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THE L^p REGULARITY PROBLEM FOR THE HEAT EQUATION IN NON-CYLINDRICAL DOMAINS

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ABSTRACT. We consider the Dirichlet problem for the heat equation in domains with a minimally smooth, time-varying boundary. Our boundary data is taken to belong to a parabolic Sobolev space having a tangential (spatial) gradient, and 1/2 of a time derivative, in L^p, $1 < p < 2 + \epsilon$. We obtain sharp L^p estimates for the parabolic non-tangential maximal function of the gradient of our solutions.

1. Introduction

In this note, we consider the regularity problem for the heat equation in certain non-smooth, time-varying domains. The class of domains which we consider are those given by the region above a time-varying graph:

$$\Omega \equiv \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \colon x_0 > A(x, t)\}. \tag{1.1}$$

Here, A is Lipschitz in x, uniformly in time, i.e.,

$$\sup_{x,t} |A(x, t) - A(x + h, t)| \leq \beta_0 |h|, \tag{1.2}$$

for some $\beta_0 < \infty$; furthermore, A satisfies a certain half-order smoothness condition in t, which we shall now describe. Following Fabes and Riviere [FR], we define the half-order time derivative

$$\tilde{\mathbb{D}}_n \equiv \frac{\partial}{\partial t} \circ \left(\frac{\partial}{\partial t} - \Delta \right)^{-\frac{1}{2}}; \tag{1.3}$$

that is, on the Fourier transform side,

$$(\tilde{\mathbb{D}}_n A)^\wedge(\xi, \tau) \equiv c \frac{\tau}{\sqrt{|\xi|^2 - i\tau}} \hat{A}(\xi, \tau),$$

where, obviously, ξ and τ denote the Fourier transform variables in space and time, respectively. We shall assume that

$$\|\tilde{\mathbb{D}}_n A\|_* \leq \beta_1 < \infty, \tag{1.4}$$

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where the parabolic BMO norm is, as usual, defined by

$$\|b\|_* = \sup_B \frac{1}{|B|} \int_B |b(x) - b_B|\, dz.$$

Here B denotes an arbitrary parabolic ball

$$B \equiv \{z \in \mathbb{R}^n : \|z - z_0\| < r\},$$

and, for non-zero $z = (x, t) \in \mathbb{R}^n$, the parabolic norm $\|z\| = \|(x, t)\|$ is defined to be the unique positive solution ρ of the equation

$$\frac{|x|^2}{\rho^2} + \frac{t^2}{\rho^4} = 1. \tag{1.5}$$

It is well known, and easy to verify, that

$$\|(x, t)\| \cong |x| + |t|^{\frac{1}{2}},$$

and that

$$\|(rx, r^2 t)\| = r\|(x, t)\|.$$

Having defined the class of domains which we shall consider, we are now in a position to define the parabolic Sobolev spaces on $\partial\Omega$, in which spaces we shall take our boundary data. For each fixed t, let Ω_t denote the cross-section

$$\Omega_t \equiv \{(x_0, x) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 > A(x, t)\}.$$

By (1.2), Ω_t is a Lipschitz domain with Lipschitz constant no larger than β_0. We define $d\sigma_t$ to be the usual surface measure on the Lipschitz graph $\partial\Omega_t$, i.e., in graph co-ordinates,

$$d\sigma_t \equiv \sqrt{1 + |\nabla_x A(x, t)|^2}\, dx.$$

We then define "surface measure" $d\sigma$ on $\partial\Omega$ as

$$d\sigma \equiv d\sigma_t dt.$$

The parabolic Sobolev spaces $L^p_{1,1/2}(\mathbb{R}^n)$ are given by $L^p_{1,1/2}(\mathbb{R}^n) \equiv (\frac{\partial}{\partial t} - \Delta)^{-1/2}(L^p(\mathbb{R}^n))$, at least for $1 < p < n + 1$ (in this paper, p will always lie in this range, and, typically $p \le 2$). By parabolic singular integral theory (see [FR]),

$$\|f\|_{L^p_{1,\frac{1}{2}}(\mathbb{R}^n)} \cong \|\nabla_x f\|_{L^p(\mathbb{R}^n)} + \|\tilde{\mathbb{D}}_n f\|_{L^p(\mathbb{R}^n)}.$$

Since A is Lipschitz, in graph co-ordinates,

$$d\sigma(x, t) \equiv d\sigma_t(x)dt \cong dxdt;$$

thus, we can naturally define $L^p_{1,\frac{1}{2}}(\partial\Omega)$ by setting

$$\|f\|_{L^p_{1,\frac{1}{2}}(\partial\Omega)} \equiv \|\tilde{f}\|_{L^p_{1,\frac{1}{2}}(\mathbb{R}^n)}$$

where $\tilde{f}(x,t) \equiv f(A(x,t),x,t)$.

In this paper, we consider the regularity problem

$$R_p \begin{cases} \Delta u - \frac{\partial u}{\partial t} = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f \in L^p_{1,\frac{1}{2}}(\partial\Omega) \\ N_*(\nabla u) \in L^p(\partial\Omega). \end{cases} \tag{1.6}$$

Here N_* denotes the parabolic non-tangential maximal operator

$$N_*(F)(A(x,t),x,t) \equiv \sup_\Gamma |F(y_0,y,s)|,$$

and $\Gamma \equiv \Gamma(A(x,t),x,t)$ is the parabolic cone

$$\Gamma \equiv \{(y_0,y,s)\colon \|(x-y,t-s)\| \leq \mu(y_0 - A(x,t))\}$$

It is not hard to see that for μ small enough, depending only on β_0 and β_1 in (1.2) and (1.4), one has $\Gamma(A(x,t),x,t) \subseteq \Omega$, at every point $(A(x,t),x,t) \in \partial\Omega$. Indeed, by a routine extension to the parabolic case of a result of Strichartz [Stz] (see also [H1] for a proof in the parabolic case), it follows that (1.2) and (1.4) imply the $\mathrm{Lip}_{1,\frac{1}{2}}$ condition

$$\sup_{x,t} |A(x,t) - A(x+h, t+h^2)| \leq C(\beta_0 + \beta_1)|h|, \tag{1.7}$$

from which the non-tangential accessibility follows easily.

The main result of this paper is the following:

THEOREM 1.8. *Given a domain Ω as in* (1.1), *which satisfies* (1.2) *and* (1.4), *then there exists $p_0 > 1$ such that the regularity problem R_p is uniquely solvable in the range $1 < p < p_0$. Here p_0 can be taken to depend only on β_0, β_1, and the dimension n.*

To put this result into context, let us review a bit of recent history. The class of domains which we consider here was introduced by the second author (Lewis) and M. Murray in [LM], although condition (1.4) of the present paper was formulated in a somewhat different fashion there; that the two formulations are equivalent was established in a previous paper of the present authors [HL]. Modulo this equivalence, in [LM], the following result was proved.

THEOREM 1.9. *Given a domain Ω as in Theorem* 1.8, *there exists $q_0 < \infty$ such that the adjoint Dirichlet problem (D_q) (defined below) is uniquely solvable in the range $q_0 < q < \infty$. Here q_0 can be taken to depend only on β_0, β_1 and dimension.*

The adjoint Dirichlet problem (D_q^*) entails finding a solution v to the following problem:

$$D_q^* \begin{cases} \Delta v + \frac{\partial v}{\partial t} = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = f \in L^q(\partial\Omega) \\ N_* v \in L^q(\partial\Omega) \end{cases}$$

(Remark—by the change of variable $t \to -t$, it is equivalent to solve D_q). The exponents p and q in Theorems 1.8 and 1.9, respectively, are dual to each other, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Indeed, our proof of Theorem 1.8, based on a technique introduced by Verchota [V] in the case of harmonic functions in a Lipschitz domain, depends on showing that the solvability of D_q^* implies that of R_p, $\frac{1}{p} + \frac{1}{q} = 1$. The converse (namely that $R_p \to D_q^*$) is easy, and has been noted in [HL]. Furthermore, in [HL] it is shown that Theorem 1.8 is optimal, in the sense that even when $\beta_0 = 0$ (i.e., the case that $A(x,t) = A(t)$), one can construct a class of domains for which solvability of R_p, for any given $p > 1$, can be made to fail by taking β_1 large enough. In other words, one can never hope to fix a p for which R_p holds in all domains Ω of the type considered here: to do so, one must impose some restriction on the size of β_1. An optimal theorem of the latter sort was proved in [HL]; namely that R_2 holds for domains of the type considered here for arbitrary $\beta_0 < \infty$, if β_1 is small enough depending only on β_0 and dimension (analogous L^2 theorems for the Dirichlet and Neumann problems were also proved in [HL]). The results of [LM] and [HL] were thus extensions to the non-cylindrical case of work of Fabes and Salsa [FS] and R. Brown [Br1], [Br2], who proved that if $\beta_1 = 0$ (i.e., $A(x,t) \equiv A(x)$), then one has solvability of D_q^* ([FS]), and R_p and the L^p Neumann problem ([Br1,2]), in the optimal ranges $2 - \epsilon < q < \infty$, $1 < p < 2 + \epsilon$. Given these theorems in the cylindrical case, and also the prior work in the harmonic case of Verchota [V], and Dahlberg and Kenig [DK], it was a reasonable conjecture that R_p, and also the L^p Neumann problem, should be solvable for p in the dual range to that of Theorem 1.9; Theorem 1.8 states that this is indeed true for R_p. What is surprising though, is that in contrast to [Br2] and [DK], this is not at all the case for the Neumann problem. Indeed, in a separate paper we shall show that the L^p Neumann problem is solvable if and only if $\beta_1 < \epsilon(p)$, where $\epsilon(p) \to 0$ as $p \to 1$. An interesting feature of the theory in non-cylindrical domains, then, is the dichotomy between the regularity and Neumann problems.

As mentioned previously, our proof of Theorem 1.8 utilizes a duality method due to Verchota [V] in the harmonic case. Verchota's method extends routinely to the case of the heat equation in cylindrical domains (although we note that this was not Brown's approach in [Br2]—rather, his proof is modeled on the H^1-atomic arguments of [DK]). When one attempts to extend the arguments of [V] to the non-cylindrical case, however, there arise certain terms which are absent in the harmonic and parabolic cylindrical cases, but which, taken individually, are terrible in the non-cylindrical case. However, by performing some algebraic manipulations, a change of variable and a few integrations by parts, one can show that, miraculously, these terrible terms sum precisely to zero. Our main contribution then, aside from some technical L^q estimates at the time-varying boundary for derivatives of caloric and adjoint caloric functions, is our unraveling of the algebra which permits this "miraculous" cancellation. Otherwise, our approach is a straightforward adaptation of the argument in [V].

The paper is organized as follows. In the next section, we give the proof of Theorem 1.8, modulo the technical L^q boundary estimates to which we had alluded. In the last section, we prove these technical estimates.

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2. Proof of Theorem 1.8

We define the single layer potential of a function f by

$$Sf(X,t) \equiv \int_{-\infty}^{t} \int_{\partial\Omega_s} W(X-Q, t-s) f(Q,s)\, d\sigma_s(Q)\, ds, \tag{2.1}$$

where $(X,t) \equiv (x_0, x, t) \in \Omega$, and

$$W(X,t) \equiv (4\pi t)^{-n/2} \exp\{-|X|^2/4t\} \chi_{\{t>0\}}$$

is the usual Gaussian in $\mathbb{R}^{n+1}$. Here, and in the sequel, we have used the notational conventions that capital Y, X denote the spatial components of points in Ω, and P, Q denote the spatial components of points on $\partial\Omega$. Our goal is to establish existence in Theorem 1.8 by obtaining a representation of our solution as the single layer potential of a suitable density function. To this end we recall that it is enough to establish, for p as in Theorem 1.8, the following inequality:

$$\|f\|_{L^p(\partial\Omega)} \le C(\beta_0, \beta_1, n, p) \|S_b f\|_{L^p_{1,\frac{1}{2}}(\partial\Omega)}, \tag{2.2}$$

where S_b denotes the boundary single layer potential

$$S_b f(P,t) \equiv \int_{-\infty}^{t} \int_{\partial\Omega_s} W(P-Q, t-s) f(Q,s)\, d\sigma_s(Q)\, ds,$$

$(P,t) \in \partial\Omega$. The singular integral estimates of [LM] and [H2], plus the method of continuity, may then be used to show that (2.2) implies the desired representation formula, and hence also existence of solutions to R_p, with non-tangential estimates. The details of the method of continuity argument may be found in the survey article of Kenig [K]. The same argument applies in the parabolic case, and with L^p in place of L^2.

To establish (2.2), we follow the strategy of Verchota [V]. For any $f \in C_c(\partial\Omega)$, set $u = Sf$, and let $u_- = S_- f$ denote the single layer potential of f in the domain $\Omega^- \equiv (\bar{\Omega})^c$. Define

$$K^* f(P,t) \equiv p.v. \int_{-\infty}^{t} \int_{\partial\Omega_s} N_{(P,t)} \cdot \nabla_P W(P-Q, t-s) f(Q,s)\, d\sigma_s(Q)\, ds,$$

where $N_{(P,t)}$ denotes the unit outer normal to $\partial\Omega_t$ at the point (P,t). In [LM], it is shown that, on $\partial\Omega$,

$$\frac{\partial u}{\partial N} = \left(\frac{1}{2}I + K^*\right) f \tag{2.3}$$

and

$$\frac{\partial u_-}{\partial N_-} = \left(\frac{1}{2}I - K^*\right) f, \tag{2.4}$$

where N_- denotes the outer unit normal to $\partial\Omega^-$ (i.e., the inner normal to $\partial\Omega$). As usual, then, to prove (2.2), it will suffice to establish the estimate

$$\left\| \frac{\partial u}{\partial N} \right\|_{L^p(\partial\Omega)} \le C(\beta_0, \beta_1, n, p) \|u\|_{L^p_{1,\frac{1}{2}}(\partial\Omega)}. \tag{2.5}$$

Indeed, interchanging the roles of Ω and Ω^-, we see that (2.5) also holds with u replaced by u_-. Since tangential derivatives of u (including $\mathbb{D}_n$) do not jump across the boundary, we immediately obtain from (2.3)-(2.5) the estimate

$$\left\| \left(\frac{1}{2} + K^*\right) f \right\|_{L^p(\partial\Omega)} + \left\| \left(\frac{1}{2}I - K^*\right) f \right\|_{L^p(\partial\Omega)} \le C(\beta_0, \beta_1, n, p) \|u\|_{L^p_{1,\frac{1}{2}}(\partial\Omega)},$$

from which (2.2) follows by the triangle inequality.

We therefore proceed to prove (2.5). Following [V], we dualize. By Theorem 1.9 (i.e., the result of [LM]), it suffices to establish the following estimate:

$$\left| \int_{-\infty}^{\infty} \int_{\partial\Omega_t} \frac{\partial u}{\partial N} v\, d\sigma_t dt \right| \le c(n, p, \beta_0, \beta_1) \|u\|_{L^p_{1,\frac{1}{2}}(\partial\Omega)} \|v\|_{L^q(\partial\Omega)}, \tag{2.6}$$

where $\Delta v + \frac{\partial v}{\partial t} = 0$, and $v|_{\partial\Omega}, N_* v \in L^q$, and $q > q_0(\beta_0, \beta_1, n)$. By density and a standard limiting argument, it suffices to prove (2.6) for $v|_{\partial\Omega} \in C_c(\partial\Omega)$. It also suffices to prove (2.6) under the a priori assumption that $A(x,t) \in C_0^\infty$. With these a priori assumptions, the function

$$w(x_0, x, t) \equiv -\int_{x_0}^{\infty} v(\alpha, x, t)\, d\alpha$$

is well defined in Ω, and is also an adjoint solution, i.e., $\Delta w + \frac{\partial \omega}{\partial t} = 0$ in Ω. Clearly, $v = \frac{\partial}{\partial x_0}\omega$. We also define

$$v_j \equiv \frac{\partial}{\partial x_j} w, \quad 0 \le j \le n-1,$$

where $v_0 \equiv v$. In the next section, we shall prove the following:

LEMMA 2.7. *Let $\Delta w + \partial w/\partial t = 0$ in Ω, and suppose that $\frac{\partial}{\partial x_0}\omega \in L^q(\partial\Omega)$. Then*

$$\text{(i)} \qquad \left\| \frac{\partial}{\partial x_j} w \right\|_{L^q(\partial\Omega)} \le C(q, \beta_0, \beta_1, n) \left\| \frac{\partial}{\partial x_0} w \right\|_{L^q(\partial\Omega)},$$

$$\text{(ii)} \qquad \|\tilde{\mathbb{D}}_n w\|_{L^q(\partial\Omega)} \le C(q, \beta_0, \beta_1, n) \left\| \frac{\partial}{\partial x_0} w \right\|_{L^q(\partial\Omega)}.$$

For the remainder of Section 2, we shall take Lemma 2.7 for granted.

Returning to the proof of (2.6), we note that by Green's Theorem (in the space variable),

$$\begin{aligned}
(2.8) \qquad \int_{-\infty}^{\infty}\int_{\partial\Omega_t} \frac{\partial u}{\partial N} v \, d\sigma_t \, dt &= \int_{-\infty}^{\infty}\int_{\partial\Omega_t} u \frac{\partial v}{\partial N} \, d\sigma_t \, dt \\
&\quad + \iint_{\Omega} (\Delta u v - u \Delta v) \, dX \, dt \\
&= \iint_{\partial\Omega} u \frac{\partial v}{\partial N} \, d\sigma_t \, dt \\
&\quad + \iint_{\Omega} \left(\frac{\partial u}{\partial t} v + u \frac{\partial v}{\partial t} \right) dX \, dt,
\end{aligned}$$

where $\frac{\partial}{\partial N}$ denotes differentiation in the direction of the outer unit normal N. Following [V], and setting $N \equiv N^0 e_0 + N^1 e_1 + \cdots + N^{n-1} e_{n-1}$ (where e_j is the unit basis vector in the x_j direction), we have

$$\begin{aligned}
\frac{\partial v}{\partial N} &\equiv \langle N, \nabla v \rangle \\
&\equiv \sum_{j=0}^{n-1} N^j \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_0} w \\
&= \sum_{j=0}^{n-1} \left[N^j \frac{\partial}{\partial x_0} - N^0 \frac{\partial}{\partial x_j} \right] \frac{\partial}{\partial x_j} w - N^0 \frac{\partial}{\partial t} w
\end{aligned}$$

(since w is an adjoint solution)

$$\equiv \sum_{j=1}^{n-1} \frac{\partial}{\partial \tau_j} v_j - N^0 \frac{\partial}{\partial t} w,$$

where $\frac{\partial}{\partial \tau_j}$, $1 \le j \le n-1$, denotes a tangential derivative. Then the right side of (2.8) equals

$$\sum_{J=1}^{n-1} \iint_{\partial\Omega} u \frac{\partial}{\partial \tau_j} v_j \, d\sigma_t \, dt + \iint_{\mathbb{R}^n} u w_t \, dx \, dt + \iint_{\Omega} (u_t v + u v_t) \, dX \, dt \tag{2.9}$$
$$\equiv I + II + III,$$

where in the middle term we have used the fact that, in graph co-ordinates, $-N^0 d\sigma_t = dx$. We integrate by parts in I to obtain

$$|I| = \left| \iint_{\partial\Omega} \frac{\partial}{\partial \tau_j} u \, v_j \right| \le \|u\|_{L^p_{1,\frac{1}{2}}(\partial\Omega)} \|v_j\|_{L^q(\partial\Omega)}$$
$$\le c(q, \beta_0, \beta_1, n) \|u\|_{L^p_{1,\frac{1}{2}}} \|v_0\|_{L^q},$$

by Lemma 2.7. In the harmonic case, we would be done now (indeed, this was the proof in [V]). Furthermore, in the case of the heat equation in cylinders, term $III = 0$, and

$$|II| = \left| \iint_{\mathbb{R}^n} H D_t^{1/2} u D_t^{1/2} w \right| dx \, dt, \tag{2.10}$$

where H denotes the Hilbert transform in t, and

$$\left(D_t^{1/2} f\right)^\wedge (\tau) \equiv |\tau|^{1/2} \hat{f}(\tau),$$

is a 1-dimensional half order derivative in t. Since $\frac{|\tau|^{1/2}}{\|(\xi,\tau)\|}$ is an L^p multiplier for $1 < p < \infty$ e.g., (see [S, Theorem 6, p. 109]), (2.10) is bounded by

$$c_p \|\mathbb{D} u\|_{L^p} \|\mathbb{D} w\|_{L^q}, \tag{2.11}$$

where $(\mathbb{D} f)^\wedge(\xi, \tau) \equiv \|(\xi, \tau)\| \hat{f}(\xi, \tau)$. But by (1.5) applied on the Fourier transform side,

$$\mathbb{D} = \sum_{j=1}^{n-1} R_j \frac{\partial}{\partial x_j} + R_n \mathbb{D}_n,$$

where

$$(\mathbb{D}_n f)^\wedge(\xi, \tau) \equiv \frac{\tau}{\|(\xi, \tau)\|} \hat{f}(\xi, \tau)$$

and

$$\begin{aligned} (R_j f)^\wedge &= \frac{\xi_j}{\|(\xi,\tau)\|}\hat{f}(\xi,\tau),\ 1 \le j \le n-1 \\ (R_n f)^\wedge(\xi) &= \frac{\tau}{\|(\xi,\tau)\|^2}\hat{f}(\xi,\tau). \end{aligned}$$

By [FR], the parabolic Riesz transforms R_j, $1 \le j \le n$, are all bounded on L^p, $1 < p < \infty$, and by [S, Theorem 6, p. 109]

$$\|\mathbb{D}_n f\|_{L^p} \cong \|\tilde{\mathbb{D}}_n f\|_{L^p}, \quad 1 < p < \infty.$$

By Lemma 2.7, (2.11) is no larger than

$$C(p, \beta_0, \beta_1, n)\|u\|_{L^p_{1,\frac{1}{2}}} \left\| \frac{\partial}{\partial x_0} w \right\|_{L^q},$$

as desired.

The problem then, in the non-cylindrical case, is to deal with the errors which arise when one attempts to treat the terms II and III in (2.9). To this end, we define a parabolic approximate identity

$$\begin{aligned} P_\lambda f(x,t) &\equiv \varphi_\lambda * f(x,t) \\ &\equiv \lambda^{-d} \iint \varphi\left(\frac{x-y}{\lambda}, \frac{t-s}{\lambda^2}\right) f(y,s)\,dy\,ds, \end{aligned}$$

where $d = n+1$ is the homogeneous dimension of parabolic $\mathbb{R}^n$, and where $\varphi \in C_0^\infty$, $\operatorname{supp}\varphi \subseteq \beta_1(0)$ (the unit ball), $\varphi \ge 0$, φ is even and $\int \varphi = 1$. Next, we choose a small (fixed) constant $\gamma > 0$, depending only on the constant $C(\beta_0 + \beta_1)$ in (1.7), such that for all $(x,t) \in \mathbb{R}^n$,

$$\frac{1}{2} \le 1 + \frac{\partial}{\partial \lambda} P_{\gamma\lambda} A(x,t) \le 3/2.$$

For such γ, the Dahlberg-Kenig-Stein mapping

$$\rho(\lambda, x, t) \equiv (\lambda + P_{\gamma\lambda}A(x,t), x, t)$$

defines a 1-1 mapping of the half-space $\mathbb{R}^{n+1}_+ \equiv \{(\lambda, x, t) = \lambda > 0, (x,t) \in \mathbb{R}^n\}$ onto Ω, and furthermore ρ: $\partial\mathbb{R}^{n+1}_+ \to \partial\Omega$. We remark that this mapping appeared first in a paper of Dahlberg [D] (although this explicit construction was due to Kenig and Stein), and has recently proven useful in [HL] and in work of Dahlberg, Kenig, Pipher and Verchota [DKPV]. Now, the term II in (2.9) equals

$$\begin{aligned}
&\int_{\mathbb{R}^n} [u \circ \rho(0,x,t)][\omega_t \circ \rho(0,x,t)]\, dx\, dt \\
&\quad = -\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial\lambda}(u\circ\rho\cdot\omega_t\circ\rho)\, d\lambda\, dx\, dt \\
&\quad = -\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial\lambda}(u\circ\rho[\omega\circ\rho]_t)\, d\lambda\, dx\, dt \\
&\qquad + \int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial\lambda}\left(u\circ\rho\,\omega_{x_0}\circ\rho\frac{\partial}{\partial t}P_{\gamma\lambda}A\right) d\lambda\, dx\, dt \\
&\quad \equiv \tilde{II} + E.
\end{aligned}$$

The term $\tilde{II}$ may be handled essentially like the term II in the cylindrical case. Indeed,

$$\begin{aligned}
|\tilde{II}| &= \left|\int_{\mathbb{R}^n} HD_t^{1/2}(uo\rho)D_t^{1/2}(wo\rho)\, dx\, dt\right| \\
&\le C_\rho \|u\|_{L^p_{1,\frac{1}{2}}} \|v\|_{L^q},
\end{aligned}$$

by exactly the same argument as in the cylindrical case (see (2.8) and the ensuing discussion).

Next, we claim that

$$E \equiv -III, \tag{2.12}$$

in which case we are done. To prove (2.12), we first observe that

$$\begin{aligned}
E &= \int_{\mathbb{R}^n}\int_0^\infty (u\circ\rho)_\lambda\omega_{x_0}\circ\rho\frac{\partial}{\partial t}P_{\gamma\lambda}A + \int_{\mathbb{R}^n}\int_0^\infty u\circ\rho(\omega_{x_0}\circ\rho)_\lambda\frac{\partial}{\partial t}P_{\gamma\lambda}A \\
&\quad + \iint u\circ\rho\; w_{x_0}\circ\rho\frac{\partial}{\partial\lambda}\frac{\partial}{\partial t}P_{\partial\lambda}A \\
&\equiv E_1 + E_2 + E_3.
\end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned}
u_t\circ\rho &= \frac{\partial}{\partial t}(u\circ\rho) - u_{x_0}\circ\rho\frac{\partial}{\partial t}P_{\gamma\lambda}A \\
v_t\circ\rho &= \frac{\partial}{\partial t}(v\circ\rho) - v_{x_0}\circ\rho\frac{\partial}{\partial t}P_{\gamma\lambda}A
\end{aligned}$$

Thus, changing variables in III in (2.9), we obtain

$$\begin{aligned}
(2.14)\qquad III &= \int_{\mathbb{R}^n}\int_0^\infty u_t\circ\rho\; v\circ\rho\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)d\lambda\,dx\,dt\\
&\quad+\int_{\mathbb{R}^n}\int_0^\infty u\circ\rho\; v_t\circ\rho\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)d\lambda\,dx\,dt\\
&\equiv \int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial t}(u\circ\rho)v\circ\rho\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\\
&\quad-\int_{\mathbb{R}^n}\int_0^\infty u_{x_0}\circ\rho\; v\circ\rho\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\frac{\partial}{\partial t}P_{\gamma\lambda}A\\
&\quad+\int_{\mathbb{R}^n}\int_0^\infty u\circ\rho\frac{\partial}{\partial t}(v\circ\rho)\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\\
&\quad-\int_{\mathbb{R}^n}\int_0^\infty u\circ\rho\; v_{x_0}\circ\rho\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\frac{\partial}{\partial t}P_{\gamma\lambda}A\\
&\equiv III_1+III_2+III_3+III_4.
\end{aligned}$$

Now,

$$III_2\equiv-\iint(u\circ\rho)_\lambda v\circ\rho\frac{\partial}{\partial t}P_{\gamma\lambda}A\equiv -E_1,$$

since $v\equiv w_{x_0}$. Also,

$$III_4\equiv-\iint u\circ\rho\;(v\circ\rho)_\lambda\frac{\partial}{\partial t}P_{\gamma\lambda}A\equiv -E_2.$$

Finally, integrating by parts in t, we obtain that

$$\begin{aligned}
III_1 &= -\int_{\mathbb{R}^n}\int_0^\infty u\circ\rho\;\frac{\partial}{\partial t}(v\circ\rho)\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)d\lambda\,dx\,dt\\
&\quad-\int_{\mathbb{R}^n}\int_0^\infty u\circ\rho\; v\circ\rho\;\frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\,d\lambda\,dx\,dt\\
&\equiv -III_3-E_3.
\end{aligned}$$

Adding (2.13) and (2.14), we get zero. Modulo the proof of Lemma 2.7, which we give in the next section, this concludes the proof of existence in Theorem 1.8. Uniqueness in Theorem 1.8 may be deduced from (2.2) as in [HL, pp. 397-399]. The argument given there for $p=2$ carries over to the case $p>1$ with minimal changes. We omit the details.

3. Proof of Lemma 2.7

We first show that (i) of Lemma 2.7 implies (ii). That is we begin by proving

$$\|\mathbb{D}_n w\|_{L^q(\partial\Omega)} \le c_q \|\nabla w\|_{L^q(\partial\Omega)}. \tag{3.1}$$

(We recall that

$$\begin{aligned}
\mathbb{D}_n f &\equiv \left(C\frac{\tau}{\|(\xi,\tau)\|}\hat{f}(\xi,\tau)\right)^{\vee} \equiv \frac{\partial}{\partial t}\circ \mathbb{D}^{-1}(f)\\
\tilde{\mathbb{D}}_n f &\equiv \left(C\frac{\tau}{\sqrt{|\xi|^2 - i\tau}}\hat{f}(\xi,\tau)\right)^{\vee} \equiv \frac{\partial}{\partial t}\circ\left(\Delta - \frac{\partial}{\partial t}\right)^{-\frac{1}{2}}(f)
\end{aligned}$$

and of course $\mathbb{D}_n$ is equivalent to $\tilde{\mathbb{D}}_n$ in the L^q norm). Let $g \in C_0^\infty$, $\|g\|_p = 1$, $\frac{1}{p} + \frac{1}{q} = 1$. To bound the left hand side of (3.1), it is enough to consider

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathbb{D}_n(\omega\circ\rho) g\,dx\,dt &= -\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial\lambda}\left(\mathbb{D}_n(w\circ\rho)P_\lambda g\right)\,d\lambda\,dx\,dt\\
&= -\int_{\mathbb{R}^n}\int_0^\infty \mathbb{D}_n(w\circ\rho)\frac{\partial}{\partial\lambda}P_\lambda g\\
&\quad -\int_{\mathbb{R}^n}\int_0^\infty \mathbb{D}_n(w\circ\rho)_\lambda P_\lambda g\\
&\equiv I + II,
\end{aligned} \tag{3.2}$$

where P_λ is the parabolic approximate identity defined in the previous section. Since $\mathbb{D}_n = \mathbb{D}^{-1}\circ\frac{\partial}{\partial t}$, where $(\mathbb{D}^{-1}f)^\wedge(\xi,\tau) = \|(\xi,\tau)\|^{-1}\hat{f}$,

$$\begin{aligned}
I &= -\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial t}(w\circ\rho)\mathbb{D}^{-1}P_\lambda g\\
&= -\int_{\mathbb{R}^n}\int_0^\infty \omega_t\circ\rho\;\mathbb{D}^{-1}\frac{\partial}{\partial\lambda}P_\lambda g - \int_{\mathbb{R}^n}\int_0^\infty \omega_{x_0}\circ\rho\;\frac{\partial}{\partial t}P_{\gamma\lambda}A\;\mathbb{D}^{-1}\frac{\partial}{\partial\lambda}P_\lambda g\\
&\equiv I_1 + I_2.
\end{aligned} \tag{3.3}$$

Since P_λ is even, and therefore has vanishing first moments, $\tilde{Q}_\lambda \equiv \mathbb{D}^{-1}\frac{\partial}{\partial\lambda}P_\lambda$ satisfies the Littlewood-Paley estimate

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty |\tilde{Q}_\lambda g|^2\frac{d\lambda}{\lambda}\right)^{p/2}dx\,dt\right)^{1/p} \le c_p\|g\|_p, \quad 1 < p < \infty.$$

Thus

$$\begin{aligned}|I_1| &\le c_p\|g\|_p \sum_{j=0}^{n-1} \left\| \left(\int_0^\infty |\omega_{x_j x_j} \circ \rho|^2 \lambda \, d\lambda \right)^{1/2} \right\|_q \\ &\le c(p, \beta_0, \beta_1)\|g\|_p \|N_*(\nabla w)\|_q \\ &\le c(p, \beta_0, \beta_1)\|g\|_p \|\nabla w\|_q,\end{aligned}$$

as desired, where in the last two inequalities we have used first the square function estimates of Brown [Br3] and then Theorem 1.9 (the result of [LM]), to remove the non-tangential maximal function, since ω_{x_j} is an adjoint solution for each $j \in \{0, 1, \dots, n-1\}$. (We remark that [Br3] treats the Lusin "S" function, rather than the "g" function which arose in the last display above, but one can use interior estimates for adjoint caloric functions to show that the former controls the latter; we omit the details.) Furthermore,

$$|I_2| \le c_p\|g\|_p \left\| \left(\int_0^\infty |\omega_{x_0} \circ \rho|^2 \left| \frac{\partial}{\partial t} P_{\gamma\lambda} A \right|^2 \lambda d\lambda \right)^{1/2} \right\|_q .$$

Since $q \ge 2$, to control the square of the last factor, it suffices to estimate

$$\int_{\mathbb{R}^n} \int_0^\infty |\omega_{x_0} \circ \rho|^2 \left| \frac{\partial}{\partial t} P_{\gamma\lambda} A \right|^2 \lambda \, d\lambda \; \nu(x, t) dx dt, \tag{3.4}$$

where $\nu \in L^r$, $\frac{1}{r} + \frac{2}{q} = 1$, $\|\nu\|_{L^r} = 1$. In the last expression, we may replace ν by the parabolic A_1 weight $\tilde{\nu} \equiv (M(|\nu|^{1+\epsilon}))^{1/1+\epsilon}$ (here M denotes the parabolic maximal function) so that (3.4) is dominated by

$$c_n \int_{\mathbb{R}^n} (N_{**}(\omega_{x_0} \circ \rho))^2 \tilde{\nu} \, dx \, dt \cdot \Phi(A, \tilde{\nu}), \tag{3.5}$$

where N_{**} denotes the non-tangential maximal operator in $\mathbb{R}^{n+1}_+$, and where

$$\Phi(A, \tilde{\nu}) \equiv \sup_B \frac{1}{|B|_{\tilde{\nu}}} \iint_B \int_0^{r(B)} \left| \frac{\partial}{\partial t} P_{\gamma\lambda} A \right|^2 \lambda \, d\lambda \; \tilde{\nu} \cdot dx \, dt. \tag{3.6}$$

Here, the sup runs over all parabolic balls B, and $r(B)$ denotes the parabolic radius of B. Also, $|B|_{\tilde{\nu}} \equiv \int_B \tilde{\nu}(x, t) \, dx \, dt$. But for ϵ chosen small enough, $\|\tilde{\nu}\|_r \le c_{n,r}\|\nu\|_r = c_{n,r}$. Hence, the first factor in (3.5) is no larger than

$$c_{n,r}\|N_{**}(\omega_{x_0} \circ \rho)\|^2_{L^q(\mathbb{R}^n)} \le c_{n,q}\|N_*(\omega_{x_0})\|^2_{L^q(\partial\Omega)},$$

where we have used the fact that

$$N_{**}(u \circ \rho) \le [N_*(u)] \circ \rho,$$

for any $u \in L^1_{\text{loc}}$ defined on Ω, as long as the cones defining N_{**} have sufficiently narrow aperture. Also, by [HL, Lemma 2.8],

$$\Phi(A, 1) \leq C(n, \gamma)\|\mathbb{D}_n A\|_* \cong C(n, \gamma)\|\tilde{\mathbb{D}}_n A\|_*,$$

and the extension to the weighted case

$$\Phi(A, \tilde{\nu}) \leq C(n, \gamma, \epsilon)\|\mathbb{D}_n A\|_* \tag{3.7}$$

is a standard exercise which we omit. We do point out however, that we have used the parabolic version of a result of Coifman and Rochberg [CR], namely that

$$\tilde{\nu} \equiv (M(|\nu|^{1+\epsilon})]^{1/1+\epsilon}$$

is an A_1 weight with A_1 constant depending only on ϵ. Since ϵ depends, in turn, only on q, and since γ had been chosen to depend only on β_0 and β_1, the desired bound for (3.4), and hence for (I_2) now follows immediately.

Next, we turn to the term II of (3.2). Integrating by parts in λ, we get

$$\begin{aligned} II &= \int_{\mathbb{R}^n}\int_0^\infty \mathbb{D}_n(\omega\circ\rho)_\lambda \frac{\partial}{\partial\lambda}P_\lambda g\lambda\, d\lambda\, dx\, dt \\ &\quad + \int_{\mathbb{R}^n}\int_0^\infty \mathbb{D}_n(\omega\circ\rho)_{\lambda\lambda} P_\lambda g\lambda\, d\lambda\, dx\, dt \\ &\equiv II_1 + II_2. \end{aligned} \tag{3.8}$$

Again, using the fact that $\mathbb{D}_n = \mathbb{D}^{-1}\circ\frac{\partial}{\partial t}$ and $\mathbb{D}^{-1}$ is self-adjoint, we obtain

$$\begin{aligned} |II_1| &= \left|\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial t}(\omega\circ\rho)_\lambda \tilde{Q}_\lambda g\ \lambda\, d\lambda\, dx\, dt\right| \\ &\equiv \left|\int_{\mathbb{R}^n}\int_0^\infty \frac{\partial}{\partial t}\left[(\omega_{x_0}\circ\rho)\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\right]\tilde{Q}_\lambda g\ \lambda\, d\lambda\, dx\, dt\right| \\ &\leq \left|\int_{\mathbb{R}^n}\int_0^\infty \omega_{x_0 t}\circ\rho\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\tilde{Q}_\lambda g\ \lambda\, d\lambda\, dx\, dt\right| \\ &\quad + \left|\int_{\mathbb{R}^n}\int_0^\infty \omega_{x_0 x_0}\circ\rho\frac{\partial}{\partial t}P_{\gamma\lambda}A\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\tilde{Q}_\lambda g\ \lambda\, d\lambda\, dx\, dt\right) \\ &\quad + \left|\int_{\mathbb{R}^n}\int_0^\infty \omega_{x_0}\circ\rho\frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\ \tilde{Q}_\lambda g\ \lambda\, d\lambda\, dx\, dt\right| \\ &\equiv II_{11} + II_{12} + II_{13}. \end{aligned}$$

Since $|1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A|\le 3/2$, for γ small enough depending only on β_0 and β_1, II_{11} can be handled like I_1 above, except that one needs to use interior estimate to remove a derivative on ω. Since $\lambda\left|\frac{\partial}{\partial t}P_{\gamma\lambda}A\right|\le C(\gamma)\|\mathbb{D}_nA\|_*$, where $\gamma=\gamma(\beta_0,\beta_1)$, II_{12} can also be estimated like I_1. And since $\lambda^3\left|\frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right|^2 d\lambda\tilde{\nu}dxdt$ satisfies a weighted Carleson measure estimate essentially like that expressed in (3.6) and (3.7), we can treat II_{13} just like I_2.

To finish the proof that (i)⇒(ii) in Lemma 2.7, it remains to consider II_2 in (3.8). But this term can also be treated by arguments similar to those we have not described, once we note that $Q_\lambda\equiv-\lambda\mathbb{D}_nP_\lambda$ satisfies standard Littlewood-Paley square function estimates, just as did $\tilde{Q}_\lambda$. The details are omitted.

Next, we turn to the proof of (i) in Lemma 2.7. Our approach here is similar in spirit to that of [DKPV]. We observe that, for $1\le j\le n-1$,

$$(\omega\circ\rho)_{x_j}=\omega_{x_j}\circ\rho+(\omega_{x_0}\circ\rho)\frac{\partial}{\partial x_j}P_{\gamma\lambda}A. \tag{3.9}$$

Thus, it suffices to prove that, for $1\le j\le n-1$,

$$\begin{aligned}\int_{\mathbb{R}^n}|(\omega\circ\rho)_{x_j}|^q\,dx\,dt\ &\le\ C(\beta_0,\beta_1,n,q)\int_{\mathbb{R}^n}|\omega_{x_0}\circ\rho|^q\,dx\,dt\\ &\quad+C_q\epsilon\int_{\mathbb{R}^n}(\omega_{x_j}\circ\rho)^q\,dx\,dt,\end{aligned} \tag{3.10}$$

because the small term can then be hidden on the left hand side of i) in Lemma 2.7. The left hand side of (3.10) equals

$$\begin{aligned}&-\int_{\mathbb{R}^n}\int_0^\infty\frac{\partial}{\partial\lambda}|(\omega\circ\rho)_{x_j}|^q\,d\lambda\,dx\,dt\\ &=\int_{\mathbb{R}^n}\int_0^\infty\left(\frac{\partial}{\partial\lambda}\right)^2|(\omega\circ\rho)_{x_j}|^q\,\lambda\,d\lambda\,dx\,dt\\ &=q(q-1)\int_{\mathbb{R}^n}\int_0^\infty|(\omega\circ\rho)_{x_j}|^{q-2}[(\omega\circ\rho)_{x_j\lambda}]^2\lambda\,d\lambda\,dx\,dt\\ &\quad+q\int_{\mathbb{R}^n}\int_0^\infty|(\omega\circ\rho)_{x_j}|^{q-1}(\omega\circ\rho)_{x_j\lambda\lambda}\lambda\,d\lambda\,dx\,dt\\ &\equiv I+II.\end{aligned} \tag{3.11}$$

Since the radial maximal function is controlled by the non-tangential maximal function, i.e. $\sup_{\lambda>0}|F(x,\lambda)| \leq N_{**}(F)(x)$, we have that

$$
\begin{aligned}
|I| &\leq c_q \int_{\mathbb{R}^n} [N_{**}((\omega\circ\rho)_{x_j})]^{q-2} \int_0^\infty ((\omega\circ\rho)_{x_j\lambda})^2 \lambda\, d\lambda\, dx\, dt \\
&\leq C_q \left(\int_{\mathbb{R}^n} (N_{**}[(\omega\circ\rho)_{x_j}])^q\right)^{\frac{q-2}{q}} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \left|(\omega\circ\rho)_{x_j\lambda}\right|^2 \lambda\, d\lambda\right)^{q/2} dx\, dt\right)^{\frac{2}{q}} \\
&\leq C_q\delta \int_{\mathbb{R}^n} N_{**}[(\omega\circ\rho)_{x_j}]^q \\
&\quad + C\left(q,\frac{1}{\delta}\right) \int_{\mathbb{R}^n} \left(\int_0^\infty |(\omega\circ\rho)_{x_j\lambda}|^2 \lambda\, d\lambda\right)^{q/2} dx\, dt \\
&\equiv I_1 + I_2.
\end{aligned}
$$

By (3.9) and Theorem 1.9 (the result of [LM]),

$$
I_1 \leq C_q\delta \int_{\mathbb{R}^n} |\omega_{x_j}\circ\rho|^q + C_q\beta_0\delta \int_{\mathbb{R}^n} |\omega_{x_0}\circ\rho|^q,
$$

which yields the desired bound (3.10) if we set $\delta = \epsilon$. Choosing ϵ small enough, and depending only on q, we can make the first summand small enough that it can be hidden on the left hand side of Lemma 2.7 (i). Next, we note that

$$
\begin{aligned}
(\omega\circ\rho)_{x_j\lambda} &= (\omega_{x_jx_0}\circ\rho)\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right) + (\omega_{x_0x_0}\circ\rho)\left(1+\frac{\partial}{\partial\lambda}P_{\gamma\lambda}A\right)\frac{\partial}{\partial x_j}P_{\gamma\lambda}A \\
&\quad + (\omega_{x_0}\circ\rho)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial x_j}P_{\gamma\lambda}A.
\end{aligned}
$$

Since δ depends only on q, and since

$$
\left|\frac{\partial}{\partial\lambda}\frac{\partial}{\partial x_j}P_{\gamma\lambda}A\right|^2 \lambda d\lambda\ \tilde{\nu}(x,t)dxdt
$$

is a weighted Carleson measure for $\tilde{\nu} \in A_1$, we can argue as above (see the proof that i)⇒ii)) to show that

$$
I_2 \leq C(q,\beta_0,\beta_1)\int_{\mathbb{R}^n} |\omega_{x_0}\circ\rho|^q\, dx\, dt,
$$

as desired. Thus the term I in (3.11) satisfies (3.10). We therefore turn to II.

Integrating by parts in x_j, we get

$$
\begin{aligned}
II &= C_q \int_{\mathbb{R}^n} \int_0^\infty |(\omega \circ \rho)_{x_j}|^{q-2} (\omega \circ \rho)_{x_j x_j} (\omega \circ \rho)_{\lambda\lambda} \lambda \, d\lambda \, dx \, dt \\
&\leq C_q \epsilon \int_{\mathbb{R}^n} \int_0^\infty |(\omega \circ \rho)_{x_j}|^{q-2} |(\omega \circ \rho)_{x_j x_j}|^2 \lambda \, d\lambda \, dx \, dt \\
&\quad + C_q \frac{1}{\epsilon} \int_{\mathbb{R}^n} \int_0^\infty |(\omega \circ \rho)_{x_j}|^{q-2} |(\omega \circ \rho)_{\lambda\lambda}|^2 \lambda \, d\lambda \, dx \, dt. \\
&\equiv II_1 + II_2
\end{aligned}
$$

Each of these terms can be treated just like I in (3.11). We obtain the bounds

$$
|II_1| \leq C_q \epsilon \int_{\mathbb{R}^n} |\omega_{x_j} \circ \rho|^q,
$$

(which can be hidden) and

$$
\begin{aligned}
|II_2| &\leq C_q \frac{1}{\epsilon} \delta \left(\int_{\mathbb{R}^n} |\omega_{x_j} \circ \rho|^q + \beta_0 \int_{\mathbb{R}^n} |\omega_{x_0} \circ \rho|^q \right) \\
&\quad + C\left(q, \frac{1}{\epsilon}, \frac{1}{\delta}\right) \int_{\mathbb{R}^n} |\omega_{x_0} \circ \rho|^q.
\end{aligned}
$$

We then set $\delta = \epsilon^2$, with ϵ depending only on q, and (3.10) follows. This concludes the proof of Lemma 2.7.

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