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INTERPOLATION BY BLOCH FUNCTIONS

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ABSTRACT. A new interpolation problem is defined for the Bloch space and some partial results are obtained.

1. Introduction

For f analytic in the unit disk $\mathbb{D} = \{z\colon |z| < 1\}$, let

$$\|f\|_{\mathbb{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|$$

and define the Bloch space $\mathbb{B}$ to be the space of analytic functions satisfying $\|f\|_{\mathbb{B}} < \infty$. Clearly, this is not a norm since the constant functions have the property that $\|f\|_{\mathbb{B}} = 0$. The Bloch space becomes a normed space if we let $\|f\| = |f(0)| + \|f\|_{\mathbb{B}}$. When discussing linear space properties of $\mathbb{B}$, it is this norm we will have in mind.

Let $\Gamma = \{z_n\}$ be an infinite sequence of distinct points in $\mathbb{D}$ having no accumulation points in $\mathbb{D}$. For technical reasons we will consider only Γ with $|z_n| \geq \frac{1}{2}$. We will show later that there is no loss of generality in doing this. To this sequence, we associate the space $\ell^\infty(\Gamma)$, which consists of all sequences $\{a_n\}$ having the property that

$$\sup_n \frac{|a_n|}{\log \frac{1}{1-|z_n|^2}} < \infty.$$

Given Γ, we consider the linear operator T, which maps an analytic function f to the sequence $\{f(z_n)\}_n$. We say that Γ is a set of interpolation for the Bloch space if T maps $\mathbb{B}$ onto $\ell^\infty(\Gamma)$.

Our ultimate goal, which we fall short of in this paper, is to find a complete geometric characterization of sets of interpolation. We do obtain some necessary and sufficient conditions, as well as discover some important properties of these sets.

The paper is organized as follows: Section 2 contains some background material and basic facts about the Bloch space. In Section 3 we examine interpolation in other spaces of analytic functions and motivate our definition for interpolation in $\mathbb{B}$. Sections 4 and 5 contain some necessary and sufficient conditions, respectively, and in Section 6 we examine sets of sampling for the Bloch space.

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2. The Bloch space

For $\zeta, z \in \mathbb{D}$, consider the Möbius transformation

$$\phi_\zeta(z) = \frac{\zeta - z}{1 - \overline{\zeta} z}.$$

An important property of the Bloch space is its Möbius invariance. It follows from the identity

$$\frac{1 - |\phi_\zeta(z)|^2}{1 - |z|^2} = |\phi'_\zeta(z)|$$

that $\|f \circ \phi_\zeta\|_{\mathbb{B}} = \|f\|_{\mathbb{B}}$ for all $f \in \mathbb{B}$ and $\zeta \in \mathbb{D}$.

The pseudo-hyperbolic metric ρ is defined by

$$\rho(z, \zeta) = |\phi_\zeta(z)|.$$

One can easily check that $\rho(\phi_\zeta(z), \phi_\zeta(w)) = \rho(z, w)$ for all $z, w, \zeta \in \mathbb{D}$. The following proposition is basic and can be found in many papers on the Bloch space. The proof below is taken directly from Zhu [19] and is included here for completeness.

PROPOSITION 2.1. *Let $f \in \mathbb{B}$ and $z, w \in \mathbb{D}$. Then*

$$|f(z) - f(w)| \leq \|f\|_{\mathbb{B}} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}.$$

Proof.

$$|f(z) - f(0)| = \left| z \int_0^1 f'(zt)\, dt \right| \leq |z| \|f\|_{\mathbb{B}} \int_0^1 \frac{dt}{1 - |z|^2 t^2} = \frac{1}{2} \|f\|_{\mathbb{B}} \log \frac{1 + |z|}{1 - |z|}.$$

Replacing f by $f \circ \phi_z$, z by $\phi_z(w)$ and by the Möbius invariance of $\|\cdot\|_{\mathbb{B}}$ and the pseudo-hyperbolic metric, we obtain the desired result.

There are some immediate consequences of this result. First, letting $w = 0$, we see the existence of a constant C such that

$$|f(z)| \leq C \|f\| \log \frac{1}{1 - |z|^2} \tag{1}$$

for all $|z| \geq \frac{1}{2}$, which means that point evaluation is a bounded linear functional on the Bloch space. This, together with the maximum principle, implies that if a sequence of functions converges in the Bloch norm, then it does so locally uniformly. In particular, if a family of Bloch functions $\{f_n\}$ satisfies $\|f_n\| \leq K$, then there is a subsequence which converges locally uniformly to a function f in $\mathbb{B}$. Another consequence of (1) is that the Bloch space is contained in every Bergman space.

It follows from an application of Schwarz' lemma and the Möbius invariance that every bounded analytic function is in the Bloch space. However, there are unbounded Bloch functions, for example $f(z) = \log(1 - z)$. For details, see Zhu [19].

For $0 < p < \infty$, the Bergman space A^p is the set of functions f analytic in the unit disk with

$$\|f\|_p^p = \frac{1}{\pi} \int_{\mathbb{D}} |f(z)|^p \, dA(z) < \infty,$$

where dA denotes Lebesgue area measure.

In the following sense, the Bloch space may be seen as the limit of A^p as p approaches infinity.

For f and g analytic in $\mathbb{D}$, consider the pairing defined by

$$\langle f, g \rangle = \lim_{t \to 1} \frac{1}{\pi} \int_{t\mathbb{D}} f(z)\overline{g(z)} \, dA(z).$$

For $1 < p < \infty$, the dual of A^p may be identified with A^q (where p and q are conjugate indices), while the dual of A^1 can be identified with the Bloch space and the dual of the little Bloch space, to be defined later, is identified with A^1. All of these results can be found in Axler [2].

3. Interpolation in other spaces

A sequence $\Gamma = \{z_n\}$ is said to be a set of interpolation for the Bergman space A^p if $T(A^p) \supseteq \ell_p(\Gamma)$, where $\ell_p(\Gamma)$ consists of those sequences $\{a_n\}$ such that

$$\sum_n (1 - |z_n|^2)^2 |a_n|^p < \infty.$$

These sets of interpolation have been described completely by Seip [14]. (See also [11] for a proof of the general case $0 < p < \infty$ and [12] for a different characterization.) In order to discuss his theorem, we first require several definitions.

The sequence Γ is said to be uniformly discrete if the points of Γ do not get to close to each other in the pseudo-hyperbolic metric, that is,

$$\delta(\Gamma) = \inf_{j \neq k} \rho(z_j, z_k) > 0.$$

For Γ uniformly discrete and $r < 1$, let

$$D(\Gamma, r) = \frac{\sum_{|z_n| < r} (1 - |z_n|)}{\log \frac{1}{1-r}}$$

and

$$D^+(\Gamma) = \limsup_{r \to 1} \sup_{\zeta \in \mathbb{D}} D(\phi_\zeta(\Gamma), r).$$

Seip's characterization is then as follows:

THEOREM 3.1. *A sequence Γ is a set of interpolation for A^p if and only if Γ is uniformly discrete and $D^+(\Gamma) < \frac{1}{p}$.*

A natural question concerns the value $p = \infty$. In other words, we would like to find a space of functions $\mathcal{A}$, such that the interpolation sets for $\mathcal{A}$ are described precisely by the condition Γ is uniformly discrete and $D^+(\Gamma) = 0$. One candidate might be H^∞, the space of bounded analytic functions. Here Γ is defined to be a set of interpolation if $T(H^\infty) \supseteq \ell^\infty$, the set of bounded sequences. This problem was solved by Carleson [5], who showed that Γ is a set of interpolation if and only if it is uniformly separated, that is, there is a positive constant δ such that

$$\prod_{j \neq k} \left| \frac{z_j - z_k}{1 - \overline{z_j} z_k} \right| \geq \delta$$

for every k.

One can easily show (see for example [10]) that if Γ is uniformly separated, then Γ is uniformly discrete and $D^+(\Gamma) = 0$. However, the converse does not hold, so it appears that H^∞ is too small for our purposes.

Since the Bloch space lies between H^∞ and every Bergman space, it might seem to be a natural candidate. The fact that, by duality, $\mathbb{B}$ is a limit of the A^p spaces provides more evidence in that direction. A natural conjecture is thus that Γ is a set of interpolation for the Bloch space if and only if Γ is uniformly discrete and $D^+(\Gamma) = 0$.

It remains to determine an appropriate definition of interpolation in this case. This means finding a sequence space W, which we require to be contained in the range of T. Since we are dealing with the case $p = \infty$, it seems reasonable to let $W = \ell^\infty$. In fact, this case has been studied by Xiao [17], who has obtained partial results.

In the Bergman space, we have

$$T(A^p) \supseteq \ell_p(\Gamma) \Rightarrow T(A^p) = \ell_p(\Gamma),$$

a fact which is enormously useful. A similar result holds in H^∞, but for every infinite sequence Γ, one can find a Bloch function f, which has the property that $\limsup_{n\to\infty} |f(z_n)| = \infty$. Therefore, it can never be the case that $T(\mathbb{B}) \subseteq \ell^\infty$.

By (1), we have $T(\mathbb{B}) \subseteq \ell^\infty(\Gamma)$. We then say that Γ is a set of interpolation for $\mathbb{B}$ if $T(\mathbb{B}) = \ell^\infty(\Gamma)$.

It turns out that the condition $D^+(\Gamma) = 0$ is not sufficient for interpolation using this definition, while we still believe the condition to be natural with the definition used in [17].

4. Some necessary conditions

The little Bloch space $\mathbb{B}_0$ consists of functions analytic in $\mathbb{D}$, which have the property that

$$\lim_{r \to 1} \sup_{|z|=r} (1 - |z|^2)|f'(z)| = 0.$$

The little Bloch space is contained in $\mathbb{B}$ and, in fact, it is the closure of the polynomials in the Bloch norm. It does not, however, have any containment relationship with H^∞.

Given Γ, let $\ell_0^\infty(\Gamma)$ be the space of sequences $\{a_n\}$ such that

$$\lim_{n \to \infty} \frac{|a_n|}{\log \frac{1}{1-|z_n|^2}} = 0.$$

Zhu [18] has shown that for a little Bloch function f,

$$|f(z)| \quad \text{is} \quad o\left(\log \frac{1}{1 - |z|^2}\right)$$

as $|z| \to 1$, so we may conclude automatically that $T(\mathbb{B}_0) \subseteq \ell_0^\infty(\Gamma)$. We then say that Γ is a set of interpolation for $\mathbb{B}_0$ if $T(\mathbb{B}_0) = \ell_0^\infty(\Gamma)$.

Our first result, the proof of which is based on an idea of Bruna and Pascuas [3], is that the interpolation sets for the Bloch space are the same as those for the little Bloch space. We will use the following functional analysis results, which may be found in [9].

THEOREM 4.1. *Let L: $X \to Y$ be a bounded linear operator between Banach spaces X and Y. Then L is surjective if and only if its adjoint L^*, acting on the dual Y^* of Y, is bounded below. Similarly, L^* maps Y^* onto X^* if and only if L is bounded below. (We say that L is bounded below if there is a constant C such that $\|x\| \le C\|Lx\|$ for all $x \in X$.)*

This allows us to prove the next result.

THEOREM 4.2. *Γ is a set of interpolation for $\mathbb{B}$ if and only if Γ is a set of interpolation for $\mathbb{B}_0$.*

Proof. Let $\ell^1(\Gamma)$ consist of sequences $\{a_n\}$ such that

$$\sum_n |a_n| \log \frac{1}{1 - |z_n|^2} < \infty.$$

It is not difficult to check that the dual of $\ell_0^\infty(\Gamma)$ can be identified with $\ell^1(\Gamma)$ and that the dual of $\ell^1(\Gamma)$ can be identified with $\ell^\infty(\Gamma)$ with respect to the pairing

$$(\{a_n\}, \{b_n\}) = \sum_n a_n \overline{b_n}.$$

For $\zeta \in \mathbb{D}$, let

$$K_\zeta(z) = (1 - \overline{\zeta} z)^{-2}$$

be the Bergman kernel function. An important property is that

$$\langle f, K_\zeta \rangle = f(\zeta)$$

for all $f \in A^1$. (See Axler [2].)

Throughout this proof, we will denote the operator T acting on the little Bloch space by t. Now consider the map s, which maps an $\ell^1(\Gamma)$ sequence $\{a_n\}$ to the function $\sum_n a_n K_{z_n}(z)$.

A calculation shows that

$$\int_{\mathbb{D}} \left| \sum_n a_n K_{z_n}(z) \right| dA(z) \leq \sum_n |a_n| \int_{\mathbb{D}} |1 - \overline{z_n} z|^{-2} \, dA(z) \leq C \sum_n |a_n| \log \frac{1}{1 - |z_n|^2},$$

which implies that s is a bounded map from $\ell^1(\Gamma)$ to A^1.

We first show that $t^* = s$. Let $f \in \mathbb{B}_0 \cap H^\infty$ and $\{a_n\} \in \ell^1(\Gamma)$. Then

$$\begin{aligned} \langle f, s(\{a_n\}) \rangle &= \lim_{t \to 1} \frac{1}{\pi} \int_{t\mathbb{D}} f(z) \overline{\sum_n a_n K_{z_n}(z)} \, dA(z) \\ &= \sum_n \overline{a_n} \lim_{t \to 1} \frac{1}{\pi} \int_{t\mathbb{D}} f(z) \overline{K_{z_n}(z)} \, dA(z) \\ &= \sum_n \overline{a_n} f(z_n) = (t(f), \{a_n\}). \end{aligned}$$

The second equality is valid because $f \in \mathbb{B}_0 \cap H^\infty$. Since $\mathbb{B}_0 \cap H^\infty$ is dense in $\mathbb{B}_0$, we see that $t^* = s$. By Theorem 4.1, Γ is a set of interpolation for $\mathbb{B}_0$ if and only if there is a constant C such that

$$\sum_n |a_n| \log \frac{1}{1 - |z_n|^2} \leq C \left\| \sum_n a_n K_{z_n} \right\|_1 \tag{2}$$

for all $\{a_n\} \in \ell^1(\Gamma)$.

To show that $s^* = T$, we prove that for any $f \in \mathbb{B}$, the n-th entry of the sequence $s^*(f)$ is equal to $f(z_n)$. Denoting by $i^{(n)}$ the sequence having a 1 in the n-th position and zeroes everywhere else, we obtain

$$(s^*(f), i^{(n)}) = \langle f, s(i^{(n)}) \rangle = \lim_{t \to 1} \frac{1}{\pi} \int_{t\mathbb{D}} f(z) \overline{K_{z_n}(z)} \, dA(z) = f(z_n).$$

This shows that $s^* = T$ and so Γ is a set of interpolation for $\mathbb{B}$ if and only if s^* is onto, which by Theorem 4.1 means precisely that (2) holds.

This characterization of interpolation sets merits being called a proposition.

PROPOSITION 4.3. *Γ is a set of interpolation for $\mathbb{B}$ if and only if there is a constant C such that*

$$\sum_n |a_n| \log \frac{1}{1-|z_n|^2} \le C \left\| \sum_n a_n K_{z_n} \right\|_1$$

for all $\{a_n\} \in \ell^1(\Gamma)$.

At several times in this paper we would like to, without loss of generality, take away finitely many points from a set of interpolation. At this point we prove that this is justified. By Theorem 4.2 it suffices to do this for the little Bloch space.

LEMMA 4.4. *Let Γ be a set of interpolation for $\mathbb{B}_0$ and suppose $\zeta \in \mathbb{D} \setminus \Gamma$. Then $\Gamma \cup \{\zeta\}$ is also a set of interpolation for $\mathbb{B}_0$.*

Proof. Note first that if $f \in \mathbb{B}_0$, then so is the function defined by $(z-a)f(z)$ for any $a \in \mathbb{C}$.

Let $\{a_n\} = \{1, 0, \dots\}$. Then there is a nontrivial function in $\mathbb{B}_0$ which vanishes on $\Gamma \setminus \{z_1\}$ and, by the above remark, we can find a $\mathbb{B}_0$ function f which vanishes on Γ.

Suppose f has a zero of order n at ζ. It is not difficult to show that $g(z) = \frac{f(z)}{(z-\zeta)^n}$ is in $\mathbb{B}_0$, vanishes on Γ and satisfies $g(\zeta) \neq 0$.

Let now $\{a_n\} \cup \{b\} \in \ell_0^\infty(\Gamma \cup \{\zeta\})$. Then $\{a_n\} \in \ell_0^\infty(\Gamma)$, so there is an $h \in \mathbb{B}_0$ such that $h(z_n) = a_n$ for all n. Let

$$l(z) = h(z) + \frac{(b - h(\zeta))}{g(\zeta)} g(z).$$

It is clear that this is a function which performs the interpolation.

Sets of interpolation for the Bergman space have to be uniformly discrete. In the case of the Bloch space, we can say a little bit more.

THEOREM 4.5. *Let Γ be a set of interpolation for $\mathbb{B}$. Then there is a positive constant L such that for every k and every $j \neq k$,*

$$\log \frac{1+\rho(z_j, z_k)}{1-\rho(z_j, z_k)} \ge L \log \frac{1}{1-|z_k|^2}. \tag{3}$$

Proof. By a standard argument involving the closed graph theorem, if Γ is a set of interpolation, there is a constant $M(\Gamma)$ such that given $\{a_n\} \in \ell^\infty(\Gamma)$, there is an $f \in \mathbb{B}$ satisfying $f(z_n) = a_n$ with $\|f\| \le M(\Gamma) \sup_n \frac{|a_n|}{\log \frac{1}{1-|z_n|^2}}$.

For each k, let $\{a_j^{(k)}\} = \{\delta_{jk}\}$ and let f_k be the solution to this interpolation problem. By Proposition 2.1,

$$1 = |f_k(z_k) - f_k(z_j)| \leq M(\Gamma) \log \frac{1+\rho(z_j, z_k)}{1-\rho(z_j, z_k)} \Big/ \log \frac{1}{1-|z_k|^2},$$

which implies that

$$\log \frac{1+\rho(z_j, z_k)}{1-\rho(z_j, z_k)} \geq \frac{1}{M(\Gamma)} \log \frac{1}{1-|z_k|^2}.$$

We obtain several results which follow directly from Theorem 4.5.

COROLLARY 4.6. *If Γ is a set of interpolation for $\mathbb{B}$ and $0 < K < 1$, there is a finite subset F of Γ such that*

$$\delta(\Gamma \setminus F) \geq K.$$

COROLLARY 4.7. *If Γ is a set of interpolation for $\mathbb{B}$, then Γ is uniformly discrete and $D^+(\Gamma) = 0$.*

In our proof of this corollary we will use the following theorem of Rochberg [8].

THEOREM 4.8. *Let $p > 0$. There is a constant $K_0(p)$ such that Γ is a set of interpolation for A^p whenever $\delta(\Gamma) \geq K_0$.*

Proof of Corollary 4.7. It is clear that sets of interpolation for $\mathbb{B}$ must be uniformly discrete.

Now suppose that $\epsilon > 0$ and let $K_0(\frac{1}{\epsilon})$ be the constant from Rochberg's theorem, so that, by Corollary 4.6, there is a finite set F, such that $\Gamma \setminus F$ is a set of interpolation for $A^{\frac{1}{\epsilon}}$. Therefore, Γ is a set of interpolation for $A^{\frac{1}{\epsilon}}$ and so, by Theorem 3.1, $D^+(\Gamma) < \epsilon$.

We remark that it is also possible to prove Corollary 4.7 directly, using the definition of $D^+(\Gamma)$.

It seems unlikely that the condition given in Theorem 4.5 will be sufficient for interpolation. It is not difficult to show that if (3) holds with $L > 2$, then the sequence $\Gamma = \{z_k\}$ must be finite, thus giving a trivial analogue of Rochberg's theorem for the Bloch space, since every finite sequence is an interpolation sequence. A more interesting result would be the existence of a constant $K < 2$ which has the property that if (3) is satisfied for $K \leq L \leq 2$, then Γ is not necessarily finite but still interpolating for $\mathbb{B}$. We will discuss a partial result of this type in §5.

One can easily prove that Rochberg's result does not hold in the Bloch space. In other words, there is no constant $K < 1$ such that $\delta(\Gamma) \geq K$ implies that Γ is an interpolation sequence for $\mathbb{B}$. Consider the following example constructed by Seip [13].

Let $a > 1, b > 0$ and let

$$\Lambda(a, b) = \{a^m(bn + i)\}_{m,n\in\mathbb{Z}},$$

where $\mathbb{Z}$ is the set of integers. $\Lambda(a, b)$ is a sequence of points in $\mathbb{H}^+$, the upper half-plane. An analytic isomorphism from $\mathbb{D}$ to $\mathbb{H}^+$ is given by

$$\psi(z) = i\left(\frac{1+z}{1-z}\right)$$

and we define

$$\Gamma(a, b) = \psi^{-1}(\Lambda(a, b)),$$

so $\Gamma(a, b)$ is a sequence of distinct points in $\mathbb{D}$. In fact, $\Gamma(a, b)$ is uniformly discrete and Seip [13] shows that

$$D^+(\Gamma(a, b)) = \frac{2\pi}{b\log a}.$$

He also shows that if $\frac{2\pi}{b\log a} > \frac{1}{p}$, then $\Gamma(a, b)$ is a set of uniqueness for A^p, that is, the only A^p function vanishing on $\Gamma(a, b)$ is the zero function. It is also not difficult to prove that

$$\delta(\Gamma(a, b)) = \min\left\{\frac{a-1}{a+1}, \frac{b}{\sqrt{b^2+4}}\right\}.$$

Suppose now that there is a constant K_0 such that $\delta(\Gamma) \geq K_0$ implies Γ is interpolating for the Bloch space. We choose a and b such that $\delta(\Gamma(a, b)) \geq K_0$. If p satisfies $\frac{2\pi}{b\log a} > \frac{1}{p}$, then $\Gamma(a, b)$ is a set of uniqueness for A^p and since $\mathbb{B} \subseteq A^p$, a set of uniqueness for $\mathbb{B}$. This is a contradiction, since sets of interpolation for $\mathbb{B}$ necessarily admit nontrivial Bloch functions which vanish on it, as can be seen by the proof of Lemma 4.4.

Another important result which pertains to Bergman space interpolation is the following theorem of Amar [1].

THEOREM 4.9. *Every uniformly discrete sequence is a finite union of sets of interpolation for A^p.*

It is easy to see that this cannot hold for the Bloch space. Take any uniformly discrete set Γ with $D^+(\Gamma) > 0$. There are plenty of examples of this. Seip's example discussed above is one. If $\Gamma = \cup_{i=1}^n \Gamma_i$, where Γ_i is interpolating for $\mathbb{B}$, Corollary 4.7 would imply

$$D^+(\Gamma) \leq \sum_{i=1}^n D^+(\Gamma_i) = 0,$$

a contradiction.

The necessary condition given by Corollary 4.6 allows us to find an example of a uniformly separated sequence which is not a set of interpolation for $\mathbb{B}$. Let $z_n = 1 - \frac{1}{2^{n-1}}$. A calculation shows that

$$\rho(z_n, z_{n+1}) = \frac{2^{n-1}}{3(2^{n-1}) - 1},$$

which does not approach 1 as $n \to \infty$, as would be necessary by Corollary 4.6.

Consider now the question of perturbing sets of interpolation. We remark that the proof of the following theorem is based on the proof of Lemma 2 in [4], p. 351.

THEOREM 4.10. *Suppose* $\Gamma = \{z_n\}$ *is a set of interpolation for* $\mathbb{B}$. *There is a* $\delta > 0$ *such that if* $\Lambda = \{\lambda_n\}$ *is such that*

$$\log \frac{1+\rho(z_n, \lambda_n)}{1-\rho(z_n, \lambda_n)} \Big/ \log \frac{1}{1-|z_n|^2} < \delta,$$

then Λ *is a set of interpolation for* $\mathbb{B}$.

Proof. Let $\{w_n\} \in \ell^\infty(\Lambda)$. Now,

$$\begin{aligned} \log \frac{(1-|z_n|)^2}{(1-|z_n|^2)(1-|\lambda_n|^2)} &\le \log \frac{|1-\overline{\lambda_n} z_n|^2}{(1-|z_n|^2)(1-|\lambda_n|^2)} = \log \frac{1}{1-\rho^2(\lambda_n, z_n)} \\ &\le \log \frac{1+\rho(\lambda_n, z_n)}{1-\rho(\lambda_n, z_n)} < \delta \log \frac{1}{1-|z_n|^2}. \end{aligned}$$

Therefore,

$$1 - |\lambda_n|^2 \ge (1-|z_n|^2)^\delta \frac{1-|z_n|}{1+|z_n|},$$

and it can be shown that the last quantity is greater than or equal to $(1-|z_n|^2)^{5+\delta}$ as long as $|z_n| \ge \frac{1}{2}$. Thus,

$$S = \sup_n \frac{|w_n|}{\log \frac{1}{1-|z_n|^2}} = \sup_n \frac{|w_n|}{\log \frac{1}{1-|\lambda_n|^2}} \frac{\log \frac{1}{1-|\lambda_n|^2}}{\log \frac{1}{1-|z_n|^2}} \le (\delta+5) \sup_n \frac{|w_n|}{\log \frac{1}{1-|\lambda_n|^2}} < \infty,$$

so that $\{w_n\} \in \ell^\infty(\Gamma)$. Therefore, there is an $f_1 \in \mathbb{B}$ such that $f_1(z_n) = w_n$ and $\|f_1\| \le SM(\Gamma)$.

Let $w_n^{(1)} = w_n - f_1(\lambda_n)$. By Proposition 2.1,

$$\frac{|w_n^{(1)}|}{\log\frac{1}{1-|z_n|^2}} = \left|\frac{f_1(z_n)}{\log\frac{1}{1-|z_n|^2}} - \frac{f_1(\lambda_n)}{\log\frac{1}{1-|z_n|^2}}\right|$$

$$\leq \|f_1\| \log\frac{1+\rho(z_n,\lambda_n)}{1-\rho(z_n,\lambda_n)}\frac{1}{\log\frac{1}{1-|z_n|^2}} < SM(\Gamma)\delta.$$

Therefore, there is an $f_2 \in \mathbb{B}$ such that $f_2(z_n) = w_n^{(1)}$ with $\|f_2\| \leq SM(\Gamma)\delta M(\Gamma)$.

Similarly, we let $w_n^{(2)} = w_n^{(1)} - f_2(\lambda_n)$ and find an f_3 such that $f_3(z_n) = w_n^{(2)}$ with $\|f_3\| \leq SM(\Gamma)(\delta M(\Gamma))^2$.

In general, we let $w_n^{(k)} = w_n^{(k-1)} - f_k(\lambda_n)$ and obtain an f_{k+1} such that $f_{k+1}(z_n) = w_n^k$ and $\|f_{k+1}\| \leq SM(\Gamma)(\delta M(\Gamma))^k$. Let $f(z) = \sum_{k=1}^{\infty} f_k(z)$. If δ is chosen so that $\delta < \frac{1}{M(\Gamma)}$, then the series will converge to a Bloch function, which, by construction, solves the interpolation problem.

COROLLARY 4.11. *Suppose $\Gamma = \{z_n\}$ is a set of interpolation. If $\Lambda = \{\lambda_n\}$ is such that $\rho(z_n,\lambda_n) \leq K < 1$ for some K, then Λ is also interpolating.*

Proof. Choose $R < 1$ such that

$$\log\frac{1+K}{1-K}\Big/\log\frac{1}{1-R^2} < \delta,$$

where δ is the constant from Theorem 4.10. There is an N such that $n \geq N$ implies that $|z_n| \geq R$. Let $\Lambda' = \{\lambda_n'\} = \{z_1, \ldots, z_{N-1}, \lambda_N, \lambda_{N+1}, \ldots\}$. Then

$$\log\frac{1+\rho(\lambda_n', z_n)}{1-\rho(\lambda_n', z_n)}\Big/\log\frac{1}{1-|z_n|^2} \leq \log\frac{1+K}{1-K}\Big/\log\frac{1}{1-R^2} < \delta.$$

Thus Λ' is a set of interpolation and therefore so is $\{\lambda_n\}_{n\geq N}$. By Lemma 4.4, this implies that Λ is interpolating.

5. A sufficient condition

In this section we will use a result of Sundberg about BMOA functions to obtain a sufficient condition for sets of interpolation.

For $g \in L^1(\mathbb{T})$, let

$$I(g) = \frac{1}{|J|}\int_J g(e^{i\theta})\,d\theta,$$

where J is a subarc of the unit circle and $|J|$ is its Lebesgue measure. We say that g is in BMO if

$$\sup_{J\subseteq\mathbb{T}} \frac{1}{|J|}\int_J |g(e^{i\theta}) - I(g)|\,d\theta < \infty.$$

An analytic function f is said to be in BMOA if it is a member of the Hardy space H^1 and its boundary function is in BMO. It is not difficult to show that BMOA is actually contained in the Bloch space. Sundberg [16] proves the following theorem about values of BMOA functions on uniformly separated sequences.

THEOREM 5.1. *Let $\{z_n\}$ be uniformly separated and let $\{a_n\} \subseteq \mathbb{C}$. There is a BMOA function f such that $f(z_n) = a_n$ if and only there is a $\lambda > 0$ and numbers $\{\beta(z)\}_{z\in\mathbb{D}}$ such that*

$$\sup_{z\in\mathbb{D}} \sum_n \exp(\lambda|a_n - \beta(z)|)(1 - \rho(z, z_n)^2) < \infty.$$

We then obtain:

THEOREM 5.2. *Suppose Γ is uniformly discrete and there is a $\gamma > 0$ such that*

$$\sup_{z\in\mathbb{D}} \sum_n (1 - |z_n|^2)^{-\gamma}(1 - \rho(z, z_n)^2) < \infty. \tag{4}$$

Then Γ is a set of interpolation for $\mathbb{B}$.

Proof. Let $\{a_n\} \in \ell^\infty(\Gamma)$. Then there is a number M such that $|a_n| \leq M \log \frac{1}{1-|z_n|^2}$, which implies in turn that $\exp(\frac{\gamma}{M}|a_n|) \leq (1 - |z_n|^2)^{-\gamma}$. Letting $\lambda = \gamma/M$, we see that

$$\sup_{z\in\mathbb{D}} \sum_n \exp(\lambda|a_n|)(1 - \rho(z, z_n)^2) \leq \sup_{z\in\mathbb{D}} \sum_n (1 - |z_n|^2)^{-\gamma}(1 - \rho(z, z_n)^2) < \infty.$$

By Sundberg's theorem, this implies that there is a BMOA (and hence a Bloch) function solving the interpolation problem.

Note that since Γ is uniformly discrete, (4) implies that Γ is uniformly separated, which justifies the application of Theorem 5.1.

We can manipulate the necessary condition given by Theorem 4.5 slightly to illustrate the gap between it and the sufficient condition of Theorem 5.2. If Γ is an interpolation sequence, then there is a constant L such that for each k and each $j \neq k$, $1 - \rho(z_j, z_k)^2 \leq 4(1 - |z_k|^2)^L$. If $L > 1$, we can choose $\gamma > 0$ such that $L - \gamma > 1$ and obtain

$$(1 - |z_k|^2)^{-\gamma}(1 - \rho(z_j, z_k)^2) \leq 4(1 - |z_k|^2)^{L-\gamma},$$

which implies that

$$\sup_j \sum_{k\neq j} (1 - |z_k|^2)^{-\gamma}(1 - \rho(z_j, z_k)^2) \leq 4 \sum_k (1 - |z_k|^2)^{L-\gamma}.$$

It is not difficult to show (see [11], for example) that if Γ is uniformly discrete, then $\sum_k (1-|z_k|^2)^a < \infty$ for any $a > 1$. We thus see that if (3) holds with $L > 1$, then

$$\sup_j \sum_{k\neq j} (1-|z_k|^2)^{-\gamma}(1-\rho(z_j,z_k)^2) < \infty.$$

One can also rewrite the necessary condition of Corollary 4.7 to say that if Γ is a set of interpolation for $\mathbb{B}$, then

$$\sup_{z\in\mathbb{D}} \sum_{\rho(z,z_n)<r} (1-\rho(z,z_n)^2) \quad \text{is} \quad o\left(\log\frac{1}{1-r}\right),$$

whereas the sufficient condition of Theorem 5.2 says that there is a $\gamma > 0$ such that

$$\sup_{z\in\mathbb{D}} \sum_{\rho(z,z_n)<r} (1-|z_n|^2)^{-\gamma}(1-\rho(z,z_n)^2) \quad \text{is} \quad O(1).$$

6. Sampling in the Bloch space

The sampling problem is in some sense the dual problem to interpolation. A sequence Γ is a set of interpolation if T, as defined in §1, is surjective, while for sampling sets we require that T be bounded below. Thus, Γ is a set of sampling for $\mathbb{B}$ if there is a positive constant K such that

$$\|f\| \leq K \sup_{z\in\Gamma} \frac{|f(z)|}{\log\frac{1}{1-|z|^2}} \tag{5}$$

for all $f \in \mathbb{B}$. Similarly, Γ is a set of sampling for the little Bloch space if (5) holds for every $f \in \mathbb{B}_0$. For technical reasons, we will consider only sequences whose members all have modulus at least 1/2.

Our first result mirrors Theorem 4.2.

THEOREM 6.1. *Γ is a set of sampling for $\mathbb{B}$ if and only if Γ is a set of sampling for $\mathbb{B}_0$.*

Proof. As mentioned above, Γ is sampling for $\mathbb{B}$ whenever T is bounded below, while it is sampling for $\mathbb{B}_0$ when t is bounded below. We showed previously that the map s satisfies $t^* = s$ and $s^* = T$. Therefore, an application of Theorem 4.1 shows that T is bounded below if and only if s is onto if and only if t is bounded below. Thus, Γ is a set of sampling for $\mathbb{B}$ (or $\mathbb{B}_0$) if for every $f \in A^1$, there exists $\{a_n\} \in \ell^1(\Gamma)$ such that $f = \sum_n a_n K_{z_n}$.

The most natural question about sets of sampling concerns their existence. It turns out that there are no sets of sampling for the Bloch space, and therefore, by the previous result, none for the little Bloch space either. We will show that there is no constant C such that

$$\|f\| \leq C \sup_{|z|\geq 1/2} \frac{|f(z)|}{\log \frac{1}{1-|z|^2}} \tag{6}$$

for all $f \in \mathbb{B}$. This clearly means there are no sampling sequences.

Denote by $H^\infty_{\log}$ the set of analytic functions with

$$\|f\|_{\log} = \sup_{|z|\geq 1/2} \frac{|f(z)|}{\log \frac{1}{1-|z|^2}} < \infty.$$

We see by (1) that $H^\infty_{\log}$ contains $\mathbb{B}$ and it is clear that multiplication by any bounded analytic function maps $H^\infty_{\log}$ to $H^\infty_{\log}$. However, Zhu [18] shows that there are bounded analytic functions not in $M(\mathbb{B})$, the set of multipliers of $\mathbb{B}$. This implies the existence of an f in $H^\infty_{\log}$, which is not in $\mathbb{B}$. Let $\{r_n\}$ be any sequence tending to 1 and define $f_n(z) = f(r_n z)$. Note that $\|f_n\|$ is an unbounded sequence. If it were bounded, a subsequence of $\{f_n\}$ would converge uniformly on compact sets to a Bloch function g by the remarks in §2. Since $\{f_n\}$ converges pointwise to f, this would imply that $f \in \mathbb{B}$, a contradiction.

It is not difficult to show that $\|f_n\|_{\log}$ is bounded. Thus, the inequality (6) fails for the family $\{f_n\}$.

7. Questions

The characterization of interpolation sequences given by Proposition 4.3 appears to yield little geometric information. One would like to be able to find a more geometric condition, such as the one obtained in Theorem 5.2.

A seemingly less difficult problem is to find the necessary and sufficient condition on the sequence $\{z_k\}$ if it lies on the positive real axis.

An application of the closed graph theorem shows that if $\{z_k\}$ is a set of interpolation, then there is a sequence $\{f_k\}$ in $\mathbb{B}$ such that

$$f_k(z_j) = \delta_{jk} \log \frac{1}{1-|z_k|^2}$$

with $\|f_k\| \leq C$. Is this sufficient for interpolation? Similar results do hold in H^∞ [6], H^p [15] and A^p [12], while the analogue of this does not hold in the Dirichlet space [7].

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