ILLINOIS JOURNAL OF MATHEMATICS
Volume 43, Number 4, Winter 1999

A BGG TYPE RESOLUTION OF HOLOMORPHIC VERMA MODULES

FLOYD WILLIAMS

ABSTRACT. For a Hermitian symmetric space $X = G/K$ of non-compact type let θ denote the Cartan involution of the semisimple Lie group G with respect to the maximal compact subgroup K of G, and let q denote a θ-stable parabolic subalgebra of the complexified Lie algebra g of G with corresponding Levi subgroup L of G. Given a finite-dimensional irreducible L module F_L we find Bernstein-Gelfand-Gelfand type resolutions of the induced $(g, L \cap K)$ module $U(g) \otimes_{U(q)} F_L$ and its Hermitian dual, the produced module $\mathrm{Hom}_{U(\overline{q})}(U(g), F_L)_{L\cap K-\text{finite}}$, where $U(\cdot)$ is the universal enveloping algebra functor and $\overline{q}$ is the complex conjugate of q. The results coupled with a Grothendick spectral sequence provide for application to certain (g, K) modules obtained by cohomological parabolic induction, and they extend results obtained initially by Stanke.

1. Introduction

Let $X = G/K$ be a Hermitian symmetric space of non-compact type where G is a non-compact connected semisimple Lie group with finite center and $K \subset G$ is a maximal compact subgroup. If g, k denote the complexifications of the Lie algebras g_0, k_0 of G, K and $h_0 \subset k_0$ is a maximal abelian subalgebra of k_0, then h_0 is a Cartan subalgebra of g_0 and we denote by Δ the set of non-zero roots of (g, h) where $h = h_0^{\mathbb{C}}$ is the complexification of h_0. The G-invariant complex structure on X corresponds to a choice $Q \subset \Delta$ of a positive system of roots such that the spaces p^+ and p^- of holomorphic and antiholomorphic tangent vectors, respectively, at $0 = 1K$ are given by

$$p^{\pm} = \sum_{\alpha \in Q \cap \Delta_n} g_{\pm\alpha} \tag{1.1}$$

for $g_\alpha \subset g$ the root space of $\alpha \in \Delta$ and for Δ_n the set of non-compact roots of Δ. For $p = p^+ \oplus p^-$, $g = k \oplus p$ is a Cartan decomposition. Denote by θ the corresponding Cartan involution of g: $\theta = 1$ on k and $\theta = -1$ on p.

Fix a θ-stable parabolic subalgebra $q = q_x = \ell + u$ of g. That is,

$$\ell = h + \sum_{\alpha(x)=0, \alpha\in\Delta} g_\alpha, \; u = \sum_{\alpha(x)>0, \alpha\in\Delta} g_\alpha \tag{1.2}$$

for some $x \in i\, h_0$; $i^2 = -1$. We assume that the structure of q is *compatible* with the G-invariant complex structure on X, i.e., for

$$\Delta(u) \stackrel{\text{def}}{=} \{\alpha \in \Delta \mid \alpha(x) > 0\}, \;\; \Delta(\ell) \stackrel{\text{def}}{=} \{\alpha \in \Delta \mid \alpha(x) = 0\}, \tag{1.3}$$

Received June 4, 1997.
1991 Mathematics Subject Classification. Primary 17B20, 22E46; Secondary 18G40.

we have

$$\Delta(u) = Q - \Delta(\ell) \tag{1.3$'$}$$

We will check below that if $b_Q = h + \sum_{\alpha \in Q} g_\alpha$ is the Borel subalgebra of g defined by Q then

$$(1.3)' \text{ holds} \iff q \supset b_Q. \tag{1.3$''$}$$

We will also denote $\Delta(\ell)$ by $r(\ell)$. Conjugation of g with respect to g_0 will be denoted by an overbar.

Let

$$L = L_x = \{a \in G \mid \mathrm{Ad}(a)x = x\} \tag{1.4}$$

be the Levi subgroup of G with Lie algebra ℓ_0 where $\ell = \ell_0^{\mathbb{C}}$ (since $\ell = \overline{\ell}$); $\ell_0 = g_o \cap \ell$ and L is connected.

In this paper, given a finite-dimensional irreducible L module F_L we find a $(g, L \cap K)$ resolution of the produced module

$$\mathrm{Pro}_{\overline{q}, L\cap K}^{g, L\cap K} F_L = \mathrm{Hom}_{U(\overline{q})}(U(g), F_L)_{L\cap K-\text{finite}} \tag{1.5}$$

where $U(\cdot)$ is the universal enveloping algebra functor and $\overline{u}$ acts trivially on F_L. The resolution is in the spirit of Bernstein, Gelfand, and Gelfand [1], [6], [7] since it has the form

$$0 \longrightarrow \mathrm{Pro}_{\overline{q}, B}^{g, B} F_L \longrightarrow V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_{\dim(\ell \cap p^+)} \longrightarrow 0 \tag{1.6}$$

for $B = L \cap K$, where

$$V_j = \sum_{\substack{w \in W^1(\ell), \\ \text{length } w = j}} \oplus \, \mathrm{Pro}_{\overline{q \cap (k + p^+)}, B}^{g, B} F_B(w) \tag{1.7}$$

is a sum over a certain subset $W^1(\ell)$ of the Weyl group of (ℓ, h) and $w \in W^1(\ell)$ indexes a particular finite-dimensional irreducible B module $F_B(w)$. See Theorem 4.23, which is a dualization of the resolution of the induced module $U(g) \otimes_{U(q)} F_L$ by a direct sum of generalized Verma modules given in Theorem 4.18. Theorems 4.18, 4.23 and applications to (g, K) modules obtained by cohomological parabolic induction (see Section 5) extend to the general Hermitian symmetric space setting results of R. Stanke obtained for the case $X = SU(n, m)/S(U(n) \times U(m))$. Compare with Proposition 4.22 of [8], for example, where for the choice of q there the sum in (1.7) here reduces to a single summand.

For the sake of completeness we verify statement (1.3)$''$. Clearly if condition (1.3)$'$ holds then $q \supset b_Q$. Assume conversely that $q \supset b_Q$. Let $\alpha \in \Delta(u)$. We have $\alpha(x) > 0 \Longrightarrow (-\alpha)(x) < 0 \Longrightarrow g_{-\alpha} \subset \overline{u}$. If $-\alpha \in Q$ then $g_{-\alpha} \subset b_Q \subset q \Longrightarrow g_{-\alpha} \subset \overline{u} \cap q = \{0\}$, which is a contradiction. That is, $-\alpha \in -Q$ or $\alpha \in Q - \Delta(\ell) \Longrightarrow$

$\Delta(u) \subset Q - \Delta(\ell)$. If $\alpha \in Q - \Delta(\ell)$ then $g_\alpha \subset b_Q \subset q$ and $\alpha \notin \Delta(\ell) \Longrightarrow \alpha(x) > 0$ of $\alpha(x) < 0$. But for $\alpha(x) < 0$, $g_\alpha \subset \bar{u} \Longrightarrow g_\alpha \subset \bar{u} \cap q = \{0\}$, which again is a contradiction. That is, $\alpha(x) > 0 \Longrightarrow \alpha \in \Delta(u) \Longrightarrow Q - \Delta(\ell) \subset \Delta(u) \Longrightarrow Q - \Delta(\ell) = \Delta(u)$, which is $(1.3)'$ as desired.

2. Some structural preliminaries

In this section some structure results needed for later applications will be dealt with and further notation will be established. Guided by Section 4 of [8] we construct a certain parabolic subalgebra p_S of the semisimple part $\ell^{ss} = [\ell, \ell]$ of ℓ whose role will be of pivotal importance.

The G-invariant almost complex structure on X and the semisimplicity of G provide for the existence of an element z in the center of k_0 which satisfies $[z, x] = \pm ix$ for $x \in p^\pm$. Since $h_0 \subset k_0$ is maximal abelian, $z \in h_0$ and one has:

$$\begin{aligned}&\alpha(z) \in \{0, \pm i\}\ \forall \alpha \in \Delta \textit{ with } \alpha(z) = 0 \Longleftrightarrow \alpha \in \Delta_k \stackrel{\text{def}}{=} \Delta - \Delta_n \\ &(\textit{the set of compact roots}),\ \alpha(z) = \pm i \Longleftrightarrow g_\alpha \subset p^\pm.\end{aligned} \tag{2.1}$$

Equation (1.1) can be written as

$$p^\pm = \sum_{\alpha(z) = \pm i, \alpha \in \Delta} g_\alpha. \tag{2.2}$$

Consider the element $x(z) \stackrel{\text{def}}{=} -iz \in i\, h_0$ and the corresponding θ-stable parabolic $q_{x(z)} = \ell_z + u_z$ it defines. By (1.2), (2.1) and (2.2), $\ell_z = h + \sum_{\alpha \in \Delta_k} g_\alpha = k$, $u_z = p^+ \Longrightarrow q_x(z) = k + p^+$. Similarly let $q_1 = \ell_1 + u_1 \stackrel{\text{def}}{=} q_{x+x(z)}$ be the θ-stable parabolic defined by $x + x(z) = x - iz \in i\, h_0$. Using the basic assumption $(1.3)'$ and the properties of z in (2.1), and writing $r(\ell) = \Delta(\ell)$, one checks the following.

Proposition 2.3. *For $\alpha \in \Delta$, $\alpha(x + x(z)) = 0 \Longleftrightarrow \alpha \in r(\ell) \cap \Delta_k$, $\alpha(x + x(z)) > 0 \Longleftrightarrow \alpha \in \Delta(u) \cup \Delta(p^+)$ where $\Delta(p^+) \stackrel{\text{def}}{=} Q \cap \Delta_n$. Hence (by (1.2)),*

$$\ell_1 = h + \sum_{\alpha \in r(\ell) \cap \Delta_k} g_\alpha = \ell \cap k,\ u_1 = \sum_{\alpha \in \Delta(u) \cup \Delta(p^+)} g_\alpha = u + p^+.$$

Also $q_1 = q \cap q_{x(z)} \supset b_Q$ and $q_1 \cap k = q \cap k$, $u_1 \cap k = u \cap k$.

Recall that subalgebras $\underline{a}$ of g which contain b_Q (i.e. the $\underline{a}$ are parabolic subalgebras) are indexed by subsets E of the system of simple roots Π_Q of Q as follows. For $\alpha \in \Delta$ write $\alpha = \Sigma_{\gamma \in \Pi_Q} n_\gamma(\alpha)\gamma$ where the $n_\gamma(\alpha)$ are integers with the same sign, with $\pm\, n_\gamma(\alpha) \geq 0$ for $\alpha \in \pm\, Q$. Define

$$\begin{aligned}&\Delta(E) = \{\alpha \in \Delta \mid n_\gamma(\alpha) = 0\ \forall \gamma \in \Pi_Q - E\} \\ &\ell_E = h + \sum_{\alpha \in \Delta(E)} g_\alpha \qquad u_E = \sum_{\alpha \in Q - \Delta(E)} g_\alpha\end{aligned} \tag{2.4}$$

Then $\underline{a} = \underline{a}(E) = \ell_E + u_E$; here ℓ_E is reductive (the Levi factor of $\underline{a}$) and $E = \Pi_Q \cap \Delta(E)$. The parabolics $q, q_{x(z)}, q_1 \stackrel{\text{def}}{=} q_{x+x(z)}$ correspond, for example, to the subsets $E \stackrel{\text{def}}{=} r(\ell) \cap \Pi_Q, \Pi_Q \cap \Delta_k$, $E_1 \stackrel{\text{def}}{=} r(\ell) \cap \Pi_Q \cap \Delta_k$, respectively, of Π_Q. In place of g, apply these remarks to the semisimple Lie algebra $\ell^{ss} \stackrel{\text{def}}{=} [\ell, \ell]$, with Cartan subalgebra $h^{ss} \stackrel{\text{def}}{=} h \cap \ell^{ss}$ and simple root system $\Pi_Q^{ss} \stackrel{\text{def}}{=}$ the set of restrictions $\{\gamma | h^{ss} | \gamma \in E = r(\ell) \cap \Pi_Q\}$. A subset S of Π_Q^{ss} thus determines a parabolic subalgebra $p_S = \ell_S + u_S$ of ℓ^{ss} which contains the Borel subalgebra $h^{ss} + \sum_{\alpha \in Q \cap r(\ell)} g_\alpha$ of ℓ^{ss}. We consider a particular subset of Π_Q^{ss}. Namely, once and for all, we choose

$$S \stackrel{\text{def}}{=} \{\gamma | h^{ss} | \gamma \in E_1 \stackrel{\text{def}}{=} \Pi_Q \cap r(\ell) \cap \Delta_k\} \tag{2.5}$$

where, as seen above, $E_1 \subset \Pi_Q$ defines q_1. Using (2.1) for example and definition (2.4) one computes that

$$\ell_S = h^{ss} + \sum_{\alpha \in r(\ell) \cap \Delta_k} g_\alpha, u_S = \sum_{\alpha \in Q \cap r(\ell) \cap \Delta_n} g_\alpha. \tag{2.6}$$

Then, by the basic assumption (1.3)$'$,

$$u_S = \ell \cap p^+. \tag{2.7}$$

Let $\mathbb{C}$ denote the field of complex numbers. For $\alpha \in \Delta$, let H_α be the unique element of $h \ni \alpha(H) = (H, H_\alpha)\ \forall H \in h$. If $h(E)$ denotes the $\mathbb{C}$ span of the elements $\{H_\alpha \mid \alpha \in E\}$ for $E \subset \Pi_Q$ then $h(E)$ is a Cartan subalgebra of $[\ell_E, \ell_E]$ and

$$[\ell_E, \ell_E] = h(E) + \sum_{\alpha \in \Delta(E)} g_\alpha \tag{2.8}$$

for ℓ_E in (2.4); note that $h(E) = h \cap [\ell_E, \ell_E]$. As $\ell \cap k = \ell_1$ is the Levi factor of $q_1 = q_1(E_1)$ for $E_1 \stackrel{\text{def}}{=} \Pi_Q \cap r(\ell) \cap \Delta_k$, as noted above, from (2.8) one obtains

$$(\ell \cap k)^{ss} \stackrel{\text{def}}{=} [\ell \cap k, \ell \cap k] = h(E_1) + \sum_{\alpha \in \Delta(E_1) = r(\ell) \cap \Delta_k} g_\alpha. \tag{2.9}$$

Similarly since $q = q(E)$ for $E \stackrel{\text{def}}{=} r(\ell) \cap \Pi_Q$, as noted above, $h^{ss} \stackrel{\text{def}}{=} h \cap \ell^{ss} = h(E)$. That is, clearly $h(E_1) \subset h(E) = h^{ss} \Longrightarrow (\ell \cap k)^{ss} + h^{ss} = h(E_1) + h^{ss} + \sum_{\alpha \in r(\ell) \cap \Delta_k} g_\alpha$ (by (2.9)) $= h^{ss} + \sum_{\alpha \in r(\ell) \cap \Delta_k} g_\alpha$. That is, we can also write (2.6) as

$$\ell_S = h^{ss} + (\ell \cap k)^{ss} \tag{2.10}$$

PROPOSITION 2.11. $[\ell_S, \ell_S] = (\ell \cap k)^{ss}$.

Proof. $[\ell_S, \ell_S[\subset (\ell \cap k)^{ss}$ by (2.10). Conversely, as $(\ell \cap k)^{ss}$ is semisimple, $(\ell \cap k)^{ss} = [(\ell \cap k)^{ss}, (\ell \cap k)^{ss}] \subset [\ell_S, \ell_S]$, again by (2.10).

Denote the center of a Lie algebra $\underline{b}$ by $z(\underline{b})$. Thus $z(\ell_E) = \{H \in h \mid \alpha(H) = 0\ \forall\alpha \in E\}$ for ℓ_E in (2.4) $\Longrightarrow z(\ell) = \{H \in h \mid \alpha(H) = 0 \forall\alpha \in r(\ell) \cap \Pi_Q\} \subset \{H \in h \mid \alpha(H) = 0\ \forall\alpha \in E_1\} = z(\ell \cap k)$ (again as $\ell \cap k = \ell_1$). Similarly $z(\ell_S) = \{H \in h^{ss} | \beta(H) = 0\ \forall\beta \in E_1\}$ (by (2.5)) $\Longrightarrow z(\ell_S) \subset z(\ell \cap k)$. Conversely, one has the following.

PROPOSITION 2.12. $z(\ell \cap k) = z(\ell) \oplus z(\ell_S)$.

Proof. $\ell = \ell^{ss} \oplus z(\ell) \Longrightarrow h = h^{ss} \oplus z(\ell)$, with $h^{ss} = h(E = r(\ell) \cap \Pi_Q)$, as noted above. Similarly $\ell_S = \ell_S^{ss} \oplus z(\ell_S) \Longrightarrow h^{ss} = h(E_1) \oplus z(\ell_S) \Longrightarrow$

$$h = h(E_1) \oplus z(\ell_S) \oplus z(\ell). \tag{2.13}$$

For $H \in z(\ell \cap k)$, write $H = H_1 + H_S + H_\ell \in h(E_1) + z(\ell_S) + z(\ell)$ by (2.13). For $\alpha \in E_1 \stackrel{\text{def}}{=} \Pi_Q \cap r(\ell) \cap \Delta_k \subset r(\ell) \cap \Pi_Q$, $\alpha(H) = 0$, $\alpha(H_S) = 0$, and $\alpha(H_\ell) = 0 \Longrightarrow \alpha(H_1) = 0$. That is, $H_1 \in h(E_1) \subset h^{ss} \ni \alpha(H_1) = 0\ \forall\alpha \in E_1 \Longrightarrow H_1 \in z(\ell_S) \Longrightarrow H = (H_1 + H_S) + H_\ell \in z(\ell_S) + z(\ell) \Longrightarrow z(\ell \cap k) \subset z(\ell_S) + z(\ell)$, as desired.

COROLLARY 2.14. $\ell \cap k = \ell_S \oplus z(\ell)$.

Proof. $\ell \cap k = (\ell \cap k)^{ss} \oplus z(\ell \cap k) = [\ell_S, \ell_S] \oplus z(\ell_S) \oplus z(\ell)$ (by Propositions 2.11, 2.12) $= \ell_S \oplus z(\ell)$.

In addition to the subalgebra p_S of ℓ^{ss} we shall need to consider the subalgebra p_ℓ of ℓ given by

$$p_\ell \stackrel{\text{def}}{=} (\ell \cap k) + u_S = (\ell \cap k) \oplus (\ell \cap p^+); \tag{2.15}$$

see (2.7). The following is easily checked.

PROPOSITION 2.16. *$p_\ell = \ell \cap q_{x(z)}$; recall $q_{x(z)} = k + p^+$. Also by Corollary 2.14, $p_\ell = \ell_S \oplus z(\ell) \oplus u_S = p_S \oplus z(\ell)$.*

PROPOSITION 2.17. *$\ell = \ell^{ss} + p_\ell$ with $\ell^{ss} \cap p_\ell = p_S$. Also $u_1 = u \oplus u_S$.*

Proof. $p_\ell = \ell \cap q_{x(z)}$, by Proposition 2.16, and $\ell^{ss} \subset \ell \Longrightarrow \ell^{ss} + p_\ell \subset \ell^{ss}$. Conversely, $\ell = \ell^{ss} + z(\ell)$ with $z(\ell) \subset h \subset \ell \cap k \subset p_\ell \Longrightarrow \ell \subset \ell^{ss} + p_\ell$. Clearly $p_S \subset p_\ell$ (cf. Corollary 2.14) and $p_S \subset \ell^{ss}$ by definition. Conversely, for $v \in \ell^{ss} \cap p_\ell$ write $v = y + z \in p_S + z(\ell)$ by Corollary 2.14, as $v \in p_\ell$. Then $y \in p_S \subset \ell^{ss}$ and $v \in \ell^{ss} \Longrightarrow z = v - y \in \ell^{ss}$; i.e., $z \in \ell^{ss} \cap z(\ell) = \{0\} \Longrightarrow v = y \in p_S \Longrightarrow \ell^{ss} \cap p_\ell = p_S$. By Proposition 2.3, $u_1 = u + p^+$, and by (2.7) $u_S = \ell \cap p^+ \Longrightarrow u + u_S \subset u_1$. Conversely, one checks that $u_1 \subset u + u_S$. Of course $u_S \subset \ell \Longrightarrow u_S \cap u = \{0\}$.

COROLLARY 2.18. $q_1 = p_\ell \oplus u$

Proof. $p_\ell + u \overset{\text{def}}{=} \ell \cap k + u_S + u = \ell \cap k + u_1$ (by Proposition 2.17) $= \ell_1 + u_1 = q_1$.

PROPOSITION 2.19. *$q = q_1 + \ell$ with $q_1 \cap \ell = p_\ell$.*

Proof. By Corollary 2.18, $q_1 = p_\ell + u \Longrightarrow q_1 + \ell = p_\ell + u + \ell = u + \ell$ (as $p_\ell \subset \ell) = q$. By Propositions 2.3, 2.16, $q_1 = q \cap q_{x(z)}$, $p_\ell = \ell \cap q_{x(z)} \Longrightarrow q_1 \cap \ell = q \cap q_{x(z)} \cap \ell = q_{x(z)} \cap \ell = p_\ell$.

Let W denote the Weyl group of (g, h) and let W_k denote the Weyl group of (k, h), the subgroup of W generated by Weyl reflections r_α as α varies over Δ_k. In (2.1), $\alpha \in Q \cap \Delta_n \Longleftrightarrow \alpha(z) = i$, $\alpha \in \Delta_k \Longleftrightarrow \alpha(z) = 0$. It follows that

$$\sigma(Q \cap \Delta_n) = Q \cap \Delta_n \quad \text{for } \sigma \in W_k. \tag{2.20}$$

In particular for $2\delta_n \overset{\text{def}}{=} \sum_{\alpha \in Q \cap \Delta_n} \alpha$,

$$\sigma\delta_n = \delta_n \text{ for } \sigma \in W_k \text{ and hence } (\delta_n, \Delta_k) = 0. \tag{2.21}$$

3. Induced and produced modules

Given the general results of Section 2, and other generalities, we can adapt most of the arguments of Section 4 of [8] and extend the results given there to arbitrary Hermitian symmetric spaces. Many of Stanke's arguments already are quite general once his notation is established. The choice of q in [8] leads to special simplifications (as indicated in the introduction) not available in general. In working with more general q's we can still move forward, since induction and production "commute" with direct summation; see Proposition 3.8. An interesting point is finding a replacement for Stanke's Proposition 4.10, which is less suited for application here. For this we make a simple but useful observation; see Theorem 3.3 and the remarks that follow its statement.

For a complex Lie algebra h containing the complexified Lie algebra of a compact Lie group B with the pair (h, B) subject to conditions (a), (b) of Definition 6.1.1 of [9], we can consider the category of (h, B) modules. In particular, choose $B \overset{\text{def}}{=} L \cap K$ for L in (1.4) and let F be a finite-dimensional C^∞ B module over $\mathbb{C}$. Then F is an $(\ell \cap k, B)$ module with $\ell \cap k$ acting on F via the differential $\dot{\Pi}$ of the representation Π of B on F. By definition (2.15), we can extend F to a (p_ℓ, B) module $\ni u_S \cdot F = 0$ (since $[\ell \cap k, u_S] \subset u_S$). Similarly, as $q_1 = \ell_1 \oplus u_1$, $\ell_1 = \ell \cap k$ and $q = \ell \oplus u$, we can extend F to a (q_1, B) module $\ni u_1 \cdot F = 0$, and we can extend the *induced* (ℓ, B) module

$$\operatorname{Ind}_{p_\ell, B}^{\ell, B} F \overset{\text{def}}{=} U(\ell) \otimes_{U(p_\ell)} F \tag{3.1}$$

(cf. Definition 6.1.5 of [9]) to a (q, B) module on which u acts trivially. With these stipulations we have the following version of Proposition 4.17 of [8].

THEOREM 3.2. *There exists a canonical* (g, B) *isomorphism* ι *of the induced module* $U(g)\otimes_{U(g)}(U(\ell)\otimes_{U(p_\ell)}F)$ *onto the induced module* $U(g)\otimes_{U(q_1)}F$. $\iota(u\otimes_{U(q)}(A\otimes_{U(p_\ell)}f)) = uA\otimes_{U(q_1)}f$ *for* $(u, A, f)\in U(g)\times U(\ell)\times F$.

We outline the proof. By Proposition 2.19, $q = q_1 + \ell$ with $q_1 \cap \ell = p_\ell$. By Proposition 5.1.14 of [2] (which we call a Mackey "subgroup" theorem), it follows that there is an ℓ module isomorphism ϕ of $U(\ell)\otimes_{U(p_\ell)} \mathrm{res}^{p_\ell, B}_{q_1, B} F$ onto $U(q)\otimes_{U(q_1)} F$ (given by $u\otimes_{U(p_\ell)} f \longrightarrow u\otimes_{U(q_1)} f$ for $(u, f)\in U(\ell)\times F$), where $\mathrm{res}^{p_\ell, B}_{q_1, B} F$ is F considered as a (p_ℓ, B) module by restricting the q_1 action to p_ℓ. ϕ is clearly a B map, where B acts via $\mathrm{Ad}\otimes\Pi$, and thus is an (ℓ, B) isomorphism. Note that we initially regarded F as a (p_ℓ, B) module with $p_\ell = (\ell\cap k)\oplus u_S$ and with u_S acting trivially. That (p_ℓ, B) module coincides with $\mathrm{res}^{p_\ell, B}_{q_1, B} F$ since $u_S\subset u_1 = u + u_S\subset q_1 = p_\ell + u$ by Proposition 2.17 and Corollary 2.18. Next, note that u acts trivially on $U(q)\otimes_{U(q_1)} F$, since $[q, u]\subset u\subset u_1\subset q_1$. By definition, u acts trivially on $U(\ell)\otimes_{U(p_\ell)} F$ in (3.1) and thus ϕ is a (q, B) isomorphism. ϕ induces a unique (g, B) isomorphism ϕ^* of $U(g)\otimes_{U(q)}(U(\ell)\otimes_{U(p_\ell)}F)$ onto $U(g)\otimes_{U(q)}(U(q)\otimes_{U(q_1)}F)$ such that $\phi^*(u\otimes_{U(q)} v) = u\otimes_{U(q)}\phi(v)$ for $(v, u)\in(U(\ell)\otimes_{U(p_\ell)}F)\times U(g)$. Finally, as $q_1\subset q\subset g$ (cf. Proposition 2.3) there is an "induction in stages" (g, B) isomorphism ψ of $U(g)\otimes_{U(q)}(U(q)\otimes_{U(q_1)}F)$ onto $U(g)\otimes_{U(q_1)}F$ given by $\psi(u\otimes_{U(q)}(A\otimes_{U(q_1)} f)) = uA\otimes_{U(q_1)} f$ for $(u, A, f)\in U(g)\times U(q)\times F$ (cf. Proposition 5.1.11 of [2]); ψ can also be constructed by repeated applications of Frobenius reciprocity (cf. (6.1.7) of [9]). $\iota \overset{\text{def}}{=} \psi\circ\phi^*$ is the desired (g, B) isomorphism.

THEOREM 3.3. *Let* $Y_1 = U(\ell)\otimes_{U(p_\ell)} F$ *be an induced* (ℓ, B) *module as in* (3.1); *thus* F *is finite dimensional with* $u_S\cdot F = 0$. *Let* Y_2 *be an* arbitrary (ℓ, B) *module. Then any* ℓ*-module map* ϕ: $Y_1 \longrightarrow Y_2$ *is automatically a* B*-module map, and hence is an* (ℓ, B) *module map.*

Remarks. A version of Theorem 3.3 is given in Proposition 4.10 of [8] where Y_2 there is also taken to be an induced module $U(\ell)\otimes_{U(p_\ell)} F_2$, and M there is B in our notation. In Theorems 3.2, 3.3 it is not necessary to assume irreducibility of F. We observe in Theorem 3.3 that Y_2 need not necessarily be an induced module, though for the proof it is certainly necessary that Y_1 be an induced module. For us this is a rather useful observation. Since B is connected and F is finite-dimensional one can adapt the proof given in [8] of Proposition 4.10, almost word for word, to prove Theorem 3.3 (even for Y_2 arbitrary).

By Proposition 2.17, $\ell = p_\ell + \ell^{ss}$ with $p_\ell\cap\ell^{ss} = p_S$. Therefore by Proposition 5.1.14 of [2] (again, as in the proof of Theorem 3.2), given a p_ℓ module F_1, there is a canonical ℓ^{ss} module isomorphism

$$U(\ell^{ss})\otimes_{U(p_S)} \mathrm{res}^{p_S}_{p_\ell} F_1 \simeq U(\ell)\otimes_{U(p_\ell)} F_1 \tag{3.4}$$

where $\mathrm{res}_{p_\ell}^{p_S} F_1 = F_1|_{p_S} = F_1$ as a p_S module. Given an $\ell \cap k$ module F, extend it to a $p_\ell = (\ell \cap k) \oplus u_S$ module such that u_S acts trivially (as before) and choose $F_1 = F$ as a p_ℓ module. Let $F_{ss} = \mathrm{res}_{\ell\cap k}^{\ell_S} F$ (where $\ell_S \subset \ell \cap k = \ell_S \oplus z(\ell)$ by Corollary 2.14) and let $\tilde{F}_{ss} = F_{ss}$ extended to a $p_S = \ell_S \oplus u_m$ module such that u_S acts trivially. Then $\mathrm{res}_{p_\ell}^{p_S} F_1 \overset{(a)}{=} \tilde{F}_{ss}$: Namely, since $F_1 = F$ as a p_ℓ module (by definition) and $\ell \cap k \subset p_\ell$ we have $\mathrm{res}_{p_\ell}^{\ell\cap k} F_1 = \mathrm{res}_{p_\ell}^{\ell\cap k} F \overset{\mathrm{def}}{=} F$ and $u_S \cdot F_1 \overset{(b)}{=} 0$. In particular $\ell_S \subset \ell \cap k \Longrightarrow \mathrm{res}_{p_\ell}^{\ell_S} F_1 = \mathrm{res}_{\ell\cap k}^{\ell_S} \mathrm{res}_{p_\ell}^{\ell\cap k} F_1 = \mathrm{res}_{\ell\cap k}^{\ell_S} F \overset{\mathrm{def}}{=} F_{ss} \Longrightarrow F_1, \tilde{F}_{ss}$ have the same ℓ_S module structure. They also have the same u_S module structure by (b) since $u_S \cdot \tilde{F}_{ss} \overset{\mathrm{def}}{=} 0$. Hence (a) holds, and by (3.4) we deduce:

PROPOSITION 3.5. *Given an $\ell \cap k$ module F (possibly infinite-dimensional) let $F_{ss} = \mathrm{res}_{\ell\cap k}^{\ell_S} F = F$ as an ℓ_S module. Then there is a canonical ℓ^{ss} module isomorphism of $U(\ell^{ss}) \otimes_{U(p_S)} F_{ss}$ onto $U(\ell) \otimes_{U(p_\ell)} F$ given by $u \otimes_{U(p_S)} f \longrightarrow u \otimes_{U(p_\ell)} f$ for $(u, f) \in U(\ell^{ss}) \times F_{ss}$, where F_{ss} and F are extended to $p_S = \ell_S \oplus u_S$ and $p_\ell = (\ell \cap k) \oplus u_S$ modules, respectively, such that u_S acts trivially.*

We refer to Definition 6.1.1 of [9] again for the notion of a general abstract (h, B) module. Here the compact Lie group B (not necessarily the specific choice $L \cap K$ above) acts on h by automorphisms $\{r(b) | b \in B\}$ which extend the adjoint action of B on its complexified Lie algebra $\underline{b} \subset h$. Let $\underline{a} \subset h$ be a complex Lie subalgebra $\ni \underline{b} \subset \underline{a}, r(b)\underline{a} \subset \underline{a} \quad \forall b \in B$. Then the category of $(\underline{a}, B)$ modules is also well defined. Given an $(\underline{a}, B)$ module X we can form in general the *induced* (h, B) module

$$\mathrm{Ind}_{\underline{a},B}^{h,B} X \overset{\mathrm{def}}{=} U(h) \otimes_{U(\underline{a})} X \tag{3.6}$$

(as in (3.1)), and the *produced* (h, B) module

$$\mathrm{Pro}_{\underline{a},B}^{h,B} X \overset{\mathrm{def}}{=} \mathrm{Hom}_{U(\underline{a})}(U(h), X)_{B-\mathrm{finite}}; \tag{3.7}$$

cf. Definition 6.1.21 of [9].

PROPOSITION 3.8. *Given finitely many $(\underline{a}, B)$ modules $X_1, \ldots, X_n$ there exists a canonical (h, B) module isomorphism of $\mathrm{Ind}_{\underline{a},B}^{h,B} \sum_{j=1}^n \oplus X_j$ onto $\sum_{j=1}^n \oplus \mathrm{Ind}_{\underline{a},B}^{h,B} X_j$ and a canonical (h, B) module isomorphism of $\mathrm{Pro}_{\underline{a},B}^{h,B} \sum_{j=1}^n \oplus X_j$ onto $\sum_{j=1}^n \mathrm{Pro}_{\underline{a},B}^{h,B} X_j$.*

Occasions arise when one is given two θ-stable parabolics $q^j = \ell^j + u^j$, $j = 1, 2$, with $q^1 \subset q^2$, $\ell^1 \subset \ell^2$, $u^1 \supset u^2$, $L^1 \cap K \subset L^2 \cap K$; cf. the notation in (1.2), (1.4). In this situation one has the following:

PROPOSITION 3.9. *If W is any $(\ell^1, L^1 \cap K)$ module extended to a $(q^1, L^1 \cap K)$ module on which u^1 acts trivially, then there is a canonical $(g, L^1 \cap K)$ isomorphism of $\mathrm{Pro}_{q^2, L^1\cap K}^{g, L^1\cap K} \mathrm{Pro}_{q^1, L^1\cap K}^{q^2, L^1\cap K} W$ onto $\mathrm{Pro}_{q^1, L^1\cap K}^{g, L^1\cap K} W$. Also u^2 acts trivially on $\mathrm{Pro}_{q^1, L^1\cap K}^{q^2, L^1\cap K} W$.*

Define

$$u^0 = u^1 \cap \ell^2 \ , \quad q^0 = \ell^1 + u^0 \subset \ell^2. \tag{3.10}$$

Then $q^1 = q^0 \oplus u^2$ and

$$q^2 = q^1 + \ell^2 \text{ with } q^1 \cap \ell^2 = q^0. \tag{3.11}$$

Given (3.11), a Mackey type "subgroup theorem" (cf. the proof of Theorem 3.2) provides

$$\mathrm{res}^{\ell^2, L^1\cap K}_{q^2, L^1\cap K} \mathrm{Pro}^{q^2, L^1\cap K}_{q^1, L^1\cap K} W = \mathrm{Pro}^{\ell^2, L^1\cap K}_{q^0, L^1\cap K} \mathrm{res}^{q^0, L^1\cap K}_{q^1, L^1\cap K} W. \tag{3.12}$$

Extend the module on the r.h.s. in (3.12) to a $(q^2, L^1 \cap K)$ module by letting u^2 act trivially. On the other hand u^2 acts trivially on $\mathrm{Pro}^{q^2, L^1\cap K}_{q^1, L^1\cap K} W$ by Proposition 3.9. Thus by (3.12)

$$\mathrm{Pro}^{q^2, L^1\cap K}_{q^1, L^1\cap K} W = \mathrm{Pro}^{\ell^2, L^1\cap K}_{q^0, L^1\cap K} \mathrm{res}^{q^0, L^1\cap K}_{q^1, L^1\cap K} W \tag{3.13}$$

is a $(q^2, L^1 \cap K)$ module equivalence and we derive the next result.

COROLLARY 3.14. *The $(g, L^1 \cap K)$ module isomorphism in Proposition* 3.9 *can be expressed as*

$$\mathrm{Pro}^{g, L^1\cap K}_{q^2, L^1\cap K} \mathrm{Pro}^{\ell^2, L^1\cap K}_{q^0, L^1\cap K} \mathrm{res}^{q^0, L^1\cap K}_{q^1, L^1\cap K} W = \mathrm{Pro}^{g, L^1\cap K}_{q^1, L^1\cap K} W \tag{3.15}$$

where $\mathrm{Pro}^{\ell^2, L^1\cap K}_{q^0, L^1\cap K} \mathrm{res}^{q^0, L^1\cap K}_{q^1, L^1\cap K} W$ is regarded as a $(q^2, L^1 \cap K)$ module on which u^2 acts trivially.

Proposition 3.9 (or Corollary 3.14) is an induction in stages result; compare equation (6.3.8) of [9], where the definition of the functor Pro differs from our (3.7) by a certain "rho shift", i.e., W is replaced by $W \otimes \wedge^{\dim u^1} u^1$. The trivial action of u^2 on $\mathrm{Pro}^{q^2, L^1\cap K}_{q^1, L^1\cap K} W$ in Proposition 3.9 compares with the trivial action of u on $U(q) \otimes_{U(q_1)} F$ in the proof of Theorem 3.2. The assumption that G/K has a Hermitian structure is not needed in Proposition 3.9, nor in Corollary 3.14 of course. Given the existence of such a structure however, using the structure theory of Section 2 we choose $q^1 = \overline{q}_1 = \overline{q \cap q_{x(z)}}$, $q^2 = \overline{q}_{x(z)}$. Then $\ell^1 = \overline{\ell \cap k} = \ell \cap k$, $u^1 = \overline{u + p^+} = \overline{u} + \overline{p^+}$, $\ell^2 = \overline{k} = k$, $u^2 = \overline{p^+} = p^-$, $L^1 \cap K = L \cap K$, $L^2 \cap K = K$ (since $\mathrm{Ad}(k)z = z\ \forall k \in K$, and K is connected). By (3.11), $q^0 = \overline{q}_1 \cap \overline{k} = \overline{q_1 \cap k} = \overline{q \cap k}$, which gives the following concrete version of Corollary 3.14 (cf. Proposition 4.24 of [8]).

COROLLARY 3.16. *Let W be an $(\ell \cap k, L \cap K)$ module extended to a $(\overline{q}_1, L \cap K)$ module on which $\overline{u} + p^-$ acts trivially. Then there is a canonical $(g, L \cap K)$ isomorphism of $\mathrm{Pro}^{g, L\cap K}_{\overline{q}_{x(z)}, L\cap K} \mathrm{Pro}^{k, L\cap K}_{\overline{q\cap k}, L\cap K} \mathrm{res}^{\overline{q\cap k}, L\cap K}_{\overline{q}_1, L\cap K} W$ onto $\mathrm{Pro}^{g, L\cap K}_{\overline{q}_1, L\cap K} W$, where $q_1 = q \cap q_{x(z)}$, $q_{x(z)} = k + p^+$, and $\mathrm{Pro}^{k, L\cap K}_{\overline{q\cap k}, L\cap K} \mathrm{res}^{\overline{q\cap k}, L\cap K}_{\overline{q}_1, L\cap K} W$ is regarded as a $(\overline{q}_{x(z)}, L \cap K)$ module on which $\overline{p^+} = p^-$ acts trivially.*

4. BGG type resolutions

The starting point for obtaining Bernstein-Gelfand-Gelfand type resolutions of induced modules is the application of Lepowsky's generalized version of BGG [6],[7] to the semisimple part ℓ^{ss} of ℓ as was done in [8]. Afterwards, one tensors appropriately the terms of the resolution and applies functorial properties of induction. The maps involved are Lie algebra module maps, a priori. Thanks to the magic of Theorem 3.3, however, they are B module maps as well. The resolution of produced modules follows by duality. Some of the arguments in [8] use the fact that $\dim z(\ell_S) = \dim z(\ell) = 1$ for the choice of q there. Since this is not the case in general we must seek alternative reasoning at various points. Given a finite-dimensional irreducible L module F_L, the goal is to construct a $(g, L\cap K)$ resolution of $U(g)\otimes_{U(q)} F_L$ by direct sums of certain generalized Verma modules for g; see Theorem 4.18. Since q satisfies (1.3)$'$ (or equivalently (1.3)$''$) we call $U(g)\otimes_{U(q)} F_L$ a *holomorphic* Verma module.

For $E_1 \overset{\text{def}}{=} \Pi_Q \cap r(\ell) \cap \Delta_k$, as before define

$$P_{E_1} = \left\{\lambda \in h^* \;\middle|\; \frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}^+ \;\; \forall\alpha \in E_1\right\} \tag{4.1}$$

where (,) denotes the Killing form of g and $\mathbb{Z}^+$ is the set of non-negative integers; h^* is the dual space of h. Let $W(\ell)$ be the subgroup of the Weyl group W of (g, h) generated by the Weyl reflections r_α as α varies over $r(\ell)$ and let

$$W^1(\ell) = \{w \in W(\ell) \mid Q\cap r(\ell) \cap \Delta_k \subset w(Q\cap r(\ell))\},$$

$$2\delta(\ell) = \sum_{\alpha\in Q\cap r(\ell)} \alpha,\ 2\delta = \sum_{\alpha\in Q} \alpha. \tag{4.2}$$

PROPOSITION 4.3. *Let $\lambda \in h^*$ be $Q\cap r(\ell)$-dominant integral: $2\frac{(\lambda,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}^+\ \forall\alpha \in Q\cap r(\ell)$. Then for $w \in W^1(\ell)$, $w(\lambda+\delta(\ell)) - \delta(\ell) = w(\lambda+\delta) - \delta \in P_{E_1}$ in* (4.1).

Note here that for any $w \in W(\ell)$ in fact, $w\delta(u) = \delta(u)$ for $2\delta(u) = \sum_{\alpha\in\Delta(u)} \alpha$; i.e. $(\alpha, \delta(u)) = 0\ \forall\alpha \in r(\ell)$.

By (1.3)$'$ and (4.2), $2\delta = 2\delta(\ell) + 2\delta(u) \Longrightarrow w(\lambda+\delta(\ell)) - \delta(\ell) = w(\lambda+\delta-\delta(u)) + \delta(u) - \delta = w(\lambda+\delta) - \delta$.

Now take $\lambda \in P_{E_1}$ and let $F(\lambda)$ be the corresponding finite-dimensional irreducible $\ell\cap k$ module. $F(\lambda)$ restricted to $(\ell\cap k)^{ss}$ is in fact $(\ell\cap k)^{ss}$-irreducible. By Proposition 2.11 and Corollary 2.14 $[\ell_S, \ell_S] = (\ell\cap k)^{ss}$ and $\ell\cap k \supset \ell_S$. We see that by restricting the $\ell\cap k$ module structure of $F(\lambda)$ to ℓ_S we obtain an irreducible ℓ_S module $F(\lambda)_{ss} \overset{\text{def}}{=} \operatorname{res}^{\ell_S}_{\ell\cap k} F(\lambda)$ since in fact $F(\lambda)_{ss}$ is $[\ell_S, \ell_S] = (\ell\cap k)^{ss}$-irreducible. Extend $F(\lambda)_{ss}$ to a $p_S = \ell_S \oplus u_S$ module on which u_S acts trivially and form the *ℓ^{ss}-generalized Verma module*

$$M_{ss}(\lambda; S) = U(\ell^{ss}) \otimes_{U(p_S)} F(\lambda)_{ss};\ \lambda \in P_{E_1}. \tag{4.4}$$

For $\lambda \in h^*$ which is $Q \cap r(\ell)$- dominant integral and $w \in W^1(\ell)$ in (4.2) let

$$\lambda_w \stackrel{\text{def}}{=} w(\lambda + \delta(\ell)) - \delta(\ell) = w(\lambda + \delta) - \delta. \tag{4.5}$$

Then by Proposition 4.3, $\lambda_w \in P_{E_1}$ and hence the generalized Verma module M_{ss} $(\lambda_w; S)$ is well defined, where λ in (4.4) is replaced more generally by λ_w; $w = 1 \Longrightarrow \lambda_w = \lambda$. For $k = 0, 1, 2, 3 \ldots, t \stackrel{\text{def}}{=} \dim u_S$ define

$$C_k = C_k(S) \stackrel{\text{def}}{=} \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus M_{ss}(\lambda_w; S) \tag{4.6}$$

for $\ell(w) \stackrel{\text{def}}{=} |\{\alpha \in Q \cap r(\ell) \mid w^{-1}\alpha \in -(Q \cap r(\ell))\}|$ equal the length of w; $|A|$ is the cardinality of a set A. Let $L_{ss}(\lambda)$ be the finite-dimensional irreducible ℓ^{ss} module with $Q \cap r(\ell)$-highest weight λ and let ε: $C_0 = M_{ss}(\lambda : S) \longrightarrow L_{ss}(\lambda)$ be the canonical surjection.

THEOREM 4.7. [6] (Generalized BGG resolution of $zL_{ss}(\lambda)$). *Let $\lambda \in h^*$ be $Q \cap r(\ell)$-dominant integral. Then there exist ℓ^{ss}-module maps $\alpha_1, \ldots, \alpha_t, t \stackrel{\text{def}}{=} \dim u_S$, such that* $0 \longrightarrow C_t(S) \xrightarrow{\alpha_t} c_{t-1}(S) \xrightarrow{\alpha_{t-1}} \cdots \longrightarrow C_0(S) \xrightarrow{\varepsilon} L_{ss}(\lambda) \longrightarrow 0$ *is an exact sequence*; *see* (4.5),(4.6).

Note that $t = \dim(\ell \cap p^+)$ by (2.7).

Since $F(\lambda)$, for $\lambda \in P_{E_1}$, is an irreducible $\ell \cap k$ module and $\ell \cap k$ is reductive, the center $z(\ell \cap k)$ of $\ell \cap k$ acts on $F(\lambda)$ by scalar operators: $\exists$ a map x_λ: $z(\ell \cap k) \longrightarrow \mathbb{C} \ni z \cdot f = x_\lambda(z) f$ for $z \in z(\ell \cap k)$, $f \in F(\lambda)$. Choose $f_0 \in F(\lambda)$ to be a non-zero λ-weight vector. Then for $z \in z(\ell \cap k) \subset h$, $\lambda(z) f_0 = z \cdot f_0 = x_\lambda(z) f_0 \Longrightarrow x_\lambda = \lambda|_{z(\ell \cap k)}$; i.e. $z \cdot f = \lambda(z) f \quad \forall z \in z(\ell \cap k), \forall f \in F(\lambda)$. Extend $F(\lambda)$ to a $p_\ell = (\ell \cap k) \oplus u_S$ module on which u_S acts trivially. Then $z \in z(\ell)$ acts on $U(\ell) \otimes_{U(p_\ell)} F(\lambda)$ via the scalar $\lambda(z)$. This follows since $z(\ell) \subset p_\ell$ (cf. Corollary 2.14) and $z(\ell) \subset z(\ell \cap k)$ (cf. Proposition 2.12). Apply these remarks to $\lambda_w \in P_{E_1}$ in place of λ, for $w \in W^1(\ell)$, λ which is $Q \cap r(\ell)$-dominant integral (see Proposition 4.3 and Definition (4.5)): $z \in z(\ell)$ acts via the scalar $\lambda_w(z)$ on $U(\ell) \otimes_{U(P_\ell)} F(\lambda_w)$. We claim however that $\lambda_w(z) = \lambda(z)$ for $z \in z(\ell)$, $w \in W(\ell)$. First, $z(\ell) = \{H \in h \mid \alpha(H) = 0 \ \forall \alpha \in r(\ell)\} \Longrightarrow wH \stackrel{(i)}{=} H$ for $w \in W(\ell)$, $H \in z(\ell)$ since for $\alpha \in r(\ell)$, $r_\alpha(H) \stackrel{\text{def}}{=} H - 2\alpha(H)(\alpha, \alpha)^{-1} H_\alpha = H$ for $H \in z(\ell)$. Also $\delta(\ell)(H) = 0$ for $H \in z(\ell)$ by Definition (4.2). Hence $\lambda_w(H) \stackrel{\text{def}}{=} [w(\lambda + \delta(\ell)) - \delta(\ell)](H) = w(\lambda + \delta(\ell))(H)$ (for $H \in z(\ell)) = (\lambda + \delta(\ell))(w^{-1}H) = (\lambda + \delta(\ell))(H)$ (by (i)) $= \lambda(H)$, as claimed. That is:

PROPOSITION 4.8. *$z \in z(\ell)$ acts on each $U(\ell) \otimes_{U(p_\ell)} F(\lambda_w)$, $w \in W^1(\ell)$, by the* same *scalar $\lambda(z)$ (which is independent of w).*

We continue to assume that $\lambda \in h^*$ is $Q \cap r(\ell)$-dominant integral; $\lambda_w \in P_{E_1}$ $\forall w \in W^1(\ell)$ and in particular $\lambda \in P_{E_1}$ (again by Proposition 4.3). By Proposition 3.5 there is a canonical ℓ^{ss} module isomorphism i_w of $M_{ss}(\lambda_w; S)$ onto $U(\ell) \otimes_{U(p_\ell)} F(\lambda_w)$ (see (4.4)) where u_S acts trivially on $F(\lambda_w)$ extended to a $p_\ell = (\ell \cap k) \oplus u_S$ module. By (4.6), there are induced ℓ^{ss} module isomorphisms

$$i_k \stackrel{\text{def}}{=} \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus i_w \colon C_k \longrightarrow \tilde{C}_k \tag{4.9}$$

for $0 \le k \le t$, where we set

$$\tilde{C}_k \stackrel{\text{def}}{=} \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus U(\ell) \otimes_{U(p_\ell)} F(\lambda_w). \tag{4.10}$$

Consequently for $k \ge 1$, $\tilde{\alpha}_k \stackrel{\text{def}}{=} i_{k-1} \circ \alpha_k \circ i_k^{-1} \colon \tilde{C}_k \longrightarrow \tilde{C}_{k-1}$ is an ℓ^{ss} module map, for α_k in Theorem 4.7. By Proposition 4.8, the action of $z \in z(\ell)$ on a summand in (4.10) is *independent* of the indexing element $w \in W^1(\ell)$ and is given by the scalar $\lambda(z)$. Hence $z \in z(\ell)$ acts on $\tilde{C}_k$ via the scalar $\lambda(z)$. That is, $\tilde{\alpha}_k \colon \tilde{C}_k \longrightarrow \tilde{C}_{k-1}$ is an ℓ^{ss} module map of $\ell = \ell^{ss} \oplus z(\ell)$ modules with $z \in z(\ell)$ acting by the *same* scalar $\lambda(z)$ on $\tilde{C}_k$ and $\tilde{C}_{k-1}$, which shows in fact that $\tilde{\alpha}_k$ is an ℓ module map.

Our main interest is the case when the $\ell \cap k$ module structure on each $F(\lambda_w)$ integrates to a B module structure: $y \cdot f = \dot{\Pi}_w(y) f$ for $y \in \ell \cap k$, $f \in F(\lambda_w)$, for a C^∞ representation Π_w of B on $F(\lambda_w)$. For this we assume in addition that $\lambda \in h^*$ is *analytically integral*: $\lambda(\Gamma) \subset 2\pi i \mathbb{Z}$ for $\Gamma \stackrel{\text{def}}{=} \{H \in h_0 \mid \exp H = 1\}$. Then each λ_w, $w \in W(\ell)$, is analytically integral, the $F(\lambda_w)$ are (p_ℓ, B) modules (cf. remarks prior to (3.1)), the

$$Y_w \stackrel{\text{def}}{=} U(\ell) \otimes_{U(p_\ell)} F(\lambda_w) \tag{4.11}$$

are (ℓ, B) modules, and consequently the $\tilde{C}_k$ are also (ℓ, B) modules. The $\tilde{\alpha}_k$, $k \ge 1$, which we have shown to be ℓ maps are B module maps. To see this let

$$F_k(\lambda) \stackrel{\text{def}}{=} \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus F(\lambda_w) \tag{4.12}$$

for $0 \le k \le t$. By Proposition 3.8 there is a canonical (ℓ, B) isomorphism

$$f_k \colon U(\ell) \otimes_{U(p_\ell)} F_k(\lambda) \xrightarrow{\text{onto}} \tilde{C}_k; \tag{4.13}$$

see (4.10). By Theorem 3.3, the ℓ module map $\tilde{\alpha}_k \circ f_k \colon U(\ell) \otimes_{U(p_\ell)} F_k(\lambda) \longrightarrow \tilde{C}_{k-1}$, $k \ge 1$, is automatically a B module map! Hence $\tilde{\alpha}_k = (\tilde{\alpha}_k \circ f_k) \circ f_k^{-1}$ is a B module map, and is thus an (ℓ, B) module map.

Now let $F_L(\lambda)$ be a C^∞ finite-dimensional irreducible L module with $Q \cap r(\ell)$-highest weight λ, say Π_L is the representation of L on $F_L(\lambda)$. More specifically, via $\dot{\Pi}_L$, $F_L(\lambda)$ is an irreducible ℓ module on which $z(\ell)$ acts by scalars, and by restriction, $F_L(\lambda)|_{\ell^{ss}}$ is ℓ^{ss} irreducible, as ℓ is reductive. We assume $L_{ss}(\lambda)$ arose via $F_L(\lambda)$: $L_{ss}(\lambda) = F_L(\lambda)|_{\ell^{ss}}$: $L_{ss}(\lambda)$ integrates to an (ℓ, L) module. By (4.4), (4.6), (4.9), (4.10) (or by Proposition 3.5 directly), i_0: $U(\ell^{ss}) \otimes_{U(p_S)} F(\lambda)_{ss} = C_0 \longrightarrow U(\ell) \otimes_{U(p_\ell)} F(\lambda) = \tilde{C}_0$ is an ℓ^{ss} module isomorphism. That is, $\tilde{\varepsilon} \overset{\text{def}}{=} \varepsilon \circ i_0^{-1}$: $\tilde{C}_0 \longrightarrow F_L(\lambda)|_{\ell^{ss}}$ is an ℓ^{ss} module map for ε in Theorem 4.7. Choosing a non-zero λ-weight vector in $F_L(\lambda)$ we see that $z \in z(\ell)$ acts on $F_L(\lambda)$ via the scalar $\lambda(z)$ (arguing as we did for the $\ell \cap k$ module $F(\lambda)$ above), which is the *same* scalar by which $z(\ell)$ acts on $\tilde{C}_0$, as noted earlier. As earlier, it follows that $\tilde{\varepsilon}$: $\tilde{C}_0 \longrightarrow F_L(\lambda)$ is an ℓ module map, and hence by Theorem 3.3, $\tilde{\varepsilon}$: $\tilde{C}_0 \longrightarrow F_L(\lambda)$ is an (ℓ, B) module map. Since the i_k are isomorphisms we deduce that

$$0 \longrightarrow \tilde{C}_t \xrightarrow{\tilde{\alpha}_t} \tilde{C}_{t-1} \xrightarrow{\tilde{\alpha}_{t-1}} \cdots \longrightarrow \tilde{C}_0 \xrightarrow{\tilde{\varepsilon}} F_L(\lambda) \longrightarrow 0 \tag{4.14}$$

is an exact sequence of (ℓ, B) modules, given the exactness expressed in Theorem 4.7.

There is a final step in constructing the desired resolution of $U(g) \otimes_{U(q)} F_L(\lambda)$. Extend each $F(\lambda_w)$ to a (q_1, B) module on which u_1 acts trivially, and each Y_w to a (q, B) module on which u acts trivially. Then $\tilde{C}_k$ is a (q, B) module such that $u \cdot \tilde{C}_k = 0$. Also, by Theorem 3.2, there is a canonical (g, B) module isomorphism

$$\delta_w\colon U(g) \otimes_{U(q)} Y_w \xrightarrow{\text{onto}} U(g) \otimes_{U(q_1)} F(\lambda_w). \tag{4.15}$$

Tensor the exact sequence of (ℓ, B) modules in (4.14) (which are now (q, B) modules) with $U(g)$ over $U(q)$ to obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow U(g) \otimes_{U(q)} \tilde{C}_t \xrightarrow{\tilde{\alpha}_t^*} U(g) \otimes_{U(q)} \tilde{C}_{t-1} \xrightarrow{\tilde{\alpha}_{t-1}^*} \\ \cdots \longrightarrow U(g) \otimes_{U(q)} \tilde{C}_0 \xrightarrow{\tilde{\varepsilon}^*} U(g) \otimes_{U(q)} F_L(\lambda) \longrightarrow 0 \end{aligned} \tag{4.16}$$

of (g, B) modules; cf. Lemma 6.1.6 of [9]. By Proposition 3.8 and equations (4.10), (4.11), (4.15), we have (g, B) module isomorphisms

$$U(g) \otimes_{U(q)} \tilde{C}_k \simeq \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus U(g) \otimes_{U(q)} Y_w \simeq \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus U(g) \otimes_{U(q_1)} F(\lambda_w) \tag{4.17}$$

which, by (4.16), lead us to the following.

THEOREM 4.18. *Let $\lambda \in h^*$ be $Q \cap r(\ell)$-dominant integral and analytically integral. Let $F_L(\lambda)$ be the smooth finite-dimensional L module with $Q \cap r(\ell)$ highest weight λ. Then there exist $(g, L \cap K)$ module maps $\phi_1, \phi_2, \ldots, \phi_t, \varepsilon$, $t = \dim(\ell \cap p^+)$, such that*

$$0 \longrightarrow \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=t}} \oplus U(g) \otimes_{U(q_1)} F(\lambda_w) \xrightarrow{\phi_t} \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=t-1}} \oplus U(g) \otimes_{U(q_1)} F(\lambda_w)$$

$$\xrightarrow{\phi_{t-1}} \sum_{\substack{w\in W^1(\ell),\\ \ell(w)=t-2}} \oplus U(g)\otimes_{U(q_1)} F(\lambda_w) \longrightarrow \cdots \xrightarrow{\phi_1} \sum_{\substack{w\in W^1(\ell),\\ \ell(w)=0}} \oplus U(g)\otimes_{U(q_1)} F(\lambda_w)$$

$$= \quad U(g)\otimes_{U(q_1)} F(\lambda) \xrightarrow{\varepsilon} U(g)\otimes_{U(q)} F_L(\lambda) \longrightarrow 0 \tag{4.19}$$

is an exact sequence of $(g, L\cap K)$ modules; see (1.1), (4.2), (4.5). *Here $\ell(w) = |\{\alpha \in Q\cap r(\ell) \mid w^{-1}\alpha \in -(Q\cap r(\ell))\}|$ is the length of w (where we have denoted $\Delta(\ell)$ in* (1.3) *by $r(\ell)$), the $F(\lambda_w)$ are irreducible smooth finite-dimensional $L\cap K$ modules ($= (q_1, L\cap K)$ modules on which $u_1 = u + p^+$ acts trivially) with $Q\cap r(\ell)\cap \Delta_k$-highest weight λ_w, and q_1 is the θ-stable parabolic $(\ell\cap k)\oplus u_1 = q\cap(k+p^+)$.*

Remarks. In Theorem 4.18, which generalizes Proposition 4.18 of [8], recall that $q = \ell \oplus u$ is any θ-stable parabolic subalgebra of g which contains the Borel subalgebra $b_Q = h + \sum_{\alpha\in Q} g_\alpha$ of g, or (equivalently) which satisfies condition $(1.3)'$. In Section 4 of [4], Enright and Wallach construct a finite resolution of a Verma module (in the category of g-modules) which differs markedly in form from that of Theorem 4.18. Also compare with Chapter 6 of [10].

For a (q, B) module Z let Z^h denote its B-finite Hermitian dual, which is a $(\overline{q}, B)$ module: Z^h is the space of B-finite vectors in the space of conjugate linear maps $Z \longrightarrow \mathbb{C}$. There is a standard (g, B) isomorphism

$$\mathrm{Pro}_{\overline{q},B}^{g,B} Z^h \simeq (U(g)\otimes_{U(q)} Z)^h; \tag{4.20}$$

see (3.7). If Z is a smooth finite-dimensional B module (and thus an $(\ell\cap k, B)$ module) regard Z as both a (q, B) and a $(\overline{q}, B)$ module on which both $u, \overline{u}$ act trivially. Then Z^h in (4.20) can be replaced by Z; cf. Lemma 4.19 of [8]. Write

$$\sum_{\substack{w\in W^1(\ell),\\ \ell(w)=k}} \oplus U(g)\otimes_{U(q_1)} F(\lambda_w) = U(g)\otimes_{U(q_1)} F_k(\lambda) \tag{4.21}$$

by Proposition 3.8 and equation (4.12). Apply (4.20) with q replaced by q_1, and apply Proposition 3.8 again:

$$(\text{l.h.s. of } (4.21))^h = \mathrm{Pro}_{\overline{q}_1,B}^{g,B} F_k(\lambda) = \sum_{\substack{w\in W^1(\ell),\\ \ell(w)=k}} \oplus \mathrm{Pro}_{\overline{q}_1,B}^{g,B} F(\lambda_w). \tag{4.22}$$

That is, if we take the B-finite Hermitian dual of the terms in (4.19), then as exactness is preserved we obtain from (4.22) the following generalization of Proposition 4.22 of [8].

THEOREM 4.23. *Let $q \supset b_Q$ be any θ-stable parabolic and let $\lambda \in h^*$, $F_L(\lambda)$ be as in Theorem* 4.18. *With the notation of that theorem, there exist $(g, L\cap K)$*

module maps $\phi_1^*, \phi_2^*, \ldots, \phi_t^*, \varepsilon^*, t = \dim(\ell \cap p^+)$, *such that for* $B = L \cap K, 1 \leq k \leq t$,

$$V_k \stackrel{\text{def}}{=} \sum_{\substack{w \in W^1(\ell), \\ \ell(w)=k}} \oplus \operatorname{Pro}_{\bar{q}_1, B}^{g, B} F(\lambda_w) \tag{4.24}$$

$$0 \longrightarrow \operatorname{Pro}_{\bar{q}, B}^{g, B} F_L(\lambda) \xrightarrow{\epsilon^*} V_0 \xrightarrow{\phi_1^*} V_1 \xrightarrow{\phi_2^*} V_2 \longrightarrow \cdots \xrightarrow{\phi_t^*} V_t \longrightarrow 0$$

is an exact sequence of (g, B) *modules.*

5. Some applications

Applications of Theorem 4.23 will be based on the folowing general result in conjunction with the Borel-Weil theorem.

THEOREM 5.1. *Let* G *be any Lie group and let* $B, K \subset G$ *be closed subgroups of* G *with* $B \subset K$, K *compact and connected. Let* g *denote the complexified Lie algebra of* G. *Given a* (g, B) *module* V, *let* $\Gamma V \subset V$ *be the corresponding* (g, K) *module given by the Zuckerman construction; see* [11] *and Chapter* 6 *of* [9]. *Suppose one has a* (g, B) *resolution* $0 \longrightarrow V \xrightarrow{\delta_0} V_0 \xrightarrow{\delta_1} V_1 \xrightarrow{\delta_2} V_2 \xrightarrow{\delta_3} \cdots$ *of* V. *Then there is a* 1*st quadrant spectral sequence* E *(i.e.,* $E_2^{r,s} = 0$ *if either* r *or* $s < 0$*) with* ∞*-terms associated to the cohomology* $H^*(T)$ *of a complex* T *given by* $H^r(T) = \Gamma^r V$ *for* $r \geq 0$, $H^r(T) = 0$ *for* $r < 0$, *where* $\Gamma^r, r \geq 0$, *is the* r^{th} *right derived functor of the Zuckerman functor* $\Gamma = \Gamma_{g,B}^{g,K}$. E *is induced by a decreasing filtration* $\{F^p T^n\}_{p \in Z}$ *of each* T^n *(the space of* n*-cochains of* T*) which satisfies the regularity conditions (i)* $F^p T^n = 0$ *for* $p > n$, *(ii)* $F^p T^n = T^n$ *for* $p \leq 0$. *The* E_1, E_2 *terms are given by* $E_1^{r,s} = \Gamma^s V_r$ *for* $r, s, \geq 0$, *with* $E_1^{r,s} = 0$ *if either* r *or* $s < 0$,

$$\begin{aligned} E_2^{r,s} &= \ker\left(\Gamma^s V_r \xrightarrow{\Gamma^s \delta_{r+1}} \Gamma^s V_{r+1}\right) / (\Gamma^s \delta_r) \Gamma^s V_{r-1} \quad \text{for } r, s, \geq 1, \\ E_2^{0,s} &= \ker\left(\Gamma^s V_0 \xrightarrow{\Gamma^s \delta_1} \Gamma^s V_1\right) \quad \text{for } s \geq 1, \\ E_2^{0,0} &= \ker\left(\Gamma V_0 \xrightarrow{\delta_1} \Gamma V_1\right). \end{aligned} \tag{5.2}$$

The differential d: $E_n \longrightarrow E_n$ *has bidegree* $(n, 1-n)$; *i.e.,* d: $E_n^{r,s} \longrightarrow E_n^{n+r,s-n+1}$.

Given that the category of (g, B) modules has enough injectives, Theorem 5.1 follows by general Grothendieck principles, e.g., the theory of resolution of a complex. See Proposition 3.2 of [8] or Appendix D of [5] for a more general result.

COROLLARY 5.3. *In Theorem* 5.1, *suppose* $\Gamma^{j_0} V \neq 0$ *for some* $j_0 \geq 0$. *Then there is an integer* $r_0 \geq 0, r_0 \leq j_0$ *such that* $\Gamma^{r_0} V_{j_0 - r_0} \neq 0$.

Proof. As $H^{j_0}(T) = \Gamma^{j_0} V \neq 0$ the regularity conditions (i) and (ii) imply that $E_1^{j_0-r_0,r_0} \neq 0$ for some integer r_0. Since $E_1^{a,b} = 0$ for a or $b < 0$ we must have $j_0 - r_0, r_0 \geq 0$. Then $\Gamma^{r_0} V_{j_0-r_0} = E_1^{j_0-r_0,r_0} \neq 0$, as desired.

The connectivity assumption on K in Theorem 5.1 is dispensable. In application of course, G will be connected semisimple as before, K will be a maximal compact subgroup of G (therefore K is connected), and we will choose, as before, $B = L \cap K$ where L (as in (1.4)) corresponds to a θ-stable parabolic in g. By Theorems 4.23 and 5.1 we therefore have the following.

THEOREM 5.4. *Let $\lambda \in h^*$, $F_L(\lambda), q$ be as in Theorem* 4.18; *thus q is any θ-stable parabolic $\supset b_Q$. Then there is a* 1*st quadrant spectral sequence E with ∞-terms associated to the cohomology $H^*(T)$ of a complex T given by $H^r(T) = \Gamma^r \mathrm{Pro}_{\overline{q},B}^{g,B} F_L(\lambda)$ for $r \geq 0$, $B \overset{\text{def}}{=} L \cap K$, $\Gamma^r = \left(\Gamma_{g,B}^{g,K}\right)^r$ with $H^r(T) = 0$ for $r < 0$. E is induced by a decreasing filtration* (*of cochains*) *which satisfies* (*i*) *and* (*ii*) *in Theorem* 5.1. *The E_1 and E_2 terms are given by $E_1^{r,s} = \Gamma^s V_r$ for $r, s \geq 0$ and by the equations of* (5.2), *where*

$$V_r \overset{\text{def}}{=} \sum_{\substack{w \in W^1(\ell),\\ \ell(w)=r}} \oplus \mathrm{Pro}_{\overline{q}_1,B}^{g,B} F(\lambda_w); \tag{5.5}$$

$\delta_0 = \varepsilon^*$, $\delta_r = \phi_r^*$, $1 \leq r \leq t = \dim(\ell \cap p^+)$ *in the notation of Theorem* 4.23. *The differential $E_n \longrightarrow E_n$ has bidegree* $(n, 1-n)$.

For $\lambda \in P_{E_1}$ in (4.1) which is analytically integral let $F(\lambda)$ be the irreducible B module with $Q \cap \Delta_k \cap \Delta(\ell)$-highest weight λ, as in Section 4, given a θ-stable $q \supset b_Q$. In fact, we have seen that $F(\lambda)$ is a (p_ℓ, B) module for $p_\ell = \ell \cap k \oplus \ell \cap p^+$ in (2.15). Using $\overline{q \cap k} = \ell \cap k \oplus \overline{u \cap k}$, regard $F(\lambda)$ as a $(\overline{q \cap k}, B)$ module on which $\overline{u \cap k}$ acts trivially. Then for $2\delta_k \overset{\text{def}}{=} \sum_{\alpha \in Q \cap \Delta_k} \alpha$ one has the following.

THEOREM 5.6 (Borel-Weil [3]). *If $(\lambda + \delta_k, \alpha) = 0$ for some $\alpha \in \Delta_k$ then $(\Gamma_{k,B}^{k,K})^j \mathrm{Pro}_{\overline{q \cap k},B}^{k,B} F(\lambda) = 0 \quad \forall j \geq 0$* (*see* (3.7)). *Assume $\lambda + \delta_k$ is Δ_k-regular: $(\lambda + \delta_k, \alpha) \neq 0 \quad \forall \alpha \in \Delta_k$. Let σ be the unique element in the Weyl group W_k of (k, h) such that $(\sigma(\lambda + \delta_k), \alpha) > 0 \ \forall \alpha \in Q \cap \Delta_k$. Then $(\Gamma_{k,B}^{k,K})^j \mathrm{Pro}_{\overline{q \cap k},B}^{k,B} F(\lambda) = 0$ for $j \neq \ell(\sigma) \overset{\text{def}}{=} |\{\alpha \in Q \cap \Delta_k \mid \sigma^{-1}\alpha \in -(Q \cap \Delta_k)\}|$, and equals the irreducible $(k.K)$ module with $Q \cap \Delta_k$-highest weight $\sigma(\lambda + \delta_k) - \delta_k$ if $j = \ell(\sigma)$.*

In the situation prior to the statement of Proposition 3.9 (there the assumption that G/K has a Hermitian structure was not needed), as in Lemma 6.3.9 of [9], one has the following.

THEOREM 5.7. *Let Y be an $(\ell^2, L^1 \cap K)$ module extended to a $(q^2, L^1 \cap K)$ module on which u^2 acts trivially. Then there is a $(g, L^2 \cap K)$ isomorphism of $\left(\Gamma^{g,L^2\cap K}_{g,L^1\cap K}\right)^r \mathrm{Pro}^{g,L^1\cap K}_{q^2,L^1\cap K} Y$ onto $\mathrm{Pro}^{g,L^2\cap K}_{q^2,L^2\cap K}\left(\Gamma^{\ell^2,L^2\cap K}_{\ell^2,L^1\cap K}\right)^r Y$ for $r \geq 0$, where $\Gamma^{\ell^2,L^2\cap K}_{\ell^2,L^1\cap K} Y$ is regarded as a $(q^2, L^2 \cap K)$ module on which u^2 acts trivially.*

Again bear in mind that the definition of the functor Pro in (3.7) differs slightly from that of [9]. For G/K Hermitian symmetric we obtain the following.

COROLLARY 5.8. *Let Y be a $(k, L \cap K)$ module extended to a $(\overline{q}_{x(z)}, L \cap K)$ module by the trivial action of p^-. Similarly extend $\Gamma^{k,K}_{k,L\cap K} Y$ to a $(\overline{q}_{x(z)}, K)$ module on which p^- acts trivially. Then there is a (g, K) module isomorphism of $\left(\Gamma^{g,K}_{g,L\cap K}\right)^r \mathrm{Pro}^{g,L\cap K}_{\overline{q}_{x(z)},L\cap K} Y$ onto $\mathrm{Pro}^{g,K}_{\overline{q}_{x(z)},K}\left(\Gamma^{k,K}_{k,L\cap K}\right)^r Y$ for $r \geq 0$; here again*

$$\overline{q}_{x(z)} = k + p^- . \tag{5.9}$$

Corollary 5.8 *follows if we choose* $q^1 = \overline{q}_1 = \overline{q \cap q_{x(z)}}$, $q^2 = \overline{q}_{x(z)}$, *exactly as we did to establish Corollary* 3.16. *Thus, as we have seen,* $\ell^2 = k, L^1 \cap K = L \cap K$, $L^2 \cap K = K$.

Using the Borel-Weil theorem we can now compute the cohomological parabolically induced (g, K) modules $\Gamma^s V_r$ appearing in the E_2 terms in (5.2) for V_r given in (5.5); compare with Proposition 4.26 of [8]. As we needed the fact that the functors Pro, and Ind commute with direct summation (Proposition 3.8) we will similarly need the fact that each Γ^s, $s \geq 0$, commutes with direct summation. As usual $q = \ell + u \supset b_Q$ is any θ-stable parabolic. Again let $2\delta_k = \sum_{\alpha\in Q\cap\Delta_k} \alpha$, let $\lambda \in P_{E_1}$ in (4.1) be analytically integral, and let $F(\lambda)$ be the corresponding irreducible $B \stackrel{\text{def}}{=} L \cap K$ module with $Q \cap \Delta_k \cap \Delta(\ell)$-highest weight λ. In Theorem 5.6, we viewed $F(\lambda)$ as a $(\overline{q \cap k}, B)$ module on which $\overline{u \cap k}$ acts trivially, using $\overline{q \cap k} = \ell \cap k \oplus \overline{u \cap k}$. Similarly we view $F(\lambda)$ as a $(\overline{q}_1, B)$ module on which $\overline{u}_1$ acts trivially, using $\overline{q}_1 = \ell \cap k \oplus \overline{u}_1$, where $\overline{u}_1 = u + p^+ = \overline{u} + p^-$.

THEOREM 5.10. *If $(\lambda+\delta_k, \alpha) = 0$ for some $\alpha \in \Delta_k$ then $\left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\overline{q}_1,B} F(\lambda) = 0$ $\forall j \geq 0$. Assume $\lambda+\delta_k$ is Δ_k-regular and let $\sigma \in W_k$ be the unique element such that $(\sigma(\lambda+\delta_k), \alpha) > 0$ $\forall\alpha \in Q\cap\Delta_k$, as in Theorem 5.6. Then $\left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\overline{q}_1,B} F(\lambda) = 0$ for $j \neq \ell(\sigma) \stackrel{\text{def}}{=} |\{\alpha \in Q \cap \Delta_k \mid \sigma^{-1}\alpha \in -(Q \cap \Delta_k)\}|$, and*

$$\left(\Gamma^{g,K}_{g,B}\right)^{\ell(\sigma)} \mathrm{Pro}^{g,B}_{\overline{q}_1,B} F(\lambda) = \mathrm{Pro}^{g,K}_{\overline{q}_{x(z)},K} F_K(\sigma(\lambda + \delta_k) - \delta_k) \tag{5.11}$$

where $F_K(\sigma(\lambda + \delta_k) - \delta_k)$ is the irreducible (k, K) module with $Q \cap \Delta_k$-highest weight $\sigma(\lambda + \delta_k) - \delta_k$, which one extends to a $(\overline{q}_{x(z)} = k + p^-, K)$ module by the trivial action of p^-.

Proof. Let $Y = \mathrm{Pro}^{k,B}_{\overline{q \cap k},B} \mathrm{res}^{\overline{q \cap k},B}_{\overline{q}_1,B} F(\lambda)$, which we extend to a $(\overline{q}_{x(z)}, B)$ module by the trivial action of p^-. Then by Corollary 3.16, $\left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\overline{q},B} F(\lambda) = \left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\overline{q}_{x(z)},B} Y$, which by Corollary 5.8 equals $\mathrm{Pro}^{g,K}_{\overline{q}_x(z),K} \left(\Gamma^{k,K}_{k,B}\right)^j Y$. Theorem 5.10 now follows from Theorem 5.6.

In (2.20) and (2.21) we observed that $\sigma(Q \cap \Delta_n) \overset{(i)}{=} Q \cap \Delta_n$ for $\sigma \in W_k$, and in particular $\sigma\delta_n \overset{(ii)}{=} \delta_n$, $(\delta_n, \Delta_k) = 0$. It follows from (4.5) that for $w \in W^1(\ell), \alpha \in \Delta_k$, $\lambda_w + \delta_k \overset{\text{def}}{=} w(\lambda+\delta) - \delta + \delta_k \overset{(iii)}{=} w(\lambda+\delta) - \delta_n \Longrightarrow (\lambda_w + \delta_k, \alpha) = (w(\lambda+\delta), \alpha)$. That is, $\lambda_w + \delta_k$ is Δ_k-regular $\Longleftrightarrow w(\lambda+\delta)$ is Δ_k-regular, in which case we define $\sigma_w \in W_k$ to be the unique element such that

$$(\sigma_w w(\lambda+\delta), \alpha) > 0 \ \forall \alpha \in Q \cap \Delta_k. \tag{5.12}$$

By (ii) and (iii), $\sigma_w(\lambda_w + \delta_k) - \delta_k = \sigma_w[w(\lambda+\delta) - \delta_n] - \delta_k = \sigma_w w(\lambda+\delta) - \delta$, and by (5.5),

$$\left(\Gamma^{g,K}_{g,B}\right)^j V_r = \sum_{\substack{w \in W^1(\ell), \\ \ell(w) = r}} \oplus \left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\overline{q}_1,B} F(\lambda_w) \tag{5.13}$$

where we now assume λ is $Q \cap r(\ell)$-dominant integral and analytically integral. Then as we have seen (by Proposition 4.3) each $\lambda_w \in P_E$ is analytically integral. Thus is (5.13) we can apply Theorem 5.10 to obtain the following.

COROLLARY 5.14. *Suppose $\lambda \in h^*$ is $Q \cap r(\ell)$-dominant integral and analytically integral, as in Theorem 5.4. Then in the formulas for the E_1, E_2 terms in Theorem 5.4 (cf. (5.2)), for $r, j, \geq 0$, and for σ_w defined in (5.12) one has*

$$\left(\Gamma^{g,K}_{g,B}\right)^j V_r = \sum_{\substack{w \in W^1(\ell), \ell(w) = r, \\ w(\lambda+\delta) \text{ is } \Delta_k\text{-regular}, \\ \ell(\sigma_w) = j}} \oplus \ \mathrm{Pro}^{g,K}_{\overline{q}_{x(z)},K} F_K(\sigma_w w(\lambda+\delta) - \delta) = E_1^{r,j}. \tag{5.15}$$

For $(\sigma, w, \tau) \in W_k \times W(\ell) \times W$ let

$$\begin{aligned} \Phi^k_\sigma &= \{\alpha \in Q \cap \Delta_k \mid \sigma^{-1}\alpha \in -(Q \cap \Delta_k)\}, \\ \Phi^\ell_w &= \{\alpha \in Q \cap \Delta(\ell) \mid w^{-1}\alpha \in -(Q \cap \Delta(\ell))\}, \\ \Phi_\tau &= \{\alpha \in Q \mid \tau^{-1}\alpha \in -Q\}. \end{aligned} \tag{5.16}$$

Then $|\Phi^k_\sigma|$ is the $Q \cap \Delta_k$-length of σ, $|\Phi^\ell_w|$ is the $Q \cap \Delta(\ell)$-length of w, and $|\Phi_\tau|$ is the Q-length (or simply the length) of τ. We have observed that $\sigma(Q \cap \Delta_n) \overset{(i)}{=} Q \cap \Delta_n$. It follows that $\Phi^k_\sigma = \Phi_\sigma$. Similarly $w\Delta(u) = \Delta(u) = Q - \Delta(\ell)$ (see (1.3)$'$)$\Longrightarrow \Phi^\ell_w = \Phi_w$. Note that $\Phi_\sigma \cap \sigma\Phi_w = \phi$: If there exists $\alpha \in \Phi_\sigma \cap \sigma\Phi_w$ then $\alpha \in Q, \sigma^{-1}\alpha \in -Q$

and $\alpha = \sigma B$ with $B \in Q$, $w^{-1}B \in -Q \Longrightarrow B = \sigma^{-1}\alpha \in -Q \cap Q$ is a contradiction. Suppose, in fact, that $w \in W^1(\ell)$: $Q \cap \Delta(\ell) \cap \Delta_k \overset{\text{(iv)}}{\subset} w(Q \cap \Delta(\ell))$ (see (4.2)); recall that the sets $\Delta(\ell)$ and $r(\ell))$ are the same in our notation. Then $\Phi_w \subset \Delta_n$: If $\alpha \in \Phi_w \cap \Delta_k$, then $\alpha \in Q \cap \Delta(\ell) \cap \Delta_k \Longrightarrow w^{-1}\alpha \in Q$ by (iv), which is a contradiction since $\alpha \in \Phi_w \Longrightarrow w^{-1}\alpha \in -Q$. Thus $\Phi_w \overset{\text{(v)}}{\subset} \Delta(\ell) \cap \Delta_n \cap Q$. In particular $\sigma\Phi_w \subset \Delta_n \cap Q$ by (i). That is, $\Phi_\sigma \cup \sigma\Phi_w \subset Q$ and since this union is disjoint

$$\sum_{\alpha \in \Phi_\sigma \cup \sigma\Phi_w} \alpha = \sum_{\alpha \in \Phi_\sigma} \alpha + \sigma \sum_{\alpha \in \Phi_w} \alpha = \delta - \sigma\delta + \sigma(\delta - w\delta) = \delta - \sigma w\delta = \sum_{\alpha \in \Phi_{\sigma w}} \alpha,$$

from whence one can conclude $\Phi_\sigma \cup \sigma\Phi_w = \Phi_{\sigma w}$. Of course one can check this equality of sets directly. This proves the following.

PROPOSITION 5.17. *For $(\sigma, w) \in W_k \times W(\ell)$, $\Phi^k_\sigma = \Phi_\sigma$ and $\Phi^\ell_w = \Phi_w$ If $w \in W^1(\ell)$, $\Phi_{\sigma w} = \Phi_\sigma \cup \sigma\Phi_w$ where the union is disjoint; see* (5.17). *In particular, in Theorems* 4.18, 4.23, 5.4, 5.6, 5.10, *and Corollary* 5.14 *we can also write $\ell(\sigma) = |\Phi_\sigma|$, $\ell(w) = |\Phi_w|$. If $w \in W^1(\ell)$, $|\Phi_{\sigma w}| = |\Phi_\sigma| + |\Phi_w|$ and $\Phi_w = \{\alpha \in Q \cap \Delta_n \cap \Delta(\ell) \mid w^{-1}\alpha \in -Q\}$ (by (v)).*

PROPOSITION 5.18. *Let $\lambda \in h^*$ be $Q \cap \Delta(\ell)$-dominant (as above) and let $(\sigma, w) \in W_k \times W^1(\ell)$. Then $\Phi_w = \{\alpha \in Q \cap \Delta_n \cap \Delta(\ell) \mid (w(\lambda + \delta), \alpha) < 0\}$. If $w(\lambda + \delta)$ is Δ_k-regular let $\sigma_w \in W_k$ be the element defined in* (5.12). *Then $\Phi^k_{\sigma_w^{-1}} = \{\alpha \in \Delta(u) \cap \Delta_k \mid (w(\lambda + \delta), \alpha) < 0\}$. Of course for any $\sigma \in W_k$, $\Phi^k_\sigma = -\sigma\Phi^k_{\sigma^{-1}}$.*

Proof. For $\alpha \in \Phi_w$, $\alpha \in Q \cap \Delta_n \cap \Delta(\ell)$ such that $-w^{-1}\alpha \in Q \cap \Delta(\ell)$ (by Proposition 5.17)$\Longrightarrow (\lambda + \delta, -w^{-1}\alpha) > 0$ as λ is $Q \cap \Delta(\ell)$-dominant; i.e., $(w(\lambda + \delta), \alpha) < 0$. Conversely, it is clear that any α in $Q \cap \Delta_n \cap \Delta(\ell)$ subject to the latter inequality belongs to Φ_w. Let $\alpha \in \Phi^k_{\sigma_w^{-1}}$: $\alpha \in Q \cap \Delta_k$, $-\sigma_w\alpha \in Q \cap \Delta_k$ by definition (5.16). If $\alpha \in \Delta(\ell)$ then $\alpha \in Q \cap \Delta_k \cap \Delta(\ell) \subset w(Q \cap \Delta(\ell))$ (since $w \in W^1(\ell)) \Longrightarrow w^{-1}\alpha \in Q \cap \Delta(\ell) \Longrightarrow (\lambda + \delta, w^{-1}\alpha) > 0$ as λ is $Q \cap \Delta(\ell)$-dominant. That is, $(\sigma_w w(\lambda + \delta), -\sigma_w\alpha) = -(w(\lambda + \delta), \alpha) = -(\lambda + \delta, w^{-1}\alpha) < 0$, which contradicts the definition of σ_w in (5.12) since $-\sigma_w\alpha \in Q \cap \Delta_k$; here we assume $w(\lambda + \delta)$ is Δ_k-regular. In other words, $\alpha \notin \Delta(\ell) \Longrightarrow \alpha \in Q - \Delta(\ell) = \Delta(u)$. That is, $\alpha \in \Delta(u) \cap \Delta_k$ and $-\sigma_w\alpha \in Q \cap \Delta_k \Longrightarrow 0 < (\sigma_w w(\lambda + \delta), -\sigma_w\alpha) = -(w(\lambda + \delta), \alpha)$. Conversely if $\alpha \in \Delta(u) \cap \Delta_k$ satisfies $(w(\lambda + \delta), \alpha) < 0$ then $(\sigma_w w(\lambda + \delta), \sigma_w\alpha) < 0$, with $\sigma_w\alpha \in \Delta_k$ and $\alpha \in Q \cap \Delta_k$, as $\Delta(u) \subset Q$. By (5.12) we must then have $\sigma_w\alpha \in -(Q \cap \Delta_k)$; i.e. $\alpha \in \Phi^k_{\sigma_w^{-1}}$, which concludes the proof.

Corollary 5.3 provides for the following vanishing theorem.

PROPOSITION 5.19. *Let $\lambda \in h^*$, $F_L(\lambda)$, q be as in Theorem* 4.18, *as usual. Suppose $\left(\Gamma^{g,K}_{g,B}\right)^j \operatorname{Pro}^{g,B}_{\bar{q},B} F_L(\lambda) \neq 0$, where $B = L \cap K$. Then there exists $w \in W^1(\ell)$*

(see (4.2)) *such that* $w(\lambda+\delta)$ *is* Δ_k*-regular. Let* $\sigma_w \in W_k$ *be the unique element such that* $(\sigma_w w(\lambda+\delta), \alpha) > 0\ \forall \alpha \in Q \cap \Delta_k$ *(as in* (5.12)). *Moreover,* $j = \ell(\sigma_w) + \ell(w)$ *with* $\ell(\sigma_w) = |\{\alpha \in \Delta_k \cap \Delta(u) \mid (w(\lambda+\delta), \alpha) < 0\}|$ *and* $\ell(w) = |\{\alpha \in Q \cap \Delta_n \cap \Delta(\ell) \mid (w(\lambda+\delta), \alpha) < 0\}|$. *We can also write* $j = \ell(\sigma_w w) \stackrel{\text{def}}{=} |\{\alpha \in Q \mid (\sigma_w w)^{-1}\alpha \in -Q\}| \leq \dim(u \cap k)$.

Proof. In Corollary 5.3, choose $V = \mathrm{Pro}^{g,B}_{\bar{q},B} F_L(\lambda)$, as in Theorem 5.4 so that the V_r are given by (5.5). Then $\left(\Gamma^{g,K}_{g,B}\right)^j V \neq 0 \Longrightarrow \left(\Gamma^{g,K}_{g,B}\right)^r V_{j-r} \neq 0$ for some r , $0 \leq r \leq j$, by Corollary 5.3. By Corollary 5.14, $\mathrm{Pro}^{g,K}_{\bar{q}_{x(z)},K} F_K(\sigma_w w(\lambda+\delta)-\delta) \neq 0$ for some $w \in W^1(\ell)$ such that $\ell(w) = j - r$, $w(\lambda+\delta)$ is Δ_k-regular, and $\ell(\sigma_w) = r$. That is, $j = \ell(\sigma_w) + \ell(w)$ and the remaining assertions of Proposition 5.19 follow from Propositions 5.17 and 5.18, except for the inequality $j \leq s \stackrel{\text{def}}{=} \dim(u \cap k)$ which is a bit deeper. This inequality follows from the generalized Blattner formula, Theorem 6.3.12 of [9]; see Corollary 6.3.21 of [9]. Strictly speaking, the latter theorem applies to our situation provided two minor points are taken into account:

(i) The cohomological parabolic induction employed here involves the opposite parabolic $\bar{q} = \ell + \bar{u}$, rather than q employed in [9].

(ii) As pointed out earlier, the functor Pro of [9] differs from ours in (3.7) by a rho shift. At any rate we can obtain the inequality $j \leq s$ by replacing the Weyl group element in W^1_K of (a) of Theorem 6.3.12 of [9] by a suitable W_k-translate of it.

As an example of the preceding proposition one has the following well-known fact. See [5], [9] and [10] for example for more general results.

COROLLARY 5.20. *Suppose that in Proposition* 5.19, λ *satisfies*

$$(\lambda+\delta, \alpha) < 0 \textit{ for every } \alpha \in \Delta(u). \tag{5.21}$$

Then $\left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\bar{q},B} F_L(\lambda) = 0$ *for* $j \neq s = \dim(u \cap k)$.

Proof. Given any $\alpha \in \Delta(u)$ (in particular given any $\alpha \in \Delta(u) \cap \Delta_k$) and $w \in W(\ell)$, one has $w^{-1}\alpha \in \Delta(u)$ so that $(w(\lambda+\delta), \alpha) = (\lambda+\delta, w^{-1}\alpha) < 0$ by (5.21). If $\Gamma^j \mathrm{Pro} F_L(\lambda) \neq 0$ we see that for a $w \in W^1(\ell)$ given by Proposition 5.19, $\ell(\sigma_w) = |\Delta_k \cap \Delta(u)| = s$. Then $j = \ell(\sigma_w) + \ell(w) \leq s$ forces $\ell(w) = 0$; i.e., $j = s$.

In the program set up in [8] to construct irreducible unitarizable (g, K) modules $\left(\Gamma^{g,K}_{g,B}\right)^j \mathrm{Pro}^{g,B}_{\bar{q},B} F_L(\lambda)$, interest is focused on the case $j < s$, which by Corollary 5.20 requires the existence of at least one α in $\Delta(u)$ for which $(\lambda+\delta, \alpha) \geq 0$. For example it is assumed eventually that $\mathrm{Pro} F_L(\lambda)$ is reducible: λ is a reduction point in the Enright-Howe-Wallach classification of unitary highest weight modules.

References

[1] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, "Differential operators on the base affine space and a study of g-modules" in *Lie groups and their representations*, Halsted Press 1975, 21–64.

[2] J. Dixmier, *Enveloping algebras*, North-Holland 1977.

[3] T. Enright and N. Wallach, *Notes on homological algebra and representations of Lie algebras*, Duke Math. J. **47** (1980), 1–15.

[4] ———, *The fundamental series of representations of a real semi-simple Lie algebra*, Acta Math. **140** (1978), 1–32.

[5] A. Knapp and D. Vogan, Jr., *Cohomological induction and unitary representations*, Princeton Math. Series, no. 45, 1995.

[6] J. Lepowsky, *A generalization of the Bernstein-Gelfand-Gelfand resolution*, J. Algebra **49** (1977), 496–511.

[7] A. Rocha-Caridi, *Splitting criteria for g-modules induced from a parabolic and the Bernstein-Gelfand-Gelfand resolution of a finite-dimensional irreducible g-module*, Trans. Amer. Math. Soc. **262** (1980), 335–366.

[8] R. Stanke, *Covariant differential operators and derived functor modules, from representation theory and harmonic analysis* (*a conference in honor of Ray A. Kunze*), Contemp. Math., Amer. Math. Soc., vol 191, 1995.

[9] D. Vogan, Jr., *Representations of real reductive Lie groups*, Birkhäuser, Progr. Math., vol. 15, 1981..

[10] N. Wallach, *Real reductive groups I*, Pure Appl. Math., vol. 132, Academic Press, 1988.

[11] G. Zuckerman, *Construction of representations via derived functors*, unpublished, 1978.

Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts, 01003-4515
williams@math.umass.edu