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CURVES OF MINIMAL DEGREE WITH PRESCRIBED PLANAR SINGULARITIES

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ABSTRACT. In this paper we study the existence of reduced and irreducible complex or real projective curves contained in an ambient normal projective variety with prescribed singularities and with "low degree". We consider germs of planar singularities of curves, up to topological or equisingular equivalence. The main result is an existence theorem for plane curves with ordinary singularities which improves previous results by Greuel, Lossen and Shustin.

Introduction

In this paper we study the existence of reduced and irreducible complex or real projective curves contained in an ambient normal projective variety X, with pre scribed singularities and with "low degree". We consider germs of planar singularities of curves, up to topological or equisingular equivalence. In section 1 we consider the case $X = \mathbf{P}^2(\mathbf{C})$. We use [3] in an essential way and a very weak form of the so-called Horace method [4]. Here the main result is the existence theorem 1.5 for plane curves with ordinary singularities. In Section 2 we use the results of the first section to prove the existence of irreducible plane curves defined over $\mathbf{R}$ and with singularities with prescribed real topological type in the sense of [6] (essentially for ordinary singularities and for unibranch singularities).Fo r the general theory of equisingularity for curve singularities on a smooth surface; see [2] and references therein. For several strong results on the existence of plane curves with prescribed singularities, see [6], [7], [1], [2], [3] and references therein.

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1. Plane curves

In Section 1 we work over an algebraically closed base field $\mathbf{K}$ with char($\mathbf{K}$) = 0. We are interested in the case $\mathbf{K} = \mathbf{C}$ and in Section 2 we will apply the results of Section 1 to the case of real algebraic curves. For every scheme Y let Y_{red} be the associated reduced algebraic variety and $Y_{\mathrm{reg}} := (P \in Y : Y \text{ is smooth at } P\}$. Let X be an integral projective variety, m a positive integer and $P \in X_{\mathrm{reg}}$. Set $n := \dim(X)$; we will use only the case $n = 2$. The $(m-1)$-th infinitesimal neighborhhood of P in X will be denoted by mP; hence mP has $(\mathbf{I}_{X,P})^m$ as ideal sheaf. Often mP is called

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a fat point; m is the multiplicity of mP and $(n+m)!/(n!m!) = h^0(mP, \mathbf{O}_{mP})$ its degree. If $s, m_1, \ldots, m_s$ are integers > 0 and $P_1, \ldots, P_s$ are distinct points of X_{reg}, the zero-dimensional scheme $\cup_{1\leq i\leq s} m_i P_i$ is called a multi-jet of X with multiplicity $\max\{m_i\}$, type $s; m_1, \ldots, m_s)$ and degree $h^0(\cup_{1\leq i\leq s} m_i P_i, \mathbf{O} \cup_{1\leq i\leq s} m_i P_i)$. For a fixed type $(s; m_1, \ldots, m_s)$, the set of all multi-jets of type $(s; m_1, \ldots, m_s)$ on X is an integral variety of dimension ns. Hence we may speak of the general multi-jet of type $(s; m_1, \ldots, m_s)$.

To find irreducible curves with prescribed singularities and no worse behaviour we will use the so-called Castelnuovo-Mumford lemma, a very useful result, which we will state in the following convenient way.

LEMMA 1.1 [5, p. 100]. *Let X be a projective scheme of dimension n, Z a 0-dimensional subscheme, L and M line bundles on X with L very ample. Assume $h^1(X, M \otimes \mathbf{I}_Z) = 0$ and $h^i(X, M \otimes L^{\otimes(-i+1)}) = 0$ for every i with $2 \leq i \leq n$. Then $\oplus_{t>0} H^0(X, M \otimes L^{\otimes t} \otimes \mathbf{I}_Z)$ is generated by $H^0(X, M \otimes L \otimes \mathbf{I}_Z)$ as an algebra over $\oplus_{t\geq 0} H^0(X, L^{\otimes t})$. In particular Z is the scheme-theoretic base locus of the linear system $|H^0(X, M \otimes L \otimes \mathbf{I}_Z)|$. Furthermore, the same is true*

We will use Lemma 1.1 and the general theory of [3] in the following form.

PROPOSITION 1.2. *Let X be a projective normal surface and $M, L \in \mathrm{Pic}(X)$ with L very ample. Fix an integer $s > 0$, s general points $P_1 \ldots, P_S$ of X_{reg}, integers $m_i > 0$, $1 \leq i \leq s$, and equisingularity types τ_i, $1 \leq i \leq s$, such that τ_i is associated to a singularity scheme Z_i with $Z_i \subseteq m_i P_i$. Let $Z := \cup_{1\leq i\leq s} M_i P_i$ and assume $h^1(X, \mathbf{I}_Z \otimes M) = h^2(X, M \otimes L^*) = 0$. Then there exists an irreducible curve $C \in |M \otimes L|$ whose only singularities are the points P_i, $1 \leq i \leq s$, and such that for each i the equisingular type of the germ (C, P_i) is τ_i.*

Proof. By Lemma 1.1 and Bertini's theorem, a general $C \in |H^0(X, M\otimes L\otimes \mathbf{I}_Z)|$ is smooth outside Z. Since $|H^0(X, M\otimes L\otimes \mathbf{I}_Z)|$ has no base components and contains all curves of the form $C' \cup C''$ with $C' \in |H^0(X, M \otimes \mathbf{I}_Z)|$ and $C'' \in |H^0(X, L)|$, we see that a general $C \in |H^0(X, M \otimes L \otimes \mathbf{I}_Z)|$ is irreducible. By Lemma 1.1, the evaluation map $H^0(X, M \otimes L \otimes \mathbf{I}_Z) \otimes \mathbf{O}_X \to \mathbf{I}_Z)|$ is surjective. Hence we conclude the proof by the general theory of the singularity schemes introduced in [3] (see in particular [3], Lemmas 2.4, 2.5 and 5.1).

Remark 1.3. Assume $X = \mathbf{P}^2$. By the proof of [5], Lemma 6.6, instead of the condition $Z_i \subseteq m_i P_i$ we may use the condition $s(\tau_i) \leq m_i - 1$, where $s(\tau_i)$ is the minimal degree, d, such that there exists an irreducible plane curve D of degree d with a unique singular point, P, with equisingular type τ_i and such that the equisingular stratum of (D,P) (among the plane curves of degree d) is smooth at P. We have $s(\tau_i) \leq \sigma(\tau_i)$ where $\sigma(\tau_i)$ is explicitly computed in [5], Lemma 6.2. For instance we have

$$\sigma(\tau_i) = [(2)^{1/2}\text{mult}(\tau_i)] - 1$$

if τ_i is ordinary and

$$\sigma(\tau_i) = [(1 + (2)^{1/2})(\deg(Z_i) + \text{mult}(\tau_i) + 1)^{1/2}] + \text{mult}(\tau_i) + 3$$

if all the branches of Z_i (i.e., of τ_i) are ordinary. Furthermore, for every singularity τ we have $(\sigma(\tau) + 1)(\sigma(\tau) + 2) \leq 196\mu(\tau)$, where $\mu(\tau)$ denotes the Milnor number of τ [5, Prop. 6.8].

To construct 0-dimensional subschemes of $\mathbf{P}^2$ with good cohomology we will use the so-called Horace method [6] in the more naive and simple way.

THEOREM 1.4. *Fix positive integers t, s, $m_1, \ldots, m_s$. Set $m := \max_{1\leq i\leq s}\{m_i\}$ and assume $(t+2)(t+1)/2 \geq t(m-1) + \sum_{1\leq i\leq s} m_i(m_i+1)/2$. Then for a general multi-jet $Z := \cup_{1\leq i\leq s} m_i P_i$ of type $(s; m_1, \ldots, m_s)$ in $\mathbf{P}^2$ we have $h^1(\mathbf{P}^2, \mathbf{I}_Z(t)) = 0$.*

Proof. We use induction on t. Fix a line $D \subset \mathbf{P}^2$. Take a general multi-jet W of type $(s; m_1, \ldots, m_s)$ with $\text{length}(D \cap W) \leq t + 1$ and with $\text{length}(D \cap W)$ as large as possible with this constraint. Note that $t + 1 - \text{length}(D \cap W) \leq m - 1$. Let E be the union of W and $t + 1 - \text{length}(D \cap W)$ general points of D. It is sufficient to show the vanishing of $h^1(\mathbf{P}^2, \mathbf{I}_E(t))$. Let $E'' := \text{Res}_D(E)$ be the residual scheme of E with respect to D. If $P \in (W_{\text{red}}) \cap D$ and W has multiplicity m_i at P, then E'' is a multi-jet which has multiplicity $m_i - 1$ at P. Since $\text{length}(D \cap E) = t + 1$ we have $H^0(\mathbf{P}^2, \mathbf{I}_E(t)) \cong H^0(\mathbf{P}^2, \mathbf{I}_{E''}(t-1))$ and $h^1(\mathbf{P}^2, \mathbf{I}_E(t)) = h^1(\mathbf{P}^2, \mathbf{I}_{E''}(t-1))$. Hence it is sufficient to show that the general multi-jet B with the same type as E'' has $h^1(\mathbf{P}^2, \mathbf{I}_B(t-1)) = 0$. This follows from the inductive assumption on t.

Theorem 1.4 and Proposition 1.2 prove the following result.

THEOREM 1.5. *Fix positive integers d, s, $m_1, \ldots, m_s$. Set $m := \max_{1\leq i\leq s}\{m_i\}$ and assume $d(d+1)/2 \geq (d-1)(m-1) + \sum_{1\leq i\leq s}(m_i+1)(m_i+2)/2$. Then for a general multi-jet $Z := \cup_{1\leq i\leq s} m_i P_i$ of type $(s; m_1, \ldots, m_s)$ in $\mathbf{P}^2$ there is an irreducible integral curve $C \subset \mathbf{P}^2$ with $\deg(C) = d$ and with each P_i, $1 \leq i \leq s$, as ordinary multiple point with multiplicity m_i. Furthermore, for all i we may take branches of C at P_i which have m_i arbitrary distinct tangents fixed arbitrarily in advance. Furthermore, the equisingular stratum of C at the multigerm $\{P_1, \ldots, P_s\}$ is smooth and of the expected dimension $(d^2 + 3d)/2 - \sum m_i(m_i + 1)/2 + 2s$.*

Theorem 1.5 is especially good if all m_i are small with respect to d. In such cases it is about two times better than the corresponding case proved in [2], Section 3.3, while the result in [2] is better for a few singularities with $m_i \approx d/4$ and many nodes. (Of course, in [3] also arbitrary singularities are treated.)

2. Real curves

In this section we discuss the existence of real curves with singularities with prescribed real topological type. All our results fit in the following framework. We have a projective variety X defined over $\mathbf{R}$, a real line bundle $M \in \mathrm{Pic}(X)(\mathbf{R})$ and a 0-dimensional subscheme Z of X_{reg}. We assume that Z is invariant under complex conjugation σ. This is allowable since the singularity schemes defined in [3] at a real point may be chosen invariant by conjugation and at a non real point P of Z the conjugate 0-dimensional subscheme supported by $\sigma(P)$ may be taken as singularity scheme. Hence all our linear systems may be taken defined over $\mathbf{R}$ and in particular invariant under the induced action, again called σ, of the complex conjugation. We fix a smooth, connected projective surface X defined over $\mathbf{R}$ and with $X(\mathbf{R}) \neq \emptyset$ (hence with $X(\mathbf{R})$ Zariski dense in $X(\mathbf{C})$). Note that for every $A \in \mathrm{Pic}(X)(\mathbf{R})$ and every σ-invariant 0-dimensional subscheme Z of $X(\mathbf{C})$ the complex vector space $H^0(X, A \otimes \mathbf{I}_Z)$ is the complexification of a real vector space. This observation and the Zariski density of both $X(\mathbf{R})$ and $X(\mathbf{C})\backslash X(\mathbf{R})$ in $X(\mathbf{C})$ immediately give the following remark.

Remark 2.1. Consider the statement of 1.5 with respect to a real line bundle. Then there is an irreducible curve C defined over $\mathbf{R}$, with the prescribed singularities and such that the number, multiplicity and complex topological type of the singularities supported at points of $X(\mathbf{R})$ is arbitrary.

Let τ be a singularity or singularity type which is σ-invariant and represented by a germ (C,P) with C a real curve in X and $P \in C(\mathbf{R})$; the real type (or real topological type) of τ in the sense of [8] is the topological type of the triad $(U, U \cap C(\mathbf{C}), U \cap C(\mathbf{R}))$ with U a small ball in $X(\mathbf{C})$ containing P.

Remark / Definition 2.2. In the particular case in which the resolution tree of a singularity τ is a chain (i.e., τ as well as all of its strict transform are unibranch) we can say much more, because the tangent cone to τ is given by a real line and the same occurs for the sequence of strict transforms of τ. We would like to call such singularities strongly unibranch. Hence there is a unique real topological type for τ. Thus under any of the assumptions considered in Remark 2.1 we construct C such that the real topological type of each strongly unibranch singularity is the assigned one, just prescribing that the support of τ is a point of $X(\mathbf{R})$.

Since $\mathbf{P}^2(\mathbf{R}) \neq \emptyset$ (and hence it is Zariski dense in $\mathbf{P}^2(\mathbf{C})$) we obtain the following corollary of Theorem 1.5.

THEOREM 2.3. *Fix positive integers d, s, $m_1, \dots, m_s$. Set $m := \max_{1\le i\le s}\{m_i\}$ and assume $d(d+1)/2 \geq (d-1)(m-1) + \sum_{1\le i\le s}(m_i+1)(m_i+2)/2$. Fix a subset A of $\{1, \dots, s\}$. For every $i \in A$ fix an integer r_i with $0 \leq r_i \leq m_i$ and $m_i - r_i$ even.*

Then for a general σ-invariant multi-jet $Z := \cup_{1 \leq i \leq s} m_i P_i$ of type $(s; m_1, \ldots, m_s)$ in $\mathbf{P}^2$ with $P_j \in \mathbf{P}^2(\mathbf{R})$ if and only if $j \in A$ there is an irreducible integral real curve $C \subset \mathbf{P}^2$ with $deg(C) = d$, with each P_i, $1 \leq i \leq s$, as ordinary multiple point with multiplicity m_i and such that for every $j \in A$ C has exactly r_j real branches at P_j.

Remark 2.4. Obviously we may combine Theorem 2.3 and Remark/Definition 2.2 to obtain existence theorems for real curves with prescribed strongly unibranch or ordinary singularities.

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