parametrization is BLH: just conjugate rotations of the circle or translations of the line; thus (b) implies (a). We present the remaining implications as Propositions 3.1, 3.3, 3.6, 3.7. Since the bounded turning hypothesis is crucial here, we recall that each of the conditions in Theorem E implies that Γ is BT; see [M, Lemme 2.1].

3.A. Bounded covering property. We say that Γ enjoys a *bounded covering* property provided there is a constant κ such that

$$N(r; \Gamma(x, y)) \le \kappa N(r; \Gamma(z, w)) \quad \text{for all } r > 0$$

whenever $x, y, z, w \in \Gamma$ satisfy $|x - y| \le |z - w|$. We abbreviate this by writing Γ is κ-BC. Such curves are a-BT with $a = 2\kappa$. First we verify that (a) implies (c) in Theorem E.

3.1. PROPOSITION. *Suppose Γ is a K-BLH, a-BT, ν-doubling Jordan curve. Then Γ is κ-BC with $\kappa = 2\nu(2)\nu(3a)\nu(2K^2)\nu(a^2K^2)$.*

Proof. Fix $x, y, z, w \in \Gamma$ with $|x - y| \le |z - w| = R$. Put $\gamma = \Gamma(x, y)$ and $\alpha = \Gamma(z, w)$. Let $r > 0$. Set $m = N(r; \alpha)$ and choose balls $B_1, \dots, B_m$ of radius r which cover α. Put $\rho = R/aK^2$ and let $x = x_0, \dots, x_n = y$ be any ρ-chain joining x, y along γ. Since Γ is a–BT, $N(aR; \gamma) = 1$, and thus by Lemma 2.1 we see that

$$n \le 2\nu(2)\nu(3a)N(\rho; \gamma) \le 2\nu(2)\nu(3a)\nu(a^2K^2).$$

Let $\gamma_i = \gamma(x_{i-1}, x_i)$ and use Fact 2.4 to select K^2-BL homeomorphisms $f_i\colon \Gamma \to \Gamma$ with $f_i(z) = x_{i-1}$ and so that $\beta_i = f_i(\alpha)$ and γ_i overlap ($i = 1, \dots, n$). As $\operatorname{diam}(\gamma_i) \le a\rho \le \operatorname{diam}(\alpha)/K^2 \le \operatorname{diam}(\beta_i)$, we must have $\gamma_i \subset \beta_i$. Now $\alpha \subset \bigcup_1^m B_j$, so the m sets $f_i(\alpha \cap B_j)$ ($j = 1, \dots, m$) cover $\beta_i \supset \gamma_i$, and each has diameter at most $2K^2r$; therefore $N(2K^2r; \gamma_i) \le m$ for $i = 1, \dots, n$. We conclude that

$$N(r; \gamma) \le \sum_1^n N(r; \gamma_i) \le \nu(2K^2)n\,m \le \kappa N(r; \alpha). \qquad \square$$

Next we confirm that (c) implies (b) in Theorem E. We require the following information from [M, Lemme 2.3,2.4].

3.2. FACT. *Let Γ be a κ-BC Jordan curve in a ν-doubling metric space. Write $\gamma = \Gamma(x, y)$ and $\beta = \Gamma(u, v)$ for points $u, v, x, y \in \Gamma$. Then there exists a constant $\mu = \mu(\kappa, \nu) > 0$ such that:*

(a) *If $0 < r \le R = |u - v| \le |x - y|$, then*

$$\frac{1}{\mu}N(r; \gamma) \le N(R; \gamma)N(r; \beta) \le \mu\, N(r; \gamma).$$

(b) *If* $N(r;\gamma) \le \lambda\, N(r;\beta)$ *for some* $\lambda > 0$ *and* $0 < r \le |u-v|$, *then*

$$|x-y| \le 2\lambda\mu|u-v|.$$

3.3. *Proposition. Let* Γ *be a* κ*-BC* ν*-doubling Jordan curve. Then* Γ *has an* η*-QH parametrization where* η *depends only on* κ *and* ν.

Proof. We consider the non-compact case; so Γ has a parametrization $\varphi\colon \mathbf{R} \to \Gamma$. See [M, p. 154] for details on adapting our argument for compact curves. We use the bounded covering property and φ to obtain parametrizations φ_n. We confirm that $\{\varphi_n\}$ is pointwise bounded and equicontinuous, and that any limit ψ is a QH parametrization of Γ.

Let $x_0 = \varphi(0)$, $y_0 = \varphi(1)$; assume $|x_0 - y_0| = 1$ and put $\gamma_0 = \Gamma(x_0, y_0)$, $\Gamma_0 = \varphi([0,+\infty))$. By relabeling a sequence of positive integers, we may assume that for each $n \in \mathbb{N}$ there exists an $r = r_n > 0$ with $N(r;\gamma_0) = n$.

Fix n and $r = r_n$. Let $\{x_i^{(n)}\colon i \in \mathbb{Z}\}$ be a doubly infinite r-chain along Γ with $x_0^{(n)} = x_0$ and $x_i^{(n)} \in \Gamma_0$ for $i \ge 0$. Let $T_i^{(n)} = [t_{i-1}^{(n)}, t_i^{(n)}] = \varphi^{-1}(\Gamma(x_{i-1}^{(n)}, x_i^{(n)}))$ and define $\varphi_n\colon \mathbf{R} \to \Gamma$ by mapping $\Lambda_i^{(n)} = [(i-1)/n, i/n]$ affinely onto $T_i^{(n)}$ and then using φ; thus $\varphi_n(i/n) = x_i^{(n)}$ and on $\Lambda_i^{(n)}$ (so here $0 \le \sigma \le 1$),

$$\varphi_n\left(\sigma\frac{i-1}{n} + (1-\sigma)\frac{i}{n}\right) = \varphi\left(\sigma t_{i-1}^{(n)} + (1-\sigma)t_i^{(n)}\right).$$

Then each $\varphi_n\colon \mathbf{R} \to \Gamma$ is a homeomorphism and $\varphi_n(\Lambda_i^{(n)}) = \Gamma(x_{i-1}^{(n)}, x_i^{(n)})$.

Our goal is to produce a convergent subsequence of $\{\varphi_n\}$ with limit a WQH parametrization of Γ; the proposition then follows from Fact 2.3. Our primary tools in this endeavor are the following two inequalities:

$$|x_{i+k}^{(n)} - x_i^{(n)}| \le M|x_{j+l}^{(n)} - x_j^{(n)}| \quad (i,j,k,l \in \mathbb{Z} \text{ with } 0 \le k \le l), \tag{3.4}$$

and

$$|x_i^{(m)} - x_0| \le M|x_1^{(n)} - x_0| \quad (m \ge n \text{ with } 0 \le i/m \le 1/n). \tag{3.5}$$

Here $m, n \in \mathbb{N}$ and $M = M(\kappa, \nu)$. Let us indicate why these inequalities are valid. Notice that $\Gamma(x_i^{(n)}, x_{i+k}^{(n)})$ contains an r-chain of length k. Thus when $0 \le k \le l$, Lemma 2.1 asserts that $N(r;\Gamma(x_i^{(n)}, x_{i+k}^{(n)}))$ and $N(r;\Gamma(x_j^{(n)}, x_{j+l}^{(n)}))$ are comparable, and then (3.4) is a consequence of Fact 3.2(b). The proof of (3.5) is similar; see [M, pp. 152–153] for more details.

Using these inequalities we easily infer that $|\varphi_n(p) - x_0| \le |p|M^2 r_1$ for any $p \in \mathbb{Z}$. From this we deduce that $\{\varphi_n\}$ is pointwise bounded: If $t \in [0,p]$ for some $p \in \mathbb{Z}$, then $\varphi_n(t) \in \Gamma(x_0, \varphi_n(p))$ and so $|\varphi_n(t) - x_0| \le \operatorname{diam}\Gamma(x_0, \varphi_n(p)) \le 2\kappa|p|M^2 r_1$.

Next we explain why $\{\varphi_n\}$ is equicontinuous. Fix $t_0 \in \mathbf{R}$ and let $\varepsilon > 0$ be given. Fix n sufficiently large so that $2\kappa M^2 r_n < \varepsilon$. We consider $m \ge 10n$. Select $i \in \mathbb{Z}$ so that i/n is a number of this form closest to t_0. Choose the largest, smallest j, k in $\mathbb{Z}$ with $j/m \leq t_0 - 1/10n$ and $k/m \geq t_0 + 1/10n$. Notice that $(k-j)/m \leq 1/n$, so by (3.5) $|x_{k-j}^{(m)} - x_0| \leq Mr_n$. Suppose $|t - t_0| < 1/10n$. Then $\varphi_m(t)$ and $\varphi_m(t_0)$ belong to $\Gamma(x_j^{(m)}, x_k^{(m)})$ and thus according to (3.4),

$$|\varphi_m(t) - \varphi_m(t_0)| \leq 2\kappa |x_j^{(m)} - x_k^{(m)}| \leq 2\kappa M |x_{k-j}^{(m)} - x_0| \leq 2\kappa M^2 r_n \leq \varepsilon.$$

Thus $\{\varphi_n\}$ is pointwise bounded and equicontinuous, so the Arzela-Ascoli theorem provides a subsequence which converges on compact subsets of $\mathbf{R}$ to some limit ψ. It follows from (3.4) that ψ is M-WQH. We claim that ψ is non-constant and maps $\mathbf{R}$ onto Γ, hence by Fact 2.3, ψ is an η-QH parametrization of Γ, and η depends only on κ and ν.

To justify that ψ is non-constant it suffices to exhibit a constant $\delta > 0$ with $|x_n^{(n)} - x_0| \geq \delta$ for all $n \in \mathbb{N}$. Since $\Gamma_n = \Gamma(x_0, x_n^{(n)})$ contains an r_n-chain of length n, Lemma 2.1 asserts that $N(r_n; \gamma_0) = n \leq 2\nu(6\kappa)N(r_n; \Gamma_n)$. We thus have $|x_n^{(n)} - x_0| \geq r_n/2\kappa = r$ and $N(r; \gamma_0) \leq 4\kappa\nu(6\kappa)N(r; \Gamma_n)$; therefore Fact 3.2(b) permits us to conclude that

$$1 = |x_0 - y_0| \leq 8\kappa\mu\nu(6\kappa)|x_n^{(n)} - x_0|.$$

Thus ψ is a QH homeomorphism from $\mathbf{R}$ to Γ. If ψ were not onto, then we could find an infinite length r-chain, $r \approx |\psi(1) - x_0|$, along a bounded subarc of Γ, which is impossible. □

3.B. Hausdorff measures and chordarc conditions. The upshot of Propositions 3.1, 3.3 is that every bilipschitz homogeneous bounded turning curve has a quasihomogeneous parametrization. Here we investigate a natural method for obtaining such a map. We begin by reformulating the problem in terms of certain measures, since a 'nice' measure μ on a curve furnishes a 'nice' parametrization via μ-arclength.

We say that a Jordan curve Γ supports a *geometric measure* μ if μ is a positive σ-finite Borel measure on Γ and there is a constant $c \geq 1$ such that

$$\mu(\Gamma(x, y)) \leq c\,\mu(\Gamma(z, w)) \quad \text{for all } x, y, z, w \in \Gamma \text{ with } |x - y| \leq |z - w|.$$

Since such curves are a-BT with $a = 2c$ (cf. [M, Lemme 2.1]), we see that a geometric measure assigns essentially the same mass to comparably sized subarcs. Next we substantiate the equivalence of (b) and (d) in Theorem E.

3.6. PROPOSITION. *There is a QH parametrization of Γ if and only if Γ supports a geometric measure.*

Proof. We assume that Γ is compact; the non-compact case is handled in a similar manner. Suppose μ is a geometric measure on Γ. Define ψ: $\mathbf{S}^1 \to \Gamma$ by the requirement that $\mu(\psi(I)) = (\mu(\Gamma)/2\pi)\ell(I)$ for each subarc I of $\mathbf{S}^1$; thus ψ is a 'μ-arclength' parametrization. Employing the homogeneous property of μ, we deduce that ψ^{-1} is K-WQH with $K = \pi c/2$; since Γ is BT and doubling, Fact 2.3 permits us to conclude that ψ is η-QH with η depending only on c and ν. Conversely, when ψ: $\mathbf{S}^1 \to \Gamma$ is QH, the measure μ defined by $\mu(E) = \mathcal{H}^1(\psi^{-1}(E)$ is a geometric measure on Γ. □

For many examples (chordarc curves, classical snowflakes, and quasi-self-similar curves) the appropriate Hausdorff measure is a geometric measure. Moreover, these curves all satisfy the chordarc condition $\mathcal{H}^\delta(\Gamma(x, y)) \approx |x-y|^\delta$ for $x, y \in \Gamma$, where δ is the Hausdorff dimension of Γ; this is simply an alternative way of saying that these curves have parametrizations ψ which satisfy $|\psi(t) - \psi(s)|^\delta \approx |s-t|$. However, for general bilipschitz homogeneous curves the corresponding Hausdorff measure can be trivial or infinite; cf. [M, p. 160] or [GH_2, 5.3]. Moreover, even when the appropriate Hausdorff measure is positive and finite, there may not exist such a nice parametrization; see [GH_2, Thm. E]. Consequently it is necessary to utilize generalized Hausdorff measures.

We call a non-decreasing δ: $(0, +\infty) \to (0, +\infty)$ with $\lim_{r\to 0} \delta(r) = 0$ a *dimension function*, and associate with δ the generalized Hausdorff measure defined by

$$\Lambda^\delta(A) = \lim_{r\to 0}\left[\inf\left\{\sum_1^\infty \delta(\operatorname{diam}(U_i)) \colon A \subset \bigcup_1^\infty U_i,\ \operatorname{diam}(U_i) \le r\right\}\right].$$

Now we explain how to specify canonical dimension functions for bilipschitz homogeneous bounded turning curves. If Γ is compact, then we simply take $\delta(r) = 1/N(r;\Gamma)$ for $r > 0$. Assume that Γ is non-compact. Fix a point $x_0 \in \Gamma$, let Γ_+ be one of the subarcs of $\Gamma\setminus\{x_0\}$, and put $\gamma_r = \Gamma(x_0, x_r)$ for $r \ge 1$, where x_r is the first point of Γ_+ with $|x_r - x_0| = r$. Then define

$$\delta(r) = \begin{cases} 1/N(r, \gamma_1) & \text{when } 0 < r \le 1, \\ N(1, \gamma_r) & \text{when } r > 1. \end{cases}$$

Obviously in the non-compact case the definition of δ depends on the choice of the arcs γ_r, but when Γ satisfies a bounded covering property (i.e., when Γ is a bilipschitz homogeneous bounded turning curve) and δ' is a dimension function corresponding to some other choice of arcs, then $\delta \approx \delta'$ and so the corresponding Hausdorff measures are comparable (having the same null sets, the same sets of infinite measure, and assigning comparable measures to sets of positive finite measure). Thus our definition gives rise to a canonical equivalence class of dimension functions where δ and δ' are equivalent exactly when $\delta \approx \delta'$—here the constant involved depends only on the bounded covering constant κ and the doubling constant ν_0. Henceforth δ always denotes the above defined canonical dimension function associated with Γ. Note that

δ satisfies the doubling condition $\delta(\sigma r) \le \rho(\sigma)\delta(r)$ for all $\sigma \ge 1$, $r > 0$ where, e.g. $\rho(\sigma) = \mu\, \nu_0\, \nu(\sigma)$ and μ is the constant from Fact 3.2.

Our next goal is to elucidate the equivalence of conditions (d) and (e) in Theorem E. We say that Γ satisfies a *generalized chordarc condition* if

$$\frac{1}{b}\,\delta(|x-y|) \le \Lambda^{\delta}(\Gamma(x,y)) \le b\,\delta(|x-y|) \quad \text{for all } x, y \in \Gamma$$

for some constant $b \ge 1$. For rectifiable curves Γ this is the usual notion of chordarc, and for curves with positive σ-finite α-dimensional Hausdorff measure this corresponds to the α-dimensional chordarc curves studied in [GH$_1$]. It is immediate that $\mu = \Lambda^{\delta}$ is a geometric measure on Γ if the above holds, which proves one of the following implications.

3.7. PROPOSITION. *There is a geometric measure supported on Γ if and only if Γ satisfies a generalized chordarc condition.*

For curves in $\mathbf{R}^2$ this is [M, Lemme 4.2], and for rectifiable curves in $\mathbf{R}^n$ this is [GH$_2$, Thm. B] (see also [GH$_1$, Thm. 4.6]). The arguments given in [M] are readily modified to verify the above as well as the following two corollaries; the crucial ingredient involves confirming that Γ satisfies a chordarc type condition with respect to μ and δ whenever μ is a geometric measure on Γ.

Proposition 3.7 furnishes two important applications. First, there is an easy way to calculate the Hausdorff dimension of a BLH BT doubling curve via the lower box-counting or Minkowski dimension.

3.8. COROLLARY. *For a bilipschitz homogeneous bounded turning Jordan curve Γ in a doubling space,* $\dim_{\mathcal{H}}(\Gamma) = \underline{\dim}_{\mathcal{M}}(\Gamma)$.

There are examples (see [M, p. 160]) showing that the lower and upper Minkowski dimensions, of a BLH BT doubling curve, may not be equal. Also, there are examples (see [GH$_2$, Thm E]) of BLH BT doubling curves which have lower δ-dimensional density zero ($\delta = \dim_{\mathcal{H}}(\Gamma)$).

Our second application is a classification of BLH BT doubling curves according to bilipschitz equivalence.

3.9. COROLLARY. *Let Γ_1 and Γ_2 be two bilipschitz homogeneous bounded turning Jordan curves which live in ambient doubling metric spaces. Suppose Γ_1, Γ_2 are either both compact or both non-compact. Then there exists a bilipschitz homeomorphism between Γ_1 and Γ_2 if and only if both curves have the same canonical dimension class.*

This result has its origin in the paper [FM] of Falconer and Marsh who verified the analog for quasi-self-similar curves. In view of this bilipschitz classification it would be nice to have a solution to the following problem: Is there a reasonable description of all classes of dimension functions that can occur? This should describe all representative bilipschitz homogeneous curves Γ^δ (with, for example, Γ^δ the usual von Koch Snowflake curve when $\delta(r) \approx r^\delta$). With such a 'dictionary' we would obtain: Every BLH BT Jordan curve Γ is bilipschitz equivalent to some curve Γ^δ.

4. Bilipschitz Group Actions

In this section we analyze bilipschitz group actions on Jordan curves and corroborate Theorems A,B,C,D in subsections 4.B, 4.C, 4.F, 4.E respectively. In addition, in 4.D we explain why Jordan plane curves which are bilipschitz homogeneous with respect to a group action must be quasicircles.

4.A. Bilipschitz groups. Here we collect some miscellaneous information regarding bilipschitz groups acting on Jordan curves in arbitrary metric spaces. Our methods were inspired by Hinkkanen's work [H] concerning quasisymmetric groups.

We begin with a result which asserts that the orientation preserving elements of such groups behave like rotations.

4.1. LEMMA. *Suppose g generates a bilipschitz group $G =< g >$ acting on a Jordan curve Γ. Then g has no fixed points, unless it is the identity.*

Proof. Assume G is a K–BL group, g is not the identity, but $g(x) = x$ for some $x \in \Gamma$. Suppose first that $g^n(y) \to \infty$ for some $y \in \Gamma$. Then

$$K|x - y| \geq |g^n(x) - g^n(y)| = |x - g^n(y)| \to \infty$$

as $n \to \infty$, which is impossible. Now suppose $\{g^n(y)\}$ is bounded for all $y \in \Gamma$

Since the set of fixed points of g is closed in Γ, we can select a subarc A of Γ whose endpoints are fixed points of g and such that g has no fixed points in the interior of A; here we include the cases where A is unbounded or $A = \Gamma \setminus \{x\}$. Let $y \in A$. Since g is orientation preserving and $\{g^n(y)\}$ is bounded, we see that $\lim_{n\to\infty} g^n(y)$ exists and is a finite point of Γ. Thus

$$0 < \frac{1}{K}|y - g(y)| \leq |g^n(y) - g^{n+1}(y)| \to 0$$

as $n \to \infty$, which is another contradiction. □

4.2. COROLLARY. *Suppose g, h are elements of a bilipschitz group G acting on a Jordan curve Γ. If $g(x) = h(x)$ for some $x \in \Gamma$, then $g = h$.*

According to the above, there is an injection from G into Γ, $g \mapsto g(x)$, where x is any given point of Γ. When G acts transitively, this provides a one-to-one correspondence between the points of Γ and the elements of G. In addition, G inherits an ordering from an orientation on Γ: $g \prec h$ for $g, h \in G$ if $g(x)$ precedes $h(x)$ on Γ, where x is any given point of Γ.

4.3. LEMMA. *For a bilipschitz group G acting on a Jordan curve Γ, the following are equivalent.*

(a) *G is cyclic.*
(b) *G is discrete.*
(c) *For all $x \in \Gamma$, $\Gamma \neq G(x)$.*
(d) *For some $x \in \Gamma$, $\Gamma \neq G(x)$.*

Proof. That (a) implies (b) implies (c) implies (d) is straightfoward; we validate the remaining implication. The crucial idea is that G is generated by any g with the property that the subarc from a given point $x \in \Gamma$ to $g(x)$ contains no other points of the orbit $G(x)$.

Assume there is some $x \in \Gamma$ with $\Gamma \backslash G(x) \neq \emptyset$. Since G is closed, so is $G(x)$ and thus $\Gamma \setminus G(x)$ consists of open subarcs of Γ. We may assume that $G(x)$ contains at least three distinct points. Let γ be a bounded component of $\Gamma \backslash G(x)$ with endpoints $u, v \in G(x)$. Choose $f \in G$ with $u = f(x)$. Let $\alpha = f^{-1}(\gamma)$. Then $\alpha \cap G(x) = \emptyset$ and α has endpoints x, $y = f^{-1}(v)$. Since $y \in G(x)$, there exists $g \in G$ with $g(x) = y$. We claim that $\Gamma = \cup_{p \in \mathbb{Z}} g^p(\bar{\alpha})$, hence $G(x) = \{g^p(x)\colon\ p \in \mathbb{Z}\}$ and so $G =< g >$. $\square$

4.4. LEMMA. *Let G be a non-discrete bilipschitz group acting on a Jordan curve Γ. Then for every $g \in G$ there exists an $h \in G$ with $h^2 = g$.*

Proof. Fix $g \in G$ and $x_0 \in \Gamma$. Since G is non-discrete, Lemma 4.3 guarantees that G acts transitively, so for each $x \in \Gamma$ there is a unique $h_x \in G$ with $h_x(x_0) = x$. Notice that the map $x \mapsto h_x(x) = h_x^2(x_0)$ is continuous in x. Consider what happens as x varies from $x = x_0$ to $x = y_0 = g(x_0)$: At $x = x_0$ we have $h_{x_0}(x_0) = x_0$ because h_{x_0} is the identity. At $x = y_0$ we have $h_x(x) = h_{y_0}(y_0) = g(y_0)$ because $h_{y_0} = g$. Since $g(y_0)$ lies beyond y_0, we conclude that there is some x between x_0 and y_0 so that $h_x^2(x_0) = h_x(x) = y_0 = g(x_0)$; then Lemma 4.1 asserts that $g = h_x^2$ as desired. $\square$

4.B. Proof of Theorem A. We recall that G is a one-parameter group provided we can write $G = \{g_t\}$ with $(t, x) \mapsto g_t(x)$ continuous where t is a real parameter, $x \in \Gamma$, and $t \mapsto g_t$ is a homomorhpism. Here G is a closed group of orientation preserving uniformly bilipschitz homeomorphisms acting on a Jordan curve Γ and we must verify that G is either cyclic (hence finite in the case of a compact curve) or a one-parameter group. According to Lemma 4.3, it suffices to consider the case when G is a non-discrete group. To demonstrate that G is a one-parameter group, we construct an increasing sequence of cyclic subgroups of G whose union has closure G.

Fix $g_1 \in G \setminus \{\mathrm{id}\}$ and let $g_{1/2}$ denote the element of G satisfying $g_{1/2}^2 = g_1$. Now choose $g_{1/2^n} \in G$ inductively such that $g_{1/2^n}^2 = g_{1/2^{n-1}}$ and let G_n be the group generated by $g_{2^{-n}}$. The sequence $\{G_n\}$ is increasing; we claim that G is the closure of $H = \cup_1^\infty G_n$. Notice that $\overline{H}$ is a closed non-discrete group acting on Γ, so by Lemma 4.3 the action is transitive. Thus given $x \in \Gamma$ and $g \in G$, we can choose $h \in \overline{H}$ with $h(x) = g(x)$; since $h \in G$, an appeal to Corollary 4.2 yields $g = h$.

We write G as a one-parameter family as follows: Suppose that t is the limit of an increasing sequence of dyadic rationals $\{k_n/2^n\}$. Then $\lim_{n\to\infty} g_{1/2^n}^{k_n}$ exists and we denote this element of G by g_t. This is independent of the approximating sequence, $t \mapsto g_t$ is continuous, and $g_t \circ g_s = g_{t+s}$.

4.C. Proof of Theorem B. Here we suppose our Jordan curve Γ is K-bilipschitz homogeneous with respect to a group G. Thanks to Theorem A we can assume that $G = \{g_t\}$ is a one-parameter group. We let Γ^1 denote either $\mathbf{S}^1$, if Γ is compact, or $\mathbf{R}$, otherwise, and we set $z_t = e^{it}$ or $z_t = t$ respectively. Fix any point x_0 in Γ. Define $\psi\colon \Gamma^1 \to \Gamma$ by $\psi(z_t) = g_t(x_0)$. Since

$$K^{-1}\,|x_0 - g_\tau(x_0)| \le |\psi(z_{t_1}) - \psi(z_{t_2})| \le K\,|x_0 - g_\tau(x_0)|$$

for all t_1, t_2 where $\tau = |t_1 - t_2|$, we see that ψ is K-VWQH. Clearly ψ conjugates G to a group of isometries of Γ^1.

4.D. Homogeneity with respect to a group. Although the following weaker result is subsumed by Bishop's work [B, Thm. 1.1] (he does *not* assume homogeneity with respect to a group action) we include it because our proof, which is based on an exercise in plane topology together with Theorem A, is quite natural and in addition provides an estimate for the bounded turning constant of the quasicircle in terms of the bilipschitz constant.

4.5. THEOREM. *A Jordan curve in the plane which is bilipschitz homogeneous with respect to a group is a quasicircle.*

We sketch a brief outline for our proof. Thanks to Fact 2.4, we can assume each group element is orientation preserving. According to Lemma 4.3 and Theorem A, we may further assume that our plane Jordan curve Γ is homogeneous with respect to a one-parameter group $G = \{g_t\}$ of K-BL homeomorphisms $g_t\colon \Gamma \to \Gamma$. Let $x, y \in \Gamma$. Put $d = |x - y|$ and assume that $\Gamma \cap [x, y] = \{x, y\}$. We demonstrate that $\mathrm{dist}(z, [x, y]) \le c\,d$ for any point $z \in \Gamma(x, y)$, where $c = c_0 K^4$ and c_0 is an absolute constant.

Fix $u \in \Gamma(x, y)$ and suppose $\mathrm{dist}(u, [x, y]) > c\,d$. Assume $u = g_\tau(x)$ for $\tau > 0$ and put $v = g_\tau(y)$. Choose $x', y' \in \Gamma \cap [u, v]$ so that $C = [x, y] \cup X \cup Y \cup [x', y']$ is a Jordan curve, where $X = \Gamma_+(x, x')$ and $Y = \Gamma_+(y, y')$. Let D denote the interior of C. Then either $\Gamma_+(x', y)$ or $\Gamma_+(y', x)$ is a Jordan arc, say Z, in D which joins $[x', y']$ to $[x, y]$. (Here $\Gamma_+(p, q)$ denotes the oriented subarc of Γ from p to q, where the orientation is given by "from x to y through $\Gamma(x, y)$".) Let $\{z, w\} = \{x', y'\}$ where z is the initial point of Z.

Notice that X and Y are 'long parallel' Jordan arcs which stay close together; in fact $x(t) = g_t(x)$, $y(t) = g_t(y)$ are nearby points on X, Y resp. Now the arc Z travels between X and Y, so there is a time t when $g_t(z)$ meets the segment $I(t) = [g_t(x), g_t(y)]$, which gives our claim with $c = 2K^2$ implying the sharp result that Γ is $a-$BT with $a = c_0K^2$. However, we have been unable to validate our algorithm for determining such a time. In lieu of this we outline the following argument which makes stronger use of the BLH property but only gives $a = c_0K^4$.

Put $E = \overline{B}([x, y]; Kd) = \{\zeta\colon \operatorname{dist}(\zeta, [x, y]) \leq Kd\}$, $E' = \overline{B}([x', y']; Kd)$, and set $\tau = \min\{\xi, \eta\}$ where $x' = x(\xi)$, $y' = y(\eta)$. Let σ_0 be the *last* time t, $0 \leq t \leq \tau$, with

$$E \cap \overline{B}(I(t); Kd) \neq \emptyset.$$

Then $0 < \sigma_0 << \tau$. Let σ_1 be the *first* time t, $t \geq \sigma_0$, with

$$E' \cap \overline{B}(I(t); Kd) \neq \emptyset.$$

Then $0 < \sigma_0 << \sigma_1 < \tau$, and for all $\sigma_0 < t < \sigma_1$ we have

$$(E \cup E') \cap \overline{B}(I(t); Kd) = \emptyset,$$

which gives useful information about 'topological pictures' for C and Γ.

Let $x_0 = x(\sigma_0)$, $y_0 = y(\sigma_1)$ and 'renormalize' so that σ_0, σ_1 correspond to 0, 1 respectively. For $0 \leq t \leq 1$ define $E(t) = B(I(t); Kd)$ and let $z(t) = g_t(z_0)$, $w(t) = g_t(w_0)$, where z_0 ($= g_\tau(x)$ say) is the last point of Z in $10\overline{E}(1)$ and $w_0 = g_\tau(y)$; thus $w(t) \in B(t) = \overline{B}(z(t); Kd)$. Now we explain why $B(t) \cap c\overline{E}(t) \neq \emptyset$ for some t, $0 \leq t \leq 1$.

Let τ_0 be the first time t, $0 \leq t \leq 1$, with $c\overline{E}(t)$ separating the points $z(t)$, z_0 in D. Then $\overline{E}(\tau_0) \cap D$ contains arcs A_0, C_0 with the following properties: A_0 is a crosscut of D which joins X, Y and separates $z(\tau_0)$ from $E \cap D$ in D; C_0 separates $z(\tau_0)$, z_0 in D. There are two possibilities for C_0: it may be a crosscut joining X, Y; or it may be a *sidecut* joining a point of X (or Y) to some 'later' point of X (or Y resp.) In all cases we have $z(\tau_0)$ 'trapped'. (For example, when A_0, C_0 are both crosscuts, $z(\tau_0)$ lies inside $A_0 \cup X_0 \cup Y_0 \cup C_0$ where X_0, Y_0 are the subarcs of X, Y joining the endpoints of A_0, C_0; when C_0 is, say, a Y-sidecut $z(\tau_0)$ lies inside $C_0 \cup Y_0$ where Y_0 is the subarc of Y joining the endpoints of C_0.) Then by selecting an appropriate set of times $\tau_0 < t_1 < t_2 < \cdots$ we find, after a finite number of steps, some t with $B(t) \cap c\overline{E}(t) \neq \emptyset$. (For example, there is a time τ_1, $\tau_0 << \tau_1 < 1$ such that $B(\tau_0) \approx \overline{E}(\tau_1)$. Then t_1 is the last time t, $\tau_0 \leq t \leq \tau_1$, with $\overline{E}(t) \cap \overline{E}(\tau_0) \neq \emptyset$, and then for $i > 1$ t_i is the first time t, $t \geq t_{i-1}$, with $c\,E(t) \cap c\,E(t_{i-1}) = \emptyset$. Instead of $B(t)$ we consider $B(t - t_1 + \tau_0)$.)

The difficulties in the above proof stem from an apparently non-trivial plane topology problem. As we mentioned, one should be able to use this approach to obtain a sharp estimate for the bounded turning constant.

4.E. Conjugation of 'one-dimensional' groups. Recall that G is an L-BL group acting on a Jordan curve Γ isometric to Γ^1 which is either $\mathbf{R}$ or $\mathbf{S}^1$. There is no harm in assuming that $\Gamma = \Gamma^1$. Now according to Theorem A, G is either cyclic or a one-parameter group. When $\Gamma^1 = \mathbf{S}^1$ we view the action of $g \in G$ as $g\colon e^{i\theta} \mapsto e^{ig(\theta)}$ where $g\colon \mathbf{R} \to \mathbf{R}$ is L-BL and 2π-periodic. Thus in all cases g' exists a.e. with $1/L \le g' \le L$.

Suppose we discover a locally integrable function φ with $1/L \le \varphi \le L$ a.e. and

$$(\varphi \circ g)g' = \varphi \quad \text{for all } g \in G.$$

A homeomorphism f with derivative $f' = \varphi$ a.e. then conjugates G to a group of isometries $f \circ G \circ f^{-1}$ and is L-BL. Thus it remains to construct such a φ in each of the four possible cases given by Theorem A (Γ compact or not and G cyclic or a one-parameter group); it is not difficult to verify that the given φ enjoys the desired conditions.

First consider the compact case $\Gamma^1 = \mathbf{S}^1$. When $G =< g >$ is a cyclic group, in particular a finite group of order say n, then we put $\varphi = \frac{1}{n}\sum_{k=0}^{n-1}(g^k)'$. If $G = \{g_t\}$ is a one-parameter group, we define $\varphi(x) = \int_0^{2\pi} g_t'(x)\,dt$; note that φ is locally integrable thanks to Tonelli's theorem.

Now look at the situation $\Gamma^1 = \mathbf{R}$. In both cases we let φ be any weak limit point of either $\{\varphi_n\colon n \in \mathbb{N}\}$ or $\{\varphi_\tau\colon \tau > 0\}$. Here $\varphi_n = \frac{1}{n}\sum_{k=0}^{n-1}(g^k)'$ if $G =< g >$ is cyclic, or $\varphi_\tau(x) = \frac{1}{\tau}\int_0^\tau g_t'(x)\,dt$ when $G = \{g_t\}$ is a one-parameter group; in the latter case we also require that $\tau_i \to \infty$ when ϕ_{τ_i} is the convergent sequence. Note that $\{\varphi_n\colon n \in \mathbb{N}\}$ and $\{\varphi_\tau\colon \tau > 0\}$ are contained in a closed ball (hence weakly compact) in $L^\infty(\mathbf{R})$.

We remark that when G is a one-parameter group, the conjugating function is unique up to post-composition with either a rotation in the case of the circle or an affine map in the case of the line. Also, in this case we can easily verify that the VWQH parametrization given by Theorem B is in fact bilipschitz, thus avoiding the above argument.

4.F. Extension of group action. Recall that G is an L-BL group acting on a K-BLH curve Γ in $\mathbf{R}^2$. According to Bishop's result [B, Thm. 1.1] Γ is BT and so by Theorem E, Γ admits a QH parametrization $\psi\colon \Gamma^1 \to \Gamma$ where Γ^1 is either $\mathbf{S}^1$ or $\mathbf{R}$. Hence the group $H = \psi^{-1}\, G\, \psi$ is a BL group acting on Γ^1, so by Theorem D, there is a BL self-homeomorphism f of Γ^1 which conjugates H to a group of isometries $k = f \circ h \circ f^{-1}$ of Γ^1; in particular, each k is an isometry of the entire plane $\mathbf{R}^2$. On the other hand, the BL map f has an extension to a BL map $\tilde{f}\colon \mathbf{R}^2 \to \mathbf{R}^2$, so we may assume that H is a BL group acting on the whole plane; here $\tilde{h} = \tilde{f}^{-1} \circ k \circ \tilde{f}$.

Next, the usual Ahlfors–Beurling–Tukia extension of ψ gives a quasiconformal map $\Psi\colon \mathbf{R}^2 \to \mathbf{R}^2$ which is bilipschitz with respect to the hyperbolic metrics in the appropriate components of $\mathbf{R}^2 \setminus \Gamma^1$ and $\mathbf{R}^2 \setminus \Gamma$. Finally, each map $\tilde{g} = \Psi \circ \tilde{h} \circ \Psi^{-1}$ is an extension of g (here $h = \psi^{-1} \circ g \circ \psi$) which is bilipschitz with respect to the hyperbolic metrics in the complementary components of Γ, so an idea due to Gehring [Ge, Thm. 2.11] guarantees that $\tilde{g}$ is M-BL where $M = M(a, K, L)$. Thus Ψ conjugates H to an M-BL group acting on $\mathbf{R}^2$ whose restriction to Γ is G.

5. Examples

Here we present examples which illustrate our results. See Bishop's paper also [B, §4]. First we mention that for any $1 \le \alpha < n$, there are α-dimensional chordarc curves in $\mathbf{R}^n$ which in particular are bilipschitz homogeneous; see [GH$_1$, Thm. A]. Next we point out that there are plenty of doubling metric spaces with BLH curves. For example, consider the plane $\mathbf{R}^2$ with the distance induced by the norm $||(x, y)|| = |x| + |y|^{1/\alpha}$ where $\alpha \ge 1$. It is easy to see that this defines a doubling metric on the plane, and the y-axis is a 1-BLH 1-BT Jordan curve with Hausdorff dimension α.

5.A. Heisenberg snowflakes. The (first) Heisenberg group is $\mathbf{H} = \mathbf{C} \times \mathbf{R}$ equipped with the Lie group product

$$(z, t)(w, s) = (z + w, t + s - 2\Im(z\overline{w})).$$

The Heisenberg distance is defined by the homogeneous norm

$$||(z, t)|| = (|z|^4 + t^2)^{1/4}.$$

This induces a left-invariant doubling metric on $\mathbf{H}$,

$$d(h, g) = ||h^{-1}g||,$$

and as above we find that the t-axis is a 1-BLH 1-BT Jordan curve with Hausdorff dimension 2. Next we explain how to construct a snowflake type curve in $\mathbf{H}$ which is the limit of a sequence of rectifiable arcs analagous to the von Koch snowflake. Our ideas originate in discussions with Seppo Rickman.

We find that

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

form a basis of left-invariant vector fields on $\mathbf{H}$, and $[X, Y] = -4T$. A vector is called *horizontal* if it belongs to the 2-dimensional *horizontal plane* spanned by X, Y. A (smooth) curve $\gamma = (x, y, u)$ is *horizontal* provided $\dot{\gamma} = (\dot{x}, \dot{y}, \dot{u})$ is horizontal, which is true if and only if $\dot{u} = 2y\dot{x} - 2x\dot{y}$. The length of a horizontal curve $\gamma(\tau) = (x(\tau), y(\tau), u(\tau))$ $(\alpha \le \tau \le \beta)$ is

$$\ell(\gamma) = \int_\alpha^\beta \left(\dot{x}^2(\tau) + \dot{y}^2(\tau)\right)^{1/2} d\tau$$

and we define the *Carnot-Carathéodory distance* between two points h, g in $\mathbf{H}$ by

$$d_C(h, g) = \inf\{\ell(\gamma)\colon\ \gamma \text{ a horizontal curve joining } h, g\}.$$

The Heisenberg and Carnot-Carathéodory distances are bilipschitz equivalent, but different, and both are compatible with the euclidean topology on $\mathbf{H}$.

We will take piecewise linear arcs in the plane and lift them to piecewise horizontals arcs in $\mathbf{H}$. In order to do the usual 'snowflake iteration' we require that the t-components return to their original values. The following information is useful.

5.1. LEMMA. *Let $\lambda = [z_1, z_2]$ be the line segment from z_1 to z_2 in $\mathbf{C}$. Let a be the signed area of the oriented triangle $[0, z_1] \cup \lambda \cup [z_2, 0]$. Choose any u_1, u_2 with $u_2 - u_1 = a$. Let γ be the d_C-geodesic joining the points $h_i = (z_i, u_i)$ in $\mathbf{H}$. Then the projection of γ onto $\mathbf{C}$ is λ.*

For example, the piecewise horizontal arc

$$\Gamma_1 = [h_0, h_1] \cup [h_1, h_2] \cup [h_2, h_3] \cup [h_3, h_4] \cup [h_4, h_5] \cup [h_5, h_6]$$

projects down to the piecewise linear arc

$$\Lambda_1 = [z_0, z_1] \cup [z_1, z_2] \cup [z_2, z_3] \cup [z_3, z_4] \cup [z_4, z_5] \cup [z_5, z_6]$$

where

$$\begin{aligned} z_0 &= 1 - \sqrt{2}, \quad z_1 = 1, \quad z_2 = 2 + i, \quad z_3 = 3, \\ & z_4 = 4 - i, \quad z_5 = 5, \quad z_6 = 5 + \sqrt{2} \end{aligned}$$

and

$$\begin{aligned} h_0 &= (1 - \sqrt{2}, 0), \quad h_1 = (1, 0), \quad h_2 = (2 + i, -2), \quad h_3 = (3, 4), \\ & h_4 = (4 - i, 10), \quad h_5 = (5, 0), \quad h_6 = (5 + \sqrt{2}, 0, 0). \end{aligned}$$

Notice that the t-components of h_0, h_1, h_5, h_6 are all zero. Of course we could replace Λ_1 by other configurations as long as we keep the 'area cancellation' property. Thanks to this 'cancellation' property, we can iterate the above process.

5.2. EXAMPLE. *There exist a self-similar, hence BLH, BT snowflake in* **H**.

Proof. Rescale Λ_1 and Γ_1 above so that their endpoints are 0, 1. Thus $z_1 = r = 1/2(1+\sqrt{2})$. Mimicking the usual von Koch snowflake construction we obtain a self-similar arc $\Lambda = \lim_{n\to\infty} \Lambda_n$ where

$$\Lambda_1 = \bigcup_{i=1}^{6} \sigma_i(\Lambda_0), \quad \Lambda_n = \bigcup_{i=1}^{6} \sigma_i(\Lambda_{n-1}) = \bigcup_{i_j=1}^{6} \sigma_{i_1,\dots,i_n}(\Lambda_0);$$

here $\Lambda_0 = [0,1]$, $\sigma_{i_1,\dots,i_n} = \sigma_{i_1} \circ \cdots \circ \sigma_{i_n}$, and σ_i are similarities of **C** with contraction factors r. In addition,

$$\Lambda = \bigcup_{i=1}^{6} \sigma_i(\Lambda) = \bigcup_{i_j=1}^{6} \sigma_{i_1,\dots,i_n}(\Lambda).$$

When we lift Λ_n to the piecewise horizontal arc $\Gamma_n \subset \mathbf{H}$ we get a similar result:

$$\Gamma_n = \bigcup_{i=1}^{6} \psi_i(\Gamma_{n-1}) = \bigcup_{i_j=1}^{6} \psi_{i_1,\dots,i_n}(\Gamma_0)$$

where $\Gamma_0 = [0,1] = [(0,0),(1,0)] \subset \mathbf{H}$, $\psi_{i_1,\dots,i_n} = \psi_{i_1} \circ \cdots \circ \psi_{i_n}$, and ψ_i are similarities of **H** with contraction factors r. Also

$$\Gamma = \lim_{n\to\infty} \Gamma_n = \bigcup_{i=1}^{6} \psi_i(\Gamma) = \bigcup_{i_j=1}^{6} \psi_{i_1,\dots,i_n}(\Gamma).$$

A routine argument employing the self-similarity of Γ shows that for each subarc $\gamma \subset \Gamma$ there is a similarity ϕ of **H** with $\phi(\gamma) \subset \Gamma$ and $\operatorname{diam}(\phi(\gamma)) \geq r^2$. Utilizing this we deduce that Γ is a−BT for some absolute constant a. Next we claim that $d(g,h) \approx r^m$ whenever $g, h \in \Gamma_n$ and $\Gamma_n(g,h)$ contains at least two adjacent vertices of Γ_m, and m is minimal with respect to this condition. Finally, this permits us to conclude that the natural parametrization $\psi\colon [0,1] \to \Gamma$ satisfies $(d(\psi(s),\psi(t)))^\alpha \approx |s-t|$ where $\alpha = \dim_{\mathcal{H}}(\Gamma) = \log(1/6)/\log(r)$. $\square$

5.B. WQH does not imply QH. As an application of Theorem E we obtain the ensuing characterization for *QH-circles*; these are the Jordan curves which possess quasihomogeneous parametrizations.

5.3. COROLLARY. *A Jordan curve is a QH-circle if and only if it is simultaneously bounded turning, bilipschitz homogeneous, and doubling.*

In this subsection we exhibit two examples of weakly quasihomogeneous maps which fail to be quasihomogeneous. Thus the doubling hypotheses in Fact 2.3 is essential. In addition our examples produce BLH, BT Jordan curves which are not doubling; we conclude that Corollary 5.3 is best possible in the sense that any two of the conditions BT, BLH, doubling need not imply the third.

First we construct a non-compact BLH, BT, bounded Jordan curve. Our ideas here were influenced by [TV, 4.12].

5.4. EXAMPLE. *Let X be the space of all doubly infinite sequences $\{x_n\}_{n\in\mathbb{Z}}$ with $x_n \in \mathbf{R}$ and $x_n \neq 0$ for at most finitely many n. Consider the distance function on X induced by the ℓ_∞ norm $|x| = \|x\|_\infty$. Let $\{e_n\}$ be the 'usual basis' for X, let Λ_n be the line segment $\Lambda_n = [e_n, e_{n+1}]$, and put $\Gamma = \bigcup_{n\in\mathbb{Z}} \Lambda_n$. Then Γ is bilipschitz homogeneous and bounded turning, but not doubling. In fact, the natural parametrization φ: $\mathbf{R} \to \Gamma$ is 4-WQH but not QH.*

Proof. To see that φ is not QH we notice that for distinct $k, l, m, n \in \mathbb{Z}$ we have $|e_k - e_l| = 1 = |e_m - e_n|$ whereas $|k - l|$ and $|m - n|$ need not be comparable, so φ^{-1} is not WQH. One way to see that φ is 4-WQH is to show that it is both 2-VWQH and 2-WQS, and then appeal to Lemma 2.2. □

Next we present a compact example which enjoys the same properties as above. Mimicking the usual snowflake construction, we obtain our Jordan arc as a limit of piecewise linear arcs: $A = \lim_{n\to\infty} A_n$. The crucial idea is to make A_n look more and more like the curve in Example 5.4. We obtain the n^{th}-generation arc A_n from A_{n-1} as follows: Let $[x, y]$ be any maximal straight line segment in A_{n-1}. Consider the n points $x_k = z + (1/2^n)e_k$ $(k = 1, \dots, n)$ where $z = (x + y)/2$ is the midpoint of $[x, y]$ and $e_1, \dots, e_n$ are "unit orthogonal basis vectors" which have not already been used. We replace the segment $[x, y]$ with $\cup_0^n [x_k, x_{k+1}]$ where $x_0 = x$ and $x_{n+1} = y$. The arc A_n is constructed by modifying every single maximal line segment of A_{n-1} in this manner. More details are presented below.

5.5. EXAMPLE. *Let X be the space c_0 of all infinite sequences $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in \mathbf{R}$, which converge to zero. Consider the distance function on X induced by the ℓ_∞ norm $|x| = \|x\|_\infty$. There is a snowflake curve Γ in X which is bilipschitz homogeneous and bounded turning, but not doubling. In fact, the natural parametrization φ: $\mathbf{S}^1 \to \Gamma$ is WQH but not QH.*

Proof. Let $\mathcal{E}$ be the set of 'usual basis' vectors for X. Let A_0 be the line segment from the origin to e_0 where e_0 is any fixed vector in $\mathcal{E}$. Let $x_0^1 = 0$, $x_1^1 = z_0 + (1/2)e_1^1$, $x_2^1 = e_0$ where $z_0 = (1/2)e_0$. Put $A_1 = I_0^1 \cup I_1^1$ where $I_k^1 = [x_k^1, x_{k+1}^1]$. Assume we have constructed $A_n = \cup_p I_p^n$ where $I_p^n = [x_p^n, x_{p+}^n]$, $p = (k_1, k_2, \dots, k_n)$ with $0 \le k_i \le i$, and $p+ = (k_1, k_2, \dots, k_n + 1)$. Define $z_p^n = (x_p^n + x_{p+}^n)/2$ (the midpoint of I_p^n) and set $x_{(p,k)}^{n+1} = z_p^n + (1/2^{n+1})e_{(p,k)}^{n+1}$ for $k = 1, \dots, n+1$ where $e_{(p,k)}^{n+1}$ are vectors in $\mathcal{E}$ which have not already been used; also: $x^{n+1}_{(p,0)} = x^n_p$, $x^{n+1}_{(p,n+2)} = x^n_{p+}$. Thus each n-cell (maximal line segment) in A_n gives rise to $n+2$ new $(n+1)$-cells $I^{n+1}_{(p,k)} = [x^{n+1}_{(p,k)}, x^{n+1}_{(p,k+1)}]$, $0 \le k \le n+1$, and $A_{n+1} = \cup_q I^{n+1}_q$ where $q = (k_1, k_2, \dots, k_{n+1})$ with $0 \le k_i \le i$. Notice that A_{n+1} consists of $(n+2)!$ $(n+1)$-cells each of diameter $1/2^{n+1}$.

Since X is complete, a standard argument reveals that there is a limit arc $A = \lim_{n\to\infty} A_n$. In fact, the natural parametrizations of A_n converge uniformly to the natural parametrization $\psi\colon [0, 1] \to A$. We claim that ψ is WQH but not QH (because ψ^{-1} is not WQH). In a similar manner we can construct a Jordan curve which is BLH and BT but not doubling.

To see that ψ^{-1} is not WQH we look at the points $x^n_{(0,\dots,0,k)} = \psi(k/(n+1)!)$ for $k = 0, 1, \dots, n$. For example, $|x^n_{(0,\dots,0,1)}| = |x^n_{(0,\dots,0,n)}|$, but the ratio $1/(n+1)!$ divided by $n/(n+1)!$ tends to zero; thus ψ is not even QS. It remains to demonstrate that ψ is WQH; again we utilize Lemma 2.2.

We employ the following terminology. The points x^n_p are called *n-vertex points* (or *n-vertices*) of A; each n-vertex is an m-vertex for all $m \ge n$, but very few $(n+1)$-vertices are also n-vertices. Note that the vertices are dense in A. An *n-arc* is a subarc of A which joins two adjacent n-vertices; the n-arcs $J^n_p = A(x^n_p, x^n_{p+})$ are 'supported by' the n-cells I^n_p.

We require the following crucial estimates concerning the distances between vertices. (The earnest reader can calculate $|w| = k/2^n$ as w traverses through the n-vertices.) First, if u and w are both n-vertices with u between 0 and w, then $|u| \le |w|$. Next, if w is any vertex on the first m-arc but w is not on the first $(m+1)$-arc, then $1/2^{m+1} < |w| \le 1/2^m$. Finally, if x, y and 0, w are equally spaced vertices (meaning that all points are n-vertices for some n and there are the same number of n-arcs separating each pair), then

$$(1/4)|w| \le |x - y| \le 2|w|.$$

That ψ is 8-VWQH follows at once from this last estimate.

Finally, we show that ψ is 8-WQS. Fix t, s, ξ with $|t - \xi| \le |t - s|$. Assume that $x = \psi(t)$, $y = \psi(s)$ and $z = \psi(\xi)$ are all n-vertices. Choose n-vertex points u and w so that 0, u and x, z are equally spaced and also 0, w and x, y are equally spaced. Then u lies between 0 and w, so by the above we obtain

$$|x - z| \le 2|u| \le 2|w| \le 8|x - y|. \qquad \square$$

5.C. BLH not BT. Here we explain how to construct a bilipschitz homogeneous Jordan curve in $\mathbf{R}^3$ which fails to be bounded turning. See Bishop's paper [B, Example 4.1] for an elegant compact version of our example, as well as numerous other interesting examples. The idea is simple, although the details are tedious to verify. We start with the usual helix Γ_1 in $\mathbf{R}^3$ parametrized by $x_1(t) = (\cos t, \sin t, t)$. Then we take a much smaller 'helix' and wrap it around Γ_1 to obtain Γ_2. Now we iterate this idea. Provided the 'size' of Γ_{i+1} is geometrically smaller than that of Γ_i, the usual argument shows that we get a limit curve Γ. By making the 'coils' wrap around tighter and tighter, Γ will not be BT, but the natural parametrization will be VWQH so Γ will be BLH.

5.6. EXAMPLE. *There is a Jordan curve in* $\mathbf{R}^3$ *which is BLH but not BT.*

Proof. We start with the helix Γ_1 parametrized by $x_1(t) = (\cos t, \sin t, t)$. Next, Γ_2 is given by $x_2(t) = x_1(t) + \varepsilon_1 z_1(t)$ where $\varepsilon_1 > 0$ is chosen sufficiently small so that x_2 is injective and smooth, and $z_1(t) = \cos(t/\varepsilon_1^2)N_1(t) + \sin(t/\varepsilon_1^2)B_1(t)$; here T_1, N_1, B_1 denotes the Frenet frame for Γ_1. Notice that Γ_2 wraps around Γ_1 ε_1^{-2} times as t goes from $t = 0$ to $t = 2\pi$. Now we iterate this construction.

In general, Γ_i is parametrized by

$$x_{i+1}(t) = x_i(t) + \varepsilon_i z_i(t) \quad , \quad z_i(t) = \cos(t/\varepsilon_i^2)N_i(t) + \sin(t/\varepsilon_i^2)B_i(t)$$

where T_i, N_i, B_i is the Frenet frame for Γ_i and $0 < \varepsilon_i << \varepsilon_{i-1}$. By standard techniques we get a limit curve Γ. Now Γ_{i+1} wraps around Γ_i one time for $0 \leq t \leq 2\pi\varepsilon_i^2$, so the diameter/distance ratio for the points $x_{i+1}(0)$ and $x_{i+1}(\varepsilon_i^2 2\pi)$ is approximately

$$\operatorname{diam}\Gamma(x_{i+1}(0), x_{i+1}(\varepsilon_i^2 2\pi))/|x_{i+1}(0) - x_{i+1}(\varepsilon_i^2 2\pi)| \approx 2\varepsilon_i/2\pi\varepsilon_i^2$$

(the distance is essentially the vertical distance which is given by the change in t) and thus we see that the limit curve Γ is not bounded turning.

We verify that Γ is BLH by checking that each $x_i(t)$ is K-VWQH with K independent of i. □

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