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# HARDY'S INEQUALITY AND EMBEDDINGS IN HOLOMORPHIC TRIEBEL-LIZORKIN SPACES

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ABSTRACT. In this work we study some properties of the holomorphic Triebel-Lizorkin spaces $HF_s^{pq}$, $0 < p, q \leq \infty$, $s \in \mathbb{R}$, in the unit ball $B$ of $\mathbb{C}^n$, motivated by some well-known properties of the Hardy-Sobolev spaces $H_s^p = HF_s^{p2}$, $0 < p < \infty$.

We show that $\sum_{n\geq 0} |a_n|/(n+1) \lesssim \| \sum_{n\geq 0} a_n z^n \|_{HF_0^{1\infty}}$, which improves the classical Hardy's inequality for holomorphic functions in the Hardy space $H^1$ in the disc. Moreover, we give a characterization of the dual of $HF_s^{1q}$, which includes the classical result $(H^1)^* = \mathrm{BMOA}$. Finally, we prove some embeddings between holomorphic Triebel-Lizorkin and Besov spaces, and we apply them to obtain some trace theorems.

## 1. Introduction

Let $B$ denote the unit ball in $\mathbb{C}^n$, and $S$ its boundary. Let $R$ denote the radial derivative and $I$ the identity operator. We recall that if $f(z) = \sum_\alpha c_\alpha z^\alpha$ is holomorphic on $B$ and $s \in \mathbb{R}$, then the operator $(I+R)^s$ is defined by $(I+R)^s f(z) = \sum_\alpha c_\alpha (1+|\alpha|)^s z^\alpha$.

The holomorphic Triebel-Lizorkin space $HF_s^{pq}$, $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, is the space of holomorphic functions $f$ on $B$ such that $\|f\|_{HF_s^{pq}} < \infty$, where

$$\|f\|_{HF_s^{pq}} = \left(\int_S \left(\int_0^1 \left|\left((I+R)^{[s]^+} f\right)(r\zeta)\right|^q (1-r^2)^{([s]^+-s)q-1}\, dr\right)^{p/q} d\sigma(\zeta)\right)^{1/p}$$

for $0 < p, q < \infty$,

$$\|f\|_{HF_s^{p\infty}} = \left(\int_S \left(\sup_{0<r<1} \left|\left((I+R)^{[s]^+} f\right)(r\zeta)\right| (1-r^2)^{[s]^+-s}\right)^p d\sigma(\zeta)\right)^{1/p},$$

and $[s]^+$ is the integer part of $s+1$.

Observe that $HF_s^{pq}$ coincides with $(I+R)^{-s} HF_0^{pq}$. Moreover, for $q = 2$ and $s = 0$, the norm in $HF_0^{p2}$ is the $L^p(S)$-norm of the Littlewood-Paley $g$-function, and therefore $HF_s^{p2}$ is the Hardy-Sobolev space $H_s^p$. For $p = q$, the space $HF_s^{pp}$

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coincides with the holomorphic Besov space $HB_s^{pp}$. These spaces are included in $HF_s^{p\infty}$.

In many situations, the properties of the Hardy-Sobolev and Besov spaces have analogies in all the scale of the Triebel-Lizorkin spaces, which permits us to give an unified treatment of the results. Therefore, it seems natural to consider the spaces $HF_s^{pq}$ by themselves. In this context we will study three classical problems.

The first one comes from the duality results $(H^1)^* =$ BMOA (with the non-isotropic metric $d(\zeta, \eta) = |1 - \bar{\zeta}\eta|^{1/2}$) ([C-R-W]) and $(HB_0^{11})^* =$ Bloch [An-C-Po]. We would like to give an analogous result for all the scale $HF_s^{1q}$, $1 \leq q < \infty$.

To solve this problem it is natural to introduce the following definition of $HF_s^{\infty q}$.

For $\zeta \in S$ and $t > 0$, let $I_{\zeta,t} = \{\eta \in S;\, |1-\bar{\zeta}\eta| < t\}$ and $\hat{I}_{\zeta,t} = \{z \in B;\, |1-\bar{\zeta}z| < t\}$.

We define $HF_s^{\infty q}$ as the space of holomorphic functions on $B$ such that, the norm given by

$$\|f\|_{HF_s^{\infty q}} = \sup_{I_{\zeta,t}} \left( \frac{1}{|I_{\zeta,t}|} \int_{\hat{I}_{\zeta,t}} |(I+R)^{[s]^+} f(z)|^q (1-|z|^2)^{([s]^+-s)q-1}\, dV(z) \right)^{1/q}, \quad \text{for } q < \infty,$$

$$\|f\|_{HF_s^{\infty\infty}} = \sup_{z\in B} |(I+R)^{[s]^+} f(z)|(1-|z|^2)^{[s]^+-s}, \quad \text{for } q = \infty, \text{ is finite.}$$

Observe that if we denote by $W^1$ the space of non-isotropic Carleson measures on $B$ then $\|f\|_{HF_s^{\infty q}}$ is just $\|\, |(I+R)^{[s]^+} f|^q (1-|z|^2)^{([s]^+-s)q-1} dV(z)\, \|_{W^1}^{1/q}$. In particular, we have $HF_0^{\infty 2} =$ BMOA.

Our first result is the following.

THEOREM A. *Let $1 \leq q < \infty$ and let $q'$ be its conjugate exponent. Then the dual of $HF_s^{1q}$ is isomorphic to $HF_s^{\infty q'}$.*

*The duality is given by the pairing*

$$(f, g) = \int_B (I+R)^k f(z)(I+\bar{R})^k \bar{g}(z)\,(1-|z|^2)^{2(k-s)-1}\, dV(z)$$

*for any $k > s$.*

The second topic that we consider in this work is the well-known Hardy's inequality

$$\sum_{n\geq 0} \frac{|a_n|}{n+1} \leq c \left\| \sum_{n\geq 0} a_n z^n \right\|_{H^1},$$

for holomorphic functions on the unit disc.

To be precise we prove the following theorem, which is a consequence of the one-dimensional case of Theorem A.

THEOREM B. *Let $D$ be the unit disc of $\mathbb{C}$, and let $f(z) = \sum_{k\geq 0} a_k z^k$ be a holomorphic function in $HF_0^{1\infty}$. Then*

$$\sum_{k\geq 0} \frac{|a_k|}{k+1} \leq c\,\|f\|_{HF_0^{1\infty}}.$$

The above result improves the classical Hardy's inequality, because $H^1 = HF_0^{12} \subset HF_0^{1\infty}$. As a consequence, we can obtain the following version for $n \geq 1$.

COROLLARY B. *Let $f$ be a holomorphic function in $HF_n^{1\infty}(B)$ and let $f(z) = \sum_{k\geq 0} f_k(z)$ be its homogeneous expansion at 0. Then*

$$\sum_{k\geq 0} \frac{\|f_k\|_{L^\infty(S)}}{(k+1)^n} \lesssim \sum_{k\geq 0} \frac{\|f_k\|_{L^1(S)}}{k+1} \lesssim \|f\|_{HF_0^{1\infty}}.$$

We point out that, for the Hardy space $H^1$ this result was obtained in [C-W] and [A-Br].

The third problem that we consider is related to a classical theorem due to Privalov. It is known that a holomorphic function $f$ on the unit disc has continuous extension to the closed unit disc with absolutely continuous boundary values if and only if $f$ is in $H_1^1$. It is also well known that for any $n$ and $t = s - n/p > 0$, the Hardy-Sobolev space $H_s^p$ is a subspace of the Lipschitz space $\Lambda_t$. In the extreme case $s - n/p = 0$, $H_s^p$ is a subspace of the space of continuous functions on $\bar{B}$ if and only if $0 < p \leq 1$. Moreover, the restriction of $f \in H^p_{n/p}$, $p \leq 1$ on smooth curves $\gamma$ of $S$ is absolutely continuous [A-Br], [B2], [Bu].

A more precise result in all the scale of holomorphic Triebel-Lizorkin spaces is given by the following theorem.

THEOREM C. *For $0 < p, q \leq \infty$ and $t = s - n/p > 0$, the space $HF_s^{pq}(B)$ is a subspace of the holomorphic Lipschitz space $HF_t^{\infty\infty}$. In the extreme case $s = n/p$, the space $HF_s^{pq}$ is a subspace of $C(\bar{B})$ if and only if $0 < p \leq 1$.*

*Moreover, for $n > 1$ and $0 < p \leq 1$, the trace of $f \in HF^{pq}_{n/p}$ on a smooth simple curve $\gamma$ on $S$ is in the Besov space $B_1^{11}(\gamma)$. In particular, it is absolutely continuous on $\gamma$.*

We obtain this result by methods different from those of the ones used in [A-Br], [B2] and [Bu] for $H_s^p$. Our proof will be a direct consequence of some embedding results that are of interest by themselves.

The paper is organized as follows. In Section 2, we start recalling some properties of the tent spaces introduced by R. R. Coifman, Y. Meyer and E. M. Stein [C-M-S] and the relations between these spaces and the spaces $HF_s^{pq}$. These relations permit us to prove Theorem A. In Section 3 we use the results of Section 2 to prove Theorem B. In Section 4, we prove some embeddings between holomorphic Triebel-Lizorkin and holomorphic Besov spaces which give Theorem C. Moreover, we complete the section showing that, for $0 < p, q < \infty$, the space of holomorphic functions in a neighbourhood of $\bar{B}$ is dense in $HF_s^{pq}$ and that $HF_s^{p\infty}$ is not separable.

We use the notation $\|f\|_p$ to denote the norm of $f$ in $L^p(S)$ with the Lebesgue measure. Moreover, we use $f \lesssim g$ if $f \leq cg$ for some constant independent of $f$ and $g$, and $f \approx g$ if $f \lesssim g \lesssim f$.

## 2. The spaces $HF_s^{\infty q}$

In this section we state some duality properties of the holomorphic Triebel-Lizorkin spaces $HF_s^{pq}$, with special attention to the case $p = 1$.

We will start recalling some results about tent spaces introduced by R. R. Coifman, Y. Meyer and E. M. Stein [C-M-S]. We point out that these results were obtained in the half-space in $\mathbb{R}^{n+1}$, but the same arguments show that they remain true for the unit ball $B$ of $\mathbb{C}^n$ with the usual non-isotropic metric.

For $\zeta \in S$, let $\Gamma_\alpha(\zeta) = \{z \in B; |1-\bar{\zeta}z| < \alpha(1-|z|^2)\}$, $I_{\zeta,t} = \{\eta \in S; |1-\bar{\zeta}\eta| < t\}$ and $\hat{I}_{\zeta,t} = \{z \in B; |1-\bar{\zeta}z| < t\}$.

For a measurable function $f$ on $B$, let

$$\begin{aligned} A_q(f)(\zeta) &= \left(\int_{\Gamma_\alpha(\zeta)} |f(z)|^q \frac{dV(z)}{(1-|z|^2)^{n+1}}\right)^{\frac{1}{q}}, \qquad q < \infty, \\ A_\infty(f)(\zeta) &= \sup\{|f(z)|; z \in \Gamma_\alpha(\zeta)\}, \\ C_q(f)(\zeta) &= \sup_t \left(\frac{1}{|I_{\zeta,t}|}\int_{\hat{I}_{\zeta,t}} |f(z)|^q \frac{dV(z)}{1-|z|^2}\right)^{\frac{1}{q}}. \end{aligned}$$

For $0 < p, q \leq \infty$, we consider the spaces

$$\begin{aligned} F^{pq}(B) &= \{f \in L^0(B); \|f\|_{F^{pq}} = \|A_q(f)\|_p < \infty\}, \qquad 0 < p, q \leq \infty, \\ T^{pq}(B) &= F^{pq}, \qquad 0 < p, q < \infty \quad \text{or} \quad p = q = \infty, \\ T^{\infty q}(B) &= \{f \in L^0(B); \|f\|_{T^{\infty q}} = \|C_q(f)\|_\infty < \infty\}, \qquad q < \infty, \end{aligned}$$

where $L^0(B)$ denotes the space of Lebesgue measurable functions on $B$.

Note that, for $0 < p < \infty$, the tent space $T^{p\infty}$ is not included in the above definitions. Following [C-M-S], this space could be defined as the closure in $F^{p\infty}$ of the subspace of continuous functions on $\bar{B}$.

Moreover, observe that if we denote by $W^1$ the space of (non-isotropic) Carleson measures on $B$, then $\|f\|_{T^{\infty q}} = \||f|^q/(1-|z|^2)\, dV(z)\|_{W^1}^{1/q}$.

Next we state two theorems that we will use later.

THEOREM 2.1. [C-M-S]. *Let $1 \leq q < \infty$, and let $q'$ be its conjugate exponent. Then*

$$\left|\int_B f(z)\bar{g}(z)\frac{dV(z)}{1-|z|^2}\right| \leq c\int_S A_{q'}(f)(\zeta)C_q(g)(\zeta)\, d\sigma(\zeta).$$

THEOREM 2.2. *Let $1 \leq p, q < \infty$ and let $p'$ and $q'$ be their conjugate exponents. Then, with the pairing*

$$((f, g)) = \int_B f(z)\bar{g}(z)\frac{dV(z)}{1-|z|^2},$$

*we have*

$$\begin{array}{lll} (1) & (F^{pq})^* = F^{p'q'}, & 1 < p < \infty, 1 \leq q < \infty \\ (2) & (F^{1q})^* = T^{\infty q'}, & 1 \leq q < \infty \\ (3) & T^{\infty 1} \subset (F^{1\infty})^*. & \end{array}$$

*Proof.* The proof of these results for the half-space in $\mathbb{R}^{n+1}$ can be found in [C-M-S], [H-T-V] and [L]. These proofs can be adapted to our case. For instance, the proof of (1) for the unit ball can be found in [O-F2]. □

The next theorem gives a characterization of the holomorphic Triebel-Lizorkin spaces in terms of the above spaces. Analogous results for the real case and $p < \infty$ can be found for instance in [T2].

THEOREM 2.3. *For $0 < p, q \leq \infty$, $s \in \mathbb{R}$ and any integer $k > s$, we have*

$$\begin{array}{ll} (1)\ HF_s^{pq}(B) = \{f \in H(B);\ \|L_s^k f\|_{F^{pq}} < \infty\}, & \text{if } 0 < p < \infty, \\ (2)\ HF_s^{\infty q}(B) = \{f \in H(B);\ \|L_s^k f\|_{T^{\infty q}} < \infty\}, & \text{if } 0 < q < \infty, \end{array}$$

*where $L_s^k f(z) = (1-|z|^2)^{k-s}(I+R)^k f(z)$.*

*Proof.* A direct proof of (1) for the non-isotropic case can be found in [O-F2]. Part (2) can be obtained from the representation formula

$$(2.1)\ (I+R)^m f(z) = c_M (I+R)^{m-k}\int_B (I+R)^k f(u)\frac{(1-|u|^2)^M}{(1-\bar{u}z)^{n+1+M}}\,dV(u),$$

with $M$ large enough, and the equivalence

$$\|\mu\|_{W^1} \approx \sup_{w\in B}\int_B \frac{(1-|w|^2)^N}{|1-\bar{z}w|^{n+N}}\,d\mu(z), \qquad \square$$

for a fixed positive $N$.

*Remark.* In [O-F2] it is shown that in the above characterization (1) of $HF_s^{pq}$, we can replace the differential operator $L_s^k$ by a sum of differential operators of type $(1-|z|^2)^k\mathbb{T} = (1-|z|^2)^k T_1 \cdots T_m$, where $m \leq 2k$, and the operators $T_j$ are complex tangential vector fields. The analogous result holds for the case $p = \infty$. For instance, for $f \in HF_0^{\infty q}$ we have

$$\|f\|_{HF_0^{\infty q}} \approx \left\|\left(|f|^q + \sum_{1\leq i<j\leq n}\left|\bar{z}_i\frac{\partial f}{\partial z_j} - \bar{z}_j\frac{\partial f}{\partial z_i}\right|^q\right)(1-|z|^2)^{q/2-1}dV(z)\right\|_{W^1}^{1/q}.$$

*Remark.* S. Krantz [K] proved that the space BMOA with the Euclidean metric, which we denote by $\mathrm{BMOA}_I$, does not coincide with the space BMOA with the non-isotropic metric defined above. In [U], D.C. Ullrich shows that the function $f(z) = f(z_1, z_2) = \sum_{n\geq 1} z_1^{10^n}$ is in $\mathrm{BMOA}(\mathbb{C}^2)$ but it is not in $\mathrm{BMOA}_I(\mathbb{C}^2)$.

The same function provides an example of a function which is in $HF_0^{\infty q}$, $0 < q < \infty$, i.e., $|(I+R)f(z)|^q(1-|z|^2)^{q-1}dV(z)$ is a Carleson measure, and that the above measure is not a Carleson measure in the Euclidean sense. We only give the scheme used to obtain this result.

Let $d\mu(z) = |\sum_{j\geq 1}(1+10^j)z_1^{10^j}|^q(1-|z|^2)^{q-1}dV(z)$. We have

$$\int_{\hat{I}_{\zeta,t}} \left|\sum_{j\geq 1}(1+10^j)z_1^{10^j}\right|^q (1-|z|^2)^{q-1}\,dV(z) \lesssim \int_{\hat{I}_{\zeta,t}} \frac{(1-|z|^2)^{q-1}}{(1-|z_1|^2)^q}\,dV(z) \lesssim t^2.$$

Then $\mu \in W^1$.

To show that $\mu$ is not a Carleson measure in the Euclidean sense, it is sufficient to show that if

$$\Omega_t = \{z = (z_1, z_2) \in B;\ |z_2| < t,\ z_1 = re^{i\theta},\ 1-t < r < 1-2t^2,\ -t < \theta < t\},$$

then

$$\int_{\Omega_t} d\mu(z) \gtrsim t^3 \log\frac{1}{t}$$

for all $0 < t < t_0$ small.

Note that for $z \in \Omega_t$, we have $1-|z|^2 \approx 1-|z_1|^2$, and that for small $\varepsilon > 0$ and $1-(1+\varepsilon)10^{-k} \leq |z_1| \leq 1-(1-\varepsilon)10^{-k}$,

$$\left|\sum_{j\geq 1}(1+10^j)z_1^{10^j}\right| \gtrsim 10^k.$$

Therefore, if $r_k = 1-(1-\varepsilon)10^{-k}$ and $r_k' = 1-(1+\varepsilon)10^{-k}$, then

$$\begin{aligned}\int_{\Omega_t} d\mu(z) &\gtrsim t^2 \sum_{\log 1/t<k<2\log 1/t} \int_{r_k'}^{r_k}\int_{-t}^{t} \left|\sum_{j\geq 1}(1+10^j)z_1^{10^j}\right|^q (1-|z|^2)^{q-1}\,d\theta dr \\ &\gtrsim t^3 \sum_{\log 1/t<k<2\log 1/t} 10^{kq}\int_{r_k'}^{r_k}(1-r^2)^{q-1}\,dr \\ &\gtrsim t^3\log 1/t. \qquad \square\end{aligned}$$

Before stating the next result we recall two technical lemmas.

LEMMA 2.4. *For* $0 < m < n+1$, $k > 0$, *and* $A > 0$,

$$(1) \int_B \frac{1}{(A + |1 - \bar{w}z|)^{n+1+k}}\, dV(w) \lesssim \frac{1}{(A + 1 - |z|^2)^k}.$$

$$(2) \int_B \frac{1}{|1 - \bar{w}u|^m |1 - \bar{w}z|^{n+1+k}}\, dV(w) \lesssim \frac{1}{(1 - |z|^2)^k |1 - \bar{u}z|^m}.$$

The proof of this lemma is standard.

LEMMA 2.5. *Let* $f$ *be a holomorphic function on* $\bar{B}$.
(1) *For* $0 < q \leq 1$ *and* $M$ *such that* $(M + n + 1)q - n > 0$,

$$\left(\int_B |f(z)| \frac{(1 - |z|^2)^M}{|1 - \bar{w}z|^{n+1+N}}\, dV(z)\right)^q \lesssim \int_B |f(z)|^q \frac{(1 - |z|^2)^{(M+n+1)q-n-1}}{|1 - \bar{w}z|^{(n+1+N)q}}\, dV(z).$$

(2) *For* $q > 1$, $\varepsilon > 0$ *and* $M > -1/q$,

$$\left(\int_B |f(z)| \frac{(1 - |z|^2)^M}{|1 - \bar{w}z|^{n+1+N}}\, dV(z)\right)^q \lesssim \int_B |f(z)|^q \frac{(1 - |z|^2)^{Mq}(1 - |w|^2)^{-\varepsilon q}}{|1 - \bar{w}z|^{n+1+(N-\varepsilon)q}}\, dV(z).$$

*Proof.* Part (1) is shown in [B1]. Part (2) follows from Hölder's inequality and the estimate (2) of Lemma 2.4 for $m = 0$. □

The next result is a duality theorem that we will use later and which includes the well-known duality $(H^1)^* = (HF_0^{12})^* = HF_0^{\infty 2} =$ BMOA and $(HF_0^{11})^* = HF_0^{\infty\infty} =$ Bloch.

THEOREM 2.6. *For* $1 \leq p, q < \infty$, *the dual of* $(HF_s^{pq})^*$ *is* $HF_s^{p'q'}$, *and* $HF_s^{\infty 1} \subset (HF_s^{1\infty})^*$. *The duality pairing is*

$$(f, g) = ((L_s^k f, L_s^k g)) = \int_B (I + R)^k f(z)(I + \bar{R})^k \bar{g}(z)\,(1 - |z|^2)^{2(k-s)-1}\, dV(z)$$

*for any* $k > s$.

*Remark.* Note that for $s = 0$ the above pairing is similar to the Cauchy pairing

$$\langle f, g\rangle = \lim_{r \to 1} \int_S f(r\zeta)\bar{g}(r\zeta)\, d\sigma(\zeta).$$

This pairing identifies the dual of $HF_s^{pq}$ as $HF_{-s}^{p'q'}$ (for instance, see [B-Bu] for Hardy spaces). We consider the pairing $(f, g)$ because, in our case, it is technically simpler.

*Proof.* The case $1 < p < \infty$ was shown in [O-F2]. Let us consider the case $p = 1$. By Theorems 2.2 and 2.3, it is clear that $HF_s^{\infty q'} \subset (HF_s^{1q})^*$. Conversely, let $q > 1$ and let $\Phi$ be a continuous functional on $HF_s^{1q}$. By the Hahn-Banach theorem there is a function $\varphi \in T^{\infty q'}$ such that

$$\Phi(f) = ((L_s^k f, \varphi)) = \int_B L_s^k f(z)\bar{\varphi}(z)\frac{dV(z)}{1-|z|^2}, \qquad k > s.$$

By representation formula (2.1) and Fubini's Theorem, we have

$$\begin{aligned}\Phi(f) &= c\int_B (I+R)^k f(w)(1-|w|^2)^{2(k-s)-1}\int_B \frac{\bar{\varphi}(z)(1-|z|^2)^{k-s-1}}{(1-\bar{w}z)^{n+2(k-s)}}\,dV(z)dV(w)\\ &= c\int_B L_s^k f(w)\bar{L}_s^k\left((I+\bar{R})^{-k}\int_B \frac{\bar{\varphi}(z)(1-|z|^2)^{k-s-1}}{(1-\bar{w}z)^{n+2(k-s)}}\,dV(z)\right)\frac{dV(w)}{1-|w|^2}.\end{aligned}$$

Therefore, it remains to show that the operator

$$T(\varphi)(w) = (I+R)^{-k}\int_B \varphi(z)\frac{(1-|z|^2)^{k-s-1}}{(1-\bar{z}w)^{n+2(k-s)}}\,dV(z) \tag{2.2}$$

maps $T^{\infty q'}$ to $HF_s^{\infty q'}$, or equivalently that

$$d\mu(w) = \left|\int_B \varphi(z)\frac{(1-|z|^2)^{k-s-1}}{(1-\bar{z}w)^{n+2(k-s)}}\,dV(z)\right|^{q'}(1-|w|^2)^{(k-s)q'-1}\,dV(w) \in W^1,$$

i.e.,

$$\sup_{u\in B}\int_B \frac{(1-|u|^2)}{|1-\bar{w}u|^{n+1}}\,d\mu(w) < \infty.$$

For $\varepsilon$ and $k$ such that $1/q' < \varepsilon < 1 < k-s$, it follows from Hölder's inequality, Fubini's theorem and Lemma 2.4 that

$$\begin{aligned}&\int_B \frac{(1-|u|^2)}{|1-\bar{w}u|^{n+1}}\,d\mu(w)\\ &\quad\lesssim \int_B \frac{(1-|u|^2)}{|1-\bar{w}u|^{n+1}}\int_B |\varphi(z)|^{q'}\frac{(1-|z|^2)^{k-s-1}(1-|w|^2)^{(k-s-\varepsilon)q'-1}}{|1-\bar{w}z|^{n+k-s+(k-s-\varepsilon)q'}}\,dV(z)\,dV(w)\\ &\quad\lesssim (1-|u|^2)\int_B |\varphi(z)|^{q'}\int_B \frac{(1-|z|^2)^{k-s-1}(1-|w|^2)^{(k-s-\varepsilon)q'-1}}{|1-\bar{w}u|^{n+1}|1-\bar{w}z|^{n+k-s+(k-s-\varepsilon)q'}}\,dV(w)\,dV(z)\\ &\quad\lesssim (1-|u|^2)\int_B |\varphi(z)|^{q'}\frac{dV(z)}{|1-\bar{z}u|^{n+1}(1-|z|^2)}\\ &\quad\lesssim \left\| |\varphi|^{q'}/(1-|z|^2)\,dV(z)\right\|_{W^1}\\ &\quad= \|\varphi\|_{T^{\infty q'}}^{q'},\end{aligned}$$

which proves the result for $p = 1, q > 1$.

The case $p = q = 1$ follows in the same way. In this case we have $F^{11} = L^1$, $F^{\infty\infty} = L^\infty$ and

$$HF_s^{\infty\infty} = \{f \in H(B); \sup_{z\in B} |L_s^k f(z)| < \infty\}.$$

Therefore, it is clear that $HF_s^{\infty\infty} \subset (HF_s^{11})^*$. To obtain the converse it is sufficient to show that the operator $T$ defined in (2.2) maps $L^\infty$ to $HF_s^{\infty\infty}$, which can be trivially verified. □

## 3. Hardy's inequality

The purpose of this section is to extent the well-known Hardy's inequality for holomorphic functions in the Hardy space on the unit disc $H^1(D)$ to the context of Triebel-Lizorkin spaces. Moreover, we give a version of this result in the unit ball of $\mathbb{C}^n$.

THEOREM 3.1. *Let $D$ be the unit disc of $\mathbb{C}$ and $f(z) = \sum_{n\geq 0} a_n z^n$ in $HF_0^{1\infty}(D)$. Then*

$$\sum_{n\geq 0} \frac{|a_n|}{n+1} \lesssim \|f\|_{HF_0^{1\infty}}.$$

To prove this result we need the following proposition.

PROPOSITION 3.2. *Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a holomorphic function on the unit disc, such that $\sup_{n\geq 0} |a_n| = M < \infty$. Then $\| \, |f(z)| dV(z) \, \|_{W^1} \lesssim M$.*

*In particular,*

$$\left\|(I+R)^{-1} f\right\|_{HF_0^{\infty 1}} = \left\| \sum_{n\geq 0} \frac{a_n}{n+1} z^n \right\|_{HF_0^{\infty 1}} \lesssim M.$$

*Proof.* Let $S_{\varepsilon,\alpha} = \{z = re^{i\theta}; 1-\varepsilon < r < 1, \alpha - \varepsilon < \theta < \alpha + \varepsilon\}$. We want to prove that

$$\sup_{\varepsilon>0, -\pi\leq\alpha\leq\pi} \frac{1}{\varepsilon} \int_{S_{\varepsilon,\alpha}} |f(z)| \, dV(z) \lesssim M.$$

Given $\varepsilon > 0$, we write $f$ as $f = h + u$, where $h(z) = \sum_{n\leq 1/\varepsilon} a_n z^n$, and we consider the measures $d\mu = |h| dV(z)$, $\quad d\nu = |u| dV(z)$.

We prove that the measures of $S_{\varepsilon,\alpha}$ with respect to $\mu$ and $\nu$ are bounded by $cM\varepsilon$.

For $\mu$, we have

$$\int_{S_{\varepsilon,\alpha}} d\mu \leq M \sum_{n\leq 1/\varepsilon} \int_{1-\varepsilon}^{1} \int_{\alpha-\varepsilon}^{\alpha+\varepsilon} r^{n+1} \, d\theta \, dr \leq 2M\varepsilon \sum_{n\leq 1/\varepsilon} \frac{1-(1-\varepsilon)^{n+2}}{n+2} \lesssim M\varepsilon.$$

To show that $\nu$ satisfies the condition, observe that for $0 < \delta < 1$, the set

$$\left\{\frac{z^n}{c_n}\right\}_{n\geq 0}, \qquad \text{where } c_n = \left(\frac{1}{2}\int_0^1 t^n(1-t)^\delta dt\right)^{1/2} \approx \frac{1}{(n+1)^{(1+\delta)/2}},$$

is an orthonormal set in the Hilbert space $L^2\left(D, (1-|z|^2)^\delta dV(z)\right)$ with the inner product

$$(f, g) = \frac{1}{2\pi}\int_D f(z)\bar{g}(z)(1-|z|^2)^\delta \, dV(z).$$

Therefore, if we let $dV_\delta(z) = (1-|z|^2)^\delta dV(z)$, we have

$$\begin{aligned}\int_{S_{\varepsilon,\alpha}} d\nu &= \sum_{n>1/\varepsilon} a_n c_n \int_{S_{\varepsilon,\alpha}} \frac{z^n}{c_n}\frac{\bar{u}(z)}{|u(z)|(1-|z|^2)^\delta} dV_\delta(z) \\ &\lesssim M\left(\sum_{n>1/\varepsilon}\frac{1}{n^{1+\delta}}\right)^{1/2}\left(\sum_{n>1/\varepsilon}\left|\int_{S_{\varepsilon,\alpha}}\frac{z^n}{c_n}\frac{\bar{u}(z)}{|u(z)|(1-|z|^2)^\delta}\, dV_\delta(z)\right|^2\right)^{1/2}.\end{aligned}$$

Now, by Bessel's inequality, we have

$$\int_{S_{\varepsilon,\alpha}} d\nu \lesssim M\varepsilon^{\delta/2}\left(\int_{S_{\varepsilon,\alpha}}(1-|z|^2)^{-\delta}\, dV(z)\right)^{1/2} \lesssim M\varepsilon. \qquad \square$$

*Proof of Theorem* 3.1. Observe that

$$\begin{aligned}\sum_{n\geq 0}\frac{|a_n|}{n+1} &= \pi\int_D\left(\sum_{n\geq 0} a_n(n+1)^{1/2}z^n\right)\left(\sum_{n\geq 0, a_n\neq 0}\frac{\bar{a}_n}{|a_n|}\frac{1}{(n+1)^{1/2}}\bar{z}^n\right) dV(z) \\ &= \pi((L^{1/2}f, L^{1/2}g))\end{aligned}$$

where $g(z) = \sum_{n\geq 0}\frac{a_n}{|a_n|}\frac{1}{n+1}z^n$.

Therefore, by duality Theorem 2.6 and Proposition 3.2,

$$\sum_{n\geq 0}\frac{|a_n|}{n+1} \lesssim \|f\|_{HF_0^{1\infty}}\|g\|_{HF_0^{\infty 1}} \lesssim \|f\|_{HF_0^{1\infty}},$$

which ends the proof. $\square$

Versions of Hardy's inequality in the unit ball were given in [C-W] and [A-Br] applying the unidimensional result to slices. The same arguments give:

COROLLARY 3.3. *Let $f$ be a holomorphic function in $HF_n^{1\infty}(B)$ and let $f(z) = \sum_{k\geq 0} f_k(z)$ be its homogeneous expansion at* 0. *Then*

$$\sum_{k\geq 0}\frac{\|f_k\|_\infty}{(k+1)^n} \lesssim \sum_{k\geq 0}\frac{\|f_k\|_1}{k+1} \lesssim \|f\|_{HF_0^{1\infty}}.$$

*Proof.* The first inequality follows from Lemma 2.2 of [A-Br]. The second inequality follows from the integration by slices formula [Ru]

$$\|f\|_{HF_0^{1\infty}} = \frac{1}{2\pi}\int_S\int_{-\pi}^{\pi} \sup_{0\le r<1} |(I+R)f(re^{i\theta}\zeta)|\,d\theta\,d\sigma(\zeta),$$

and Theorem 3.1 applied to the function $f_\zeta(\lambda) = f(\lambda\zeta)$, $|\lambda| < 1$. □

*Remark.* Observe that the above theorem shows that if $f \in HF_n^{1\infty}(B)$, then

$$(I+R)^n f(z) = \sum_{k\ge 0}(1+k)^n f_k(z) \in HF_0^{1\infty}(B),$$

and therefore

$$\sum_{k\ge 0} \|f_k\|_\infty \lesssim \|f\|_{HF_n^{1\infty}}.$$

Hence, $HF_n^{1\infty}$ is a subspace of the ball algebra $HC(\bar{B})$.

In the next section, we improve this result using other techniques.

## 4. Embedding theorems

The purpose of this section consists to give some embeddings between holomorphic Triebel-Lizorkin and Besov spaces, which in particular will permit us to obtain analogous results to the Privalov's theorem for $n \ge 1$.

We recall that the holomorphic Besov space $HB_s^{pq}(B)$, $0 < p, q \le \infty$, $s \in \mathbb{R}$, is the subspace of holomorphic functions on $B$ such that the norm

$$\|f\|_{HB_s^{pq}} = \left(\int_0^1 \left(\int_S |(I+R)^{[s]^+} f(r\zeta)|^p\,d\sigma(\zeta)\right)^{q/p} (1-r^2)^{([s]^+-s)q-1}\,dr\right)^{1/q}$$

is finite, with the usual conventions for $p = \infty$ or $q = \infty$.

It is known that if we replace $[s]^+$ by $k > s$, we obtain equivalent norms.

The next results give some embeddings between these spaces. For $r < \infty$, parts (1), (2), (3), (4) and (6) of the following theorem can be found in [B-Bu] for $p = q$ or $q = 2$, or in [O-F1] and [O-F2]. We include them for completeness.

THEOREM 4.1. *For* $0 < p, q \le \infty$ *and* $s \in \mathbb{R}$,

(1) $HF_s^{pq} \subset HF_s^{pm}$, $\quad 0 < q \le m \le \infty$,
(2) $HB_s^{pq} \subset HB_s^{pm}$, $\quad 0 < q \le m \le \infty$,
(3) $HF_s^{pq} \subset HB_s^{pq}$, $\quad 0 < p \le q \le \infty$,
(4) $HB_s^{pq} \subset HF_s^{pq}$, $\quad 0 < q \le p \le \infty$,
(5) $HF_s^{pq} \subset HF_t^{rm}$, $\quad 0 < p < r \le \infty, s - n/p = t - n/r$
(6) $HB_s^{pq} \subset HB_t^{rq}$, $\quad 0 < p < r \le \infty, s - n/p = t - n/r$
(7) $HF_s^{pq} \subset HB_t^{rp}$, $\quad 0 < p < r \le \infty, s - n/p = t - n/r$

*Proof.* First, we consider the case $p < \infty$.

To show (5), we will prove that

(a) $\|f\|_{HF^{\infty m}_{s-n/p}} \lesssim \|f\|_{HF^{p\infty}_s}$, $\qquad 0 < m < p$

(b) $\|f\|_{HF^{rm}_t} \lesssim \|f\|^{p/r}_{HF^{p\infty}_s} \|f\|^{1-p/r}_{HF^{\infty l}_{s-n/p}}$, $\qquad 0 < m < p < r < \infty$, $l = m(1-p/r)$.

Then (5) follows from (1), (a) and (b). To obtain (a), observe that for $k > s$ and $N > 0$,

$$
\begin{aligned}
&\|f\|^m_{HF^{\infty m}_{s-n/p}} \\
&\quad\lesssim \sup_{w\in B} \int_B |L^k_{s-n/p} f(z)|^m \frac{(1-|w|^2)^N}{|1-\bar{z}w|^{n+N}} \frac{dV(z)}{1-|z|^2} \\
&\quad\lesssim \sup_{w\in B} \int_S \int_{\Gamma_\alpha(\zeta)} |L^k_s f(z)|^m \frac{(1-|w|^2)^N (1-|z|^2)^{nm/p-n-1}}{|1-\bar{z}w|^{n+N}} \, dV(z)\, d\sigma(\zeta) \\
&\quad\lesssim \sup_{w\in B} \int_S A_\infty(|L^k_s f|)^m(\zeta) \int_{\Gamma_\alpha(\zeta)} \frac{(1-|w|^2)^N (1-|z|^2)^{nm/p-n-1}}{|1-\bar{z}w|^{n+N}} \, dV(z)\, d\sigma(\zeta).
\end{aligned}
$$

For $z \in \Gamma_\zeta$, we have $|1-\bar{\zeta}z| \lesssim (1-|z|) \leq |1-\bar{z}w|$, so Lemma 2.4 gives

$$
\begin{aligned}
\int_{\Gamma_\alpha(\zeta)} \frac{(1-|z|^2)^{nm/p-n-1}}{|1-\bar{z}w|^{n+N}} \, dV(z) &\lesssim \int_B \frac{(1-|z|^2)^{nm/p-n-1}}{(|1-\bar{\zeta}w| + |1-\bar{z}w|)^{n+N}} \, dV(z) \\
&\lesssim \frac{1}{|1-\bar{\zeta}w|^{n+N-nm/p}}.
\end{aligned}
$$

Using this estimate, Hölder's inequality and Lemma 2.4, we obtain

$$
\begin{aligned}
\|f\|^m_{HF^{\infty m}_{s-n/p}} &\lesssim \sup_{w\in B} \int_S A_\infty(|L^k_s f|)^m(\zeta) \frac{(1-|w|^2)^N}{|1-\bar{\zeta}w|^{n+N-nm/p}} \, d\sigma(\zeta). \\
&\lesssim \|f\|_{HF^{p\infty}_s}.
\end{aligned}
$$

To prove (b), we have

$$
\begin{aligned}
\|f\|^m_{HF^{rm}_t} &\approx \left( \int_S \left( \int_{\Gamma_\alpha(\zeta)} |L^k_t f(z)|^m \tfrac{dV(z)}{(1-|z|^2)^{n+1}} \right)^{r/m} d\sigma(\zeta) \right)^{m/r} \\
&= \sup_{\|\varphi\|_{(r/m)'}=1} \int_S \int_{\Gamma_\alpha(\zeta)} |L^k_t f(z)|^m \tfrac{dV(z)}{(1-|z|^2)^{n+1}} |\varphi(\zeta)| \, d\sigma(\zeta) \\
&= \sup_{\|\varphi\|_{(r/m)'}=1} \int_B |L^k_t f(z)|^m \int_S \chi_{\Gamma_\alpha(\zeta)}(z) |\varphi(\zeta)| \, d\sigma(\zeta) \tfrac{dV(z)}{(1-|z|^2)^{n+1}}.
\end{aligned}
$$

Observe that

$$
|L^k_t f(z)|^m = |L^k_s f(z)|^{mp/r} |L^k_{s-n/p} f(z)|^{m(1-p/r)}
$$

and that for $\eta \in S$,

$$\sup_{z\in\Gamma_\eta} \frac{1}{(1-|z|^2)^n} \int_S \chi_{\Gamma_\alpha(\zeta)}(z)|\varphi(\zeta)|\, d\sigma(\zeta) \lesssim M_{H-L}(|\varphi|)(\eta),$$

where $M_{H-L}(|\varphi|)$ denotes the Hardy-Littlewood maximal function

$$M_{H-L}(f)(\eta) = \sup_{t>0} \frac{1}{|I_{\eta,t}|} \int_{I_{\eta,t}} |f(\zeta)|\, d\sigma(\zeta).$$

Thus, by Theorem 2.1,

$$\|f\|^m_{HF_t^{rm}} \lesssim \sup_{\|\varphi\|_{(r/m)'}=1} \int_S A_\infty(|L_s^k f(z)|)^{mp/r}(\eta) M_{H-L}(|\varphi|)(\eta)\, d\sigma(\eta) \|f\|^l_{HF^{\infty l}_{s-n/p}}.$$

Clearly, (b) follows from Hölder's inequality and the fact that the Hardy-Littlewood maximal operator is continuous from $L^d(S)$ to $L^d(S)$ for $d > 1$.

Now we prove (6) for $r = \infty$. Let $m < \min(p, q, 1)$. By the representation formula (2.1) and Lemma 2.5, we have

$$\begin{aligned} &\|f\|^q_{B^{\infty q}_{s-n/p}} \\ &\approx \int_0^1 \sup_{\zeta\in S} |(I+R)^k f(x\zeta)|^q (1-x)^{(k-s+n/p)q-1}\, dx \\ &\lesssim \int_0^1 \sup_{\zeta\in S} \left( \int_0^1 \int_S |(I+R)^k f(y\eta)|^m \frac{(1-y^2)^{(N+n+1)m-n-1}}{|1-y\bar\eta x\zeta|^{(n+1+N)m}} y^{2n-1}\, d\sigma(\eta)\, dy \right)^{q/m} \\ &\qquad \times (1-x^2)^{(k-s+n/p)q-1}\, dx. \end{aligned}$$

For $N$ large enough, two applications of Hölder's inequality, give

$$\begin{aligned} \|f\|^q_{B^{\infty q}_{s-n/p}} &\lesssim \int_0^1 \left( \int_0^1 \left( \int_S |(I+R)^k f(y\eta)|^p\, d\sigma(\eta) \right)^{m/p} \right. \\ &\qquad \left. \times \frac{(1-y)^{(N+n+1)m-n-1}(1-x)^{(k-s+n/p)m}}{|1-y+1-x|^{(n+1+N)m-n(p-m)/p}} y^{2n-1}\, dy \right)^{q/m} \frac{dx}{1-x} \\ &\lesssim \|f\|^q_{B_s^{pq}}. \end{aligned}$$

Now we prove (7). By (1) and (6) it is clear that it is sufficient to obtain the result for $r < \infty$ and $q = \infty$. Using the duality between $L^1(L^{r/p})$ mixed-norm spaces (see [Be-P]), we have

$$\begin{aligned} \|f\|^p_{B_t^{rp}} &\approx \left( \int_0^1 \left( \int_S |(I+R)^k f(x\zeta)|^r\, d\sigma(\zeta) \right)^{p/r} (1-x^2)^{(k-t)p-1} x^{2n-1} dx \right)^{p/r} \\ &= \sup_\psi \int_0^1 \int_S |(I+R)^k f(x\zeta)|^p\, d\sigma(\zeta)(1-x)^{(k-t)p-1} |\psi(x\zeta)| x^{2n-1}\, dx \end{aligned}$$

where the supremum is taken over all the functions $\psi$ satisfying

$$\sup_{0<x<1}\left(\int_S |\psi(x\zeta)|^{(r/p)'}\,d\sigma(\zeta)\right)^{1/(r/p)'} = 1.$$

Therefore,

$$\begin{aligned}\|f\|^p_{B^{rp}_t} &\lesssim \sup_{\psi}\int_B |L^k_s f(z)|^p(1-|z|^2)^{n-np/r-1}|\psi(z)|\,dV(z)\\ &\lesssim \sup_{\psi}\int_S A_\infty(|L^k_s f|(\zeta))^p C_1((1-|z|^2)^{n-np/r}|\psi|)(\zeta)\,d\sigma(\zeta).\end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned}C_1\left((1-|z|^2)^{n-np/r}|\psi|\right)(\zeta) &\lesssim \sup_{I_{\zeta,t}} t^{-n}\int_{1-t}^1\int_{|1-\bar\zeta\eta|<t}|\psi(x\eta)|d\sigma(\eta)(1-x)^{n-np/r-1}dx\\ &\lesssim \sup_{0<t<1} t^{np/r-n}\int_{1-t}^1 (1-x)^{n-np/r-1}dx \le c<\infty,\end{aligned}$$

which concludes the proof of (7).

To finish the proof of the theorem we consider the case $p=\infty$.

To prove part (1), first we show that $HF^{\infty q}_s \subset HF^{\infty\infty}_s$. For $q>1$ and $0<\varepsilon<1/q$, the representation formula (2.1) and part (2) of Lemma 2.5 give

$$\begin{aligned}\|f\|_{HF^{\infty\infty}_s} &\approx \sup_{z\in B}(1-|z|^2)^{k-s}|(I+R)^k f(z)|\\ &\lesssim \sup_{z\in B}\left(\int_B |(I+R)^k f(w)|^q\frac{(1-|w|^2)^{Nq}(1-|z|^2)^{(k-s-\varepsilon)q}}{|1-\bar w z|^{n+1+(N-\varepsilon)q}}\,dV(w)\right)^{1/q}\\ &\lesssim \||(I+R)^k f(z)|^q(1-|z|^2)^{(k-s)q-1}\,dV(z)\|^{1/q}_{W^1}\\ &\approx \|f\|_{HF^{\infty q}_s}.\end{aligned}$$

The case $0<q\le 1$ follows in the same way.

Observe that the above result shows that if $f$ is in $HF^{\infty q}_s$, then the function $|(I+R)^k f(z)|(1-|z|^2)^{k-s}$ is bounded. The case $q<m<\infty$ follows trivially from this result.

An analogous argument shows that $HB^{\infty q}_s \subset HF^{\infty\infty}_s$. Therefore, if $f\in HB^{\infty q}_s$, then $|(I+R)^k f(z)|(1-|z|^2)^{k-s}$ is bounded. Clearly, (2) follows from this fact.

Part (3) is trivial. Finally (4) follows from

$$\begin{aligned}\|f\|^q_{HF^{\infty q}_s} &\approx \sup_{w\in B}\int_B |(I+R)^k f(z)|^q(1-|z|^2)^{(k-s)q-1}\frac{(1-|w|^2)}{|1-\bar z w|^{n+1}}\,dV(z)\\ &\lesssim \int_0^1 \sup_{\zeta\in S}|(I+R)^k f(x\zeta)|^q(1-|x|^2)^{(k-s)q-1}x^{2n-1}\,dx,\end{aligned}$$

which concludes the proof of the theorem. □

From this theorem, we generalize a result of P. Ahern and J. Bruna [A-Br],and F. Beatrous [B2] and J. Burbea [Bu], about the boundary continuity of the functions $f$ in $H_s^p(B)$, for the extreme cases $s = n/p$, $0 < p \leq 1$. These authors showed that the Hardy-Sobolev space $H_n^1(B)$ is a subspace of $C(\bar{B})$ and that the trace of the $H_n^1(B)$ on a smooth curve $\gamma \subset S$ is a subspace of the space of absolutely continuous functions on $\gamma$, which we will denote by $AC(\gamma)$. The next theorems show that for $n \geq 1$ and $0 < p \leq 1$, $HF_{n/p}^{pq}$ is a subspace of the ball algebra, and that for $n > 1$ the trace $HF_n^{1\infty}$ on smooth curves $\gamma \subset S$ is absolutely continuous.

THEOREM 4.2. *The space $HB_0^{\infty 1}$ is a subspace of $C(\bar{B})$.*

*Proof.* We will prove that if $f \in HB_0^{\infty 1}$ and $f_t(z) = f(tz)$, then $\|f_t - f\|_\infty \to 0$ when $t \to 1$.

By the representation formula,

$$\begin{aligned}
\|f_t - f\|_\infty &\lesssim \sup_{z \in B} \int_B |(I+R)f(w)| \left| \frac{(1-|w|^2)^N}{(1-\bar{w}tz)^{n+N}} - \frac{(1-|w|^2)^N}{(1-\bar{w}z)^{n+N}} \right| dV(w) \\
&\lesssim \sup_{z \in B} \int_B |(I+R)f(w)| \frac{(1-t)(1-|w|^2)^N}{|1-\bar{w}tz|\,|1-\bar{w}z|^{n+N}}\, dV(w) \\
&\lesssim \int_0^1 \sup_{\zeta \in S} |(I+R)f(x\zeta)| \frac{1-t}{1-x+1-t}\, dx,
\end{aligned}$$

which tends to zero by the Lebesgue dominated convergence theorem. □

*Remark.* The above result fails if we replace $q = 1$ by $q > 1$. For instance, for $n = 1$ and $0 < t < 1 - 1/q$, the function $f(z) = \log^t \frac{1}{1-z}$ is in $HB_0^{\infty q}$, and it is not bounded.

THEOREM 4.3. *The space $HF_s^{pq}(B)$ is a subspace of $C(\bar{B})$ if and only if $s - n/p > 0$ or $s - n/p = 0$ and $0 < p \leq 1$.*

*For $n > 1$ and $p$, $s$ satisfying the above conditions, the restriction of $HF_s^{pq}$ on smooth curves $\gamma$ of $S$ is a subspace of $B_1^{11}(\gamma)$ and therefore of the space of absolutely continuous functions on $\gamma$.*

*Proof.* If $t = s - n/p > 0$ then $HF_s^{pq}$ is a subspace of the holomorphic Lipschitz space $HF_t^{\infty\infty}$, and therefore, the result is obvious.

If $s = n/p$, $0 < p \leq 1$ and $0 < t < 1 - 1/p$, then $HF_s^{pq} \subset HF_n^{1\infty} \subset HB_0^{\infty 1} \subset C(\bar{B})$.

For $p > 1$ and $s = n/p$, the function $f_t(z) = \log^t \frac{1}{1-z_1}$ is in $HF_s^{pq}$, and it is not bounded.

The fact that the restriction of $HF_n^{1\infty}$ on $\gamma$ be absolutely continuous follows from $HF_n^{1\infty}(B)|_\gamma \subset B_1^{11}(\gamma) \subset L_1^1(\gamma) = AC(\gamma)$. The first embedding is proved in Section 3 of [Br-O]. □

*Remark.* In some sense the above result is sharp. If $\gamma$ is a complex-tangential curve (i.e., $\gamma(t)\bar{\gamma}'(t) = 0$ for all $t$,) then the trace of $HF_n^{1\infty}$ on $\gamma$ is exactly $B_1^{11}(\gamma)$ (see Section 3 of [Br-O]).

Observe that, as a consequence of the above results, if $f \in HF_n^{1\infty}$, then the functions $f_t(z) = f(tz)$ converge uniformly to $f(z)$. The next theorem extends this result for $f$ in $HF_s^{pq}$ and $p, q < \infty$.

PROPOSITION 4.4. *For* $0 < p, q < \infty$, $s \in \mathbb{R}$, *the space of holomorphic functions on a neighbourhood of* $\bar{B}$ *is dense in* $HF_s^{pq}$.

*Proof.* Since $HF_s^{pq}$ is isomorphic to $HF_t^{pq}$, we can assume that $-sq - 1 > 0$. We prove that the functions $f_t(z) = f(tz)$ satisfy $\|f_t - f\|_{HF_s^{pq}} \to 0$, when $t \to 1$, i.e.,

$$\int_S \left( \int_0^1 |f(tr\zeta) - f(r\zeta)|^q (1-r)^{-sq-1}\, dr \right)^{p/q} d\sigma(\zeta) \to 0, \quad t \to 1.$$

Let

$$\varphi_t(\zeta) = \left( \int_0^1 |f(tr\zeta) - f(r\zeta)|^q (1-r)^{-sq-1}\, dr \right)^{p/q}.$$

We want to prove that for $1/2 < t < 1$,

$$(a)\ \varphi_t(\zeta) \lesssim \left( \int_0^1 |f(r\zeta)|^q (1-r)^{-sq-1}\, dr \right)^{p/q} \in L^1(S),$$
$$(b)\ \varphi_t(\zeta) \to 0 \text{ if } t \to 1.$$

Clearly, (a), (b) and the Lebesgue dominated convergence theorem prove the proposition.

Now, we prove (a). Note that

$$\varphi_t(\zeta) \lesssim \left( \int_0^1 |f(tr\zeta)|^q (1-r)^{-sq-1}\, dr \right)^{p/q} + \left( \int_0^1 |f(r\zeta)|^q (1-r)^{-sq-1}\, dr \right)^{p/q}.$$

By the change of variables $tr = u$ in the first integral, we obtain

$$\begin{aligned} \int_0^1 |f(tr\zeta)|^q (1-r)^{-sq-1}\, dr &= t^{sq} \int_0^t |f(u\zeta)|^q (t-u)^{-sq-1}\, du \\ &\le 2^{-sq} \int_0^1 |f(u\zeta)|^q (1-u)^{-sq-1}\, du, \end{aligned}$$

which proves (a).

To prove (b), we recall that, as a consequence of Egorov's theorem, if $0 < m < \infty$, $\mu$ is a positive measure, $\|h_t\|_{L^m(d\mu)} \to \|h\|_{L^m(d\mu)}$ and $h_t(x) \to h(x)$, $\mu$-a.e., then $\|h_t - h\|_{L^m(d\mu)} \to 0$. Therefore (b) follows from the fact that $f_t(r\zeta) \to f(r\zeta)$ for $0 \le r < 1$, and

$$\int_0^1 |f(tr\zeta)|^q (1-r)^{-sq-1}\,dr \to \int_0^1 |f(r\zeta)|^q (1-r)^{-sq-1}\,dr$$

when $t \to 1$. $\square$

The result of Proposition 4.4 for $HF_s^{p\infty}$ is false. To show this, we prove that the space $HF_s^{p\infty}$ on the unit disc is not separable. For the Bloch space $HF_0^{\infty\infty}$, this result is shown in [Ca-Ci-P].

PROPOSITION 4.5. *Let $D$ be the unit disc. Then $HF_s^{p\infty}(D)$ is not separable.*

*Proof.* It is sufficient to obtain the result for $s = 0$. Consider the set

$$\mathcal{F} = \left\{ \sum_{k \ge k_0} a_k z^{2^k};\ a_k = 0, 1 \right\},$$

for some $k_0$ which we will later.

Observe that $\mathcal{F}$ is not enumerable. Therefore, to show that $HF_0^{p\infty}$ is not separable, it is sufficient to prove that $\mathcal{F} \subset HF_0^{p\infty}$, and that $\|f - g\|_{HF_0^{p\infty}} \ge c > 0$, for $f, g \in \mathcal{F}$ and $f \ne g$.

It is clear that

$$\sup_{z \in D}(1 - |z|^2) \left| (I + R) \sum_{k \ge k_0} a_k z^{2^k} \right| \le \sup_{z \in D}(1 - |z|^2) \sum_{k \ge 0} (1 + 2^k)|z|^{2^k} < \infty.$$

Therefore, $\mathcal{F} \subset HF_0^{\infty\infty} \subset HF_0^{p\infty}$.

To prove that $\|f - g\|_{HF_0^{p\infty}} \ge c > 0$, we have

$$\left\| z^{2^m} + \sum_{k > m} a_k z^{2^k} \right\|_{HF_0^{p\infty}} \ge \sup_{0<r<1} \left( (1 + 2^m) r^{2^m} - \sum_{k > m} (1 + 2^k) r^{2^k} \right)(1 - r).$$

Clearly, it is sufficient to show that for $m \ge k_0$,

$$\sup_{0<r<1} \left( (1 + 2^m) r^{2^m} - \sum_{k > m} (1 + 2^k) r^{2^k} \right)(1 - r) \ge c > 0.$$

Observe that for $0 \le x < 1$,

$$S(x) = \sum_{l \ge 1} 2^l x^{2^l} \le x^2 + 2x^4 + \sum_{l \ge 5} x^l + \frac{1}{2} S(x)$$

and thus $S(x) \le 2x^2 + 4x^4 + 2x^5/(1-x)$.

Therefore, for $x = r^{2^m}$, we have

$$\begin{aligned}
&\left((1+2^m)r^{2^m} - \sum_{k>m}(1+2^k)r^{2^k}\right)(1-r) \\
&\quad \ge (1+2^m)\left(x - \sum_{k>m} 2^{k-m} x^{2^{k-m}}\right)(1-r) \\
&\quad \ge (1+2^m)\left(x - 2x^2 - 4x^4 - \frac{2x^5}{1-x}\right)(1-r).
\end{aligned}$$

To conclude the proof, note that for $r = 1 - 2^{-m}$,

$$\lim_{m \to \infty} (1+2^m)\left(x - 2x^2 - 4x^4 - \frac{2x^5}{1-x}\right)(1-r) = e^{-1} - 2e^{-2} - 4e^{-4} - \frac{2e^{-4}}{e-1} > 0. \quad \square$$

COROLLARY 4.6. *The space of holomorphic functions on a neighbourhood of $\bar{D}$ is not dense in $HF_s^{p\infty}$.*

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