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HARDY-LITTLEWOOD THEOREMS FOR A-HARMONIC TENSORS

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ABSTRACT. Conjugate A-harmonic tensors are generalizations of conjugate harmonic functions to differential forms. They share common analytical properties such as integrability and Holder continuity. Applications to quasiregular mappings follow.

1. Introduction

The theory of conjugate harmonic functions plays a central role in such areas of mathematics as potential theory, harmonic analysis and the theory of H^p-spaces. Conjugate harmonic functions have many analytical properties in common, among which are global L^p-integrability and Hölder continuity. These discoveries essentially began with the work of Hardy and Littlewood in the 1930's; see [HL1] and [HL2]. See [P] for an earlier reference on Hölder continuity.

Here we mention three specific results.

THEOREM A. *For each $p > 0$, there is a constant C such that*

$$\int_{\mathbb{D}} |u - u(0)|^p \, dx \, dy \leq C \int_{\mathbb{D}} |v - v(0)|^p \, dx \, dy$$

for all analytic functions $u + iv$ in the unit disk $\mathbb{D}$.

THEOREM B. *For each $0 < k \leq 1$, there is a constant C such that*

$$\|u\|_{\mathrm{Lip}_k, \mathbb{D}} \leq C \|v\|_{\mathrm{Lip}_k, \mathbb{D}}$$

for all analytic functions $u + iv$ in $\mathbb{D}$. Here $\|\cdot\|_{\mathrm{Lip}_k, \mathbb{D}}$ is the usual Lipschitz norm over $\mathbb{D}$.

THEOREM C. *There is a constant C such that*

$$\|u\|_{\mathrm{BMO}, \mathbb{D}} \leq C \|v\|_{\mathrm{BMO}, \mathbb{D}}$$

for all analytic functions $u + iv$ in $\mathbb{D}$. Here $\|\cdot\|_{\mathrm{BMO}, \mathbb{D}}$ is the usual BMO norm over $\mathbb{D}$.

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Conjugate A-harmonic tensors are interesting and important generalizations of conjugate harmonic functions and p-harmonic functions, $p > 1$. See Definition 2.17. They have recently found important applications in areas such as quasiregular mappings and the theory of elasticity; see [I] and [IM]. The main results of this paper, Theorems 4.2, 5.5 and 6.7, generalize Theorems A, B and C to conjugate A-harmonic tensors defined in domains in $\mathbb{R}^n$ which possess an appropriate geometry. Examples show that in many ways the results are best possible.

For example, a p-harmonic function is a solution u to the p-harmonic equation

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

with $p > 1$. Its conjugate in the plane is a q-harmonic function v (often referred to as the "stream function", see [A1]), with $\frac{1}{p} + \frac{1}{q} = 1$, which satisfies

$$|\nabla u|^{p-2} \nabla u = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right). \tag{1.1}$$

Notice that when $p = q = 2$ we have the usual conjugate harmonic functions.

In the theory of elasticity as well as the theory of quasiregular mappings, the phenomenon of p, q-conjugacy arises naturally for solutions to certain elliptic equations for differential forms. More specifically, if u is a solution to

$$d^\star A(x, du) = 0$$

in $\mathbb{R}^n$, then its conjugate is a tensor v such that

$$A(x, du) = d^\star v. \tag{1.2}$$

As such, v is a solution to

$$dA^{-1}(x, d^\star v) = 0.$$

If $A(x, \xi) \cong |\xi|^p$, then $A^{-1}(x, \xi) \cong |\xi|^q$ with $\frac{1}{p} + \frac{1}{q} = 1$. See Section 2 for details. Notice that (1.1) can be rewritten to produce an example of (1.2) in $\mathbb{R}^2$.

A sharp regularity theorem for quasiregular mappings was recently proved using certain conjugate A-harmonic tensors u_I and v_J associated with a quasiregular mapping. See [I] and [IM]. As such, the results of this paper yield corollaries for the quasiregular tensors u_I and v_J. We discuss a few of these which are particularly interesting in Section 7.

In Section 3 we prove the local integrability result for conjugate A-harmonic tensors. This is the basic estimate, a result of the conjugacy, which is used throughout the paper to obtain global results.

The next two sections investigate the global L^s-integrability of conjugate A-harmonic tensors. Here the global geometry of the domains of integration is important. The integrability exponents and the conjugacy exponents p and q directly determine each other.

In Section 4 the first main result, Theorem 4.2, appears. This is a generalization of Theorem A to conjugate A-harmonic tensors in John domains in $\mathbb{R}^n$.

In Section 5 we deal with more general domains than John domains, so-called L^s-averaging domains and obtain a weaker result than Theorem 4.2. Theorem 5.5 seems to be new even in the case of conjugate harmonic functions in the plane.

In Section 6 we treat the local Lipschitz spaces and **BMO** spaces and obtain generalizations of Theorems B and C. We use the integrability result in [Me] for Lipschitz functions to extend the definition of local Lipschitz spaces to differential forms. Again the relationship between p, q and the Lipschitz exponents is seen to be the best possible. A global Lipschitz result holds in the plane for so-called $\mathrm{Lip}_{k,k'}$-extension domains.

2. Exterior algebra, Sobolev spaces and elliptic equations

Let $e_1, e_2, \dots, e_n$ denote the standard unit basis of $\mathbb{R}^n$. For $\ell = 0, 1, \dots, n$, the linear space of ℓ-vectors, spanned by the exterior products $e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_\ell}$ corresponding to all ordered ℓ-tuples $I = (i_1, i_2, \dots, i_\ell)$, $1 \le i_1 < i_2 < \cdots < i_\ell \le n$, is denoted by $\Lambda^\ell = \Lambda^\ell(\mathbb{R}^n)$. The Grassman algebra $\Lambda = \oplus \Lambda^\ell$ is a graded algebra with respect to the exterior product. For $\alpha = \sum \alpha_I e_I \in \Lambda$ and $\beta = \sum \beta_I e_I \in \Lambda$, the inner product in Λ is given by

$$\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I$$

with summation over all ℓ-tuples $I = (i_1, \dots, i_\ell)$ and all integers $\ell = 0, 1, \dots, n$. We define the Hodge star operator $\star\colon \Lambda \to \Lambda$ by the rule

$$\star 1 = e_1 \wedge e_2 \wedge \cdots \wedge e_n$$

and

$$\alpha \wedge \star \beta = \beta \wedge \star \alpha = \langle \alpha, \beta \rangle (\star 1)$$

for all $\alpha, \beta \in \Lambda^\ell$, $\ell = 1, 2, \dots, n$. Then the norm of $\alpha \in \Lambda$ is

$$|\alpha|^2 = \langle \alpha, \alpha \rangle = \star(\alpha \wedge \star \alpha) \in \Lambda^0 = \mathbb{R}.$$

The Hodge star is an isometric isomorphism on Λ with $\star\colon \Lambda^\ell \to \Lambda^{n-\ell}$ and $\star\star$ $(-1)^{\ell(n-\ell)}\colon \Lambda^\ell \to \Lambda^\ell$.

Throughout this paper Ω is an open subset of $\mathbb{R}^n$. We write $L^p(\Omega, \mathbb{R})$, $0 < p \le \infty$, for the usual L^p space of real-valued functions with respect to Lebesgue measure. The norm of $f \in L^p(\Omega, \mathbb{R})$ is denoted

$$\|f\|_{p,\Omega} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}$$

for $0 < p < \infty$ and

$$\|f\|_{\infty,\Omega} = \operatorname{esssup}\{|f(x)| \,\big|\, x \in \Omega\}.$$

We also write

$$L^p_{\mathrm{loc}}(\Omega, \mathbb{R}) = \cap L^p(\Omega', \mathbb{R})$$

where the intersection is over all Ω' compactly contained in Ω. The Sobolev space $W^1_p(\Omega, \mathbb{R})$ is the subspace of $L^p(\Omega, \mathbb{R})$ whose distributional first derivatives are also in $L^p(\Omega, \mathbb{R})$. Similarly we have the local space $W^1_{p,\mathrm{loc}}(\Omega, \mathbb{R})$.

A differential ℓ-form ω on Ω is a Schwartz distribution on Ω with values in $\Lambda^\ell(\mathbb{R}^n)$. We denote the space of differential ℓ-forms by $\mathcal{D}'(\Omega, \Lambda^\ell)$. We write $L^p(\Omega, \Lambda^\ell)$ for the ℓ-forms $\omega(x) = \sum_I \omega_I(x)\, dx_I = \sum \omega_{i_1 i_2 \cdots i_\ell}(x)\, dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_\ell}$ with $\omega_I \in L^p(\Omega, \mathbb{R})$ for all ordered ℓ-tuples I. Thus $L^p(\Omega, \Lambda^\ell)$ is a Banach space with norm

$$\begin{aligned}
\|\omega\|_{p,\Omega} &= \left(\int_\Omega |\omega(x)|^p \, dx \right)^{1/p} \\
&= \left(\int_\Omega \left(\sum_I |\omega_I(x)|^2 \right)^{p/2} dx \right)^{1/p}.
\end{aligned}$$

Similarly $W^1_p(\Omega, \Lambda^\ell)$ are those differential ℓ-forms on Ω whose coefficients are in $W^1_p(\Omega, \mathbb{R})$. The notations $W^1_{p,\mathrm{loc}}(\Omega, \mathbb{R})$ and $W^1_{p,\mathrm{loc}}(\Omega, \Lambda^\ell)$ are self-explanatory. We denote the exterior derivative by

$$d\colon \mathcal{D}'(\Omega, \Lambda^\ell) \to \mathcal{D}'(\Omega, \Lambda^{\ell+1})$$

for $\ell = 0, 1, 2, \ldots, n$. Its formal adjoint (the Hodge codifferential) is the operator

$$d^\star\colon \mathcal{D}'(\Omega, \Lambda^{\ell+1}) \to \mathcal{D}'(\Omega, \Lambda^\ell)$$

given by

$$d^\star = (-1)^{n\ell+1} \star d \star$$

on $\mathcal{D}'(\Omega, \Lambda^{\ell+1})$, $\ell = 0, 1, \ldots, n$. We require a version of the Poincaré inequality for differential forms.

The details of the following constructions and results can be found in [IL]. Given $d\omega \in L^p(Q, \Lambda^\ell)$, $1 \le p < \infty$, we construct the closed ℓ-form $\omega_Q \in \mathcal{D}'(Q, \Lambda^\ell)$ used below. When $\ell = 0$, ω_Q is the average value of ω over the cube Q. Otherwise it plays a similar role in the Poincaré-Sobolev inequalities.

THEOREM 2.1. *If* $\omega \in \mathcal{D}'(Q, \Lambda^\ell)$ *and* $d\omega \in L^p(Q', \Lambda^{\ell+1})$, *then* $\omega - \omega_Q \in W_p^1(Q, \Lambda^\ell)$ *and*

$$\|\omega - \omega_Q\|_{p,Q} \leq C(n, p) \operatorname{diam} Q \|d\omega\|_{p,Q}. \tag{2.2}$$

for $1 < p < \infty$. *Moreover,*

$$\|\omega_Q\|_{p,Q} \leq C_2(n, p)\|\omega\|_{p,Q}. \tag{2.3}$$

As in the case when ω_Q is an average value, we have the following lemma.

LEMMA 2.4. *There exists a constant* C, *depending only on* n *and* p, *such that*

$$\|\omega - \omega_Q\|_{p,Q} \leq C\|\omega - c\|_{p,Q} \tag{2.5}$$

for all $\omega \in L^p(Q, \Lambda^\ell)$ *and all* $c \in \mathcal{D}'(Q, \Lambda^\ell)$ *with* $dc = 0$. *Here* $1 < p < \infty$. *When* $p = 1$ *we have*

$$\|\omega - \omega_Q\|_{p,Q} \leq C(n) \operatorname{diam} Q\|\omega - c\|_{p,Q}. \tag{2.6}$$

Proof. When c is closed, $c_Q = c$; see [IL]. Using (2.3) we obtain

$$\begin{aligned} \|\omega - \omega_Q\|_{p,Q} &= \|(\omega - c) - (\omega_Q - c_Q)\|_{p,Q} \\ &= \|(\omega - c) - (\omega - c)_Q\|_{p,Q} \\ &\leq (1 + c(n, p))\|\omega - c\|_{p,Q}. \end{aligned}$$

We consider solutions to equations of the form

$$d^\star A(x, d\omega) = 0. \tag{2.7}$$

Here $A\colon \Omega \times \Lambda^\ell(\mathbb{R}^n) \to \Lambda^\ell(\mathbb{R}^n)$ satisfies the assumptions

$$\begin{aligned} |A(x, \xi)| &\leq a|\xi|^{p-1} \\ \langle A(x, \xi), \xi\rangle &\geq |\xi|^p \end{aligned} \tag{2.8}$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^\ell(\mathbb{R}^n)$. Here $a > 0$ is constant and $1 < p < \infty$ is a fixed exponent associated with (2.7). The exponent p will denote this exponent throughout the rest of this paper. A solution to (2.7) is an element of the Sobolev space $W_{p,\mathrm{loc}}^1(\Omega, \Lambda^{\ell-1})$ such that

$$\int_\Omega \langle A(x, d\omega), d\varphi\rangle = 0$$

for all $\varphi \in W_p^1(\Omega, \Lambda^{\ell-1})$ with compact support. Such differential forms are called A-harmonic tensors; see [I] and [IL].

It is important that the Euler-Lagrange equations of certain variational integrals are of the form (2.7).

Let $\eta \in C_0^\infty(\sigma Q)$, $\eta \equiv 1$ in Q and $|\nabla\eta| \le \frac{C}{(\sigma-1)}|Q|^{-1/n}$. Using the test form $\varphi = -\omega\eta^p$ for (2.7) and using the inequalities (2.8) we get Theorem 2.9.

THEOREM 2.9. *Let ω be a solution to* (2.7) *in Ω and let $\sigma > 1$. There exists a constant C, depending only on a, p and n, such that*

$$\|d\omega\|_{p,Q} \le \frac{C}{(\sigma-1)\operatorname{diam} Q}\|\omega - c\|_{p,\sigma Q} \tag{2.10}$$

for all cubes Q with $\sigma Q \subset \Omega$ and all closed forms c.

We extend this result for the positive and negative parts of the form ω to obtain the weak-reverse Hölder inequality (2.15). Our proofs are modifications of the proofs of theorems in [HKM]. We present the details of those modifications. First, if $u \in \Lambda^0(\mathbb{R})$ we write

$$\begin{aligned} u^+ &= \max\{u, 0\}, \\ u^- &= \min\{u, 0\}. \end{aligned}$$

Also we write

$$\begin{aligned} \omega^+ &= \sum_I \omega_I^+ dx_I, \\ \omega^- &= \sum_I \omega_I^- dx_I. \end{aligned}$$

THEOREM 2.11. *Let ω be a solution to* (2.7) *in Ω and $q > 0$. There exists a constant C, depending only on a, p, q and n, such that*

$$\int_\Omega |\omega^+|^q|d\omega^+|^p\eta^p \le C\int_\Omega |\omega^+|^{q+p}|\nabla\eta|^p \tag{2.12}$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. Also, (2.12) *holds with ω^- in place of ω^+.*

Proof. Using the test form $\varphi = -\omega^+\eta^p$ for (2.7) we get

$$\int_\Omega |d\omega^+|^p\eta^p \le C\int_\Omega |\omega^+|^p|\nabla\eta|^p. \tag{2.13}$$

See [HKM], Lemma 3.27. Next, let $T = \sum_I t\,dx_I$ where $t > 0$. Then $\omega - T$ is also a solution to (2.7) and as such satisfies (2.13) as well.

For $t > 0$, consider the sets

$$\begin{aligned} A &= \{x \mid |(\omega - T)^+| \ne 0\}, \\ B &= \bigcup_I \{x \mid (\omega_I - t)^+ > 0\}, \\ C &= \{x \mid |\omega^+| > t\}, \\ D_I &= \{x \mid \omega_I^+ > t\}. \end{aligned}$$

Now $D_I \subset B = A \subset C$ for all I. Hence with $d\nu = |d\omega^+|^p \eta^p$, and using (2.13) we get

$$\begin{aligned}
\int_\Omega |\omega^+|^q \, d\nu &\le C_1 \sum_I \int_\Omega (\omega_I^+)^q \, d\nu \\
&= C_1 \sum_I \left\{ q \int_0^\infty t^{q-1} \int_{D_I} d\nu \, dt \right\} \\
&\le C_1 \sum_I \left\{ q \int_0^\infty t^{q-1} \int_B |d(\omega - T)^+|^p \eta^p \, dt \right\} \\
&\le C_2 \int_0^\infty t^{q-1} \int_B |d(\omega - T)^+|^p \eta^p \, dt \\
&\le C_2 \int_0^\infty t^{q-1} \int_C |\omega^+|^p |\nabla \eta|^p \, dt \\
&\le C_2 \int_\Omega |\omega^+|^{q+p} |\nabla \eta|^p .
\end{aligned}$$

THEOREM 2.14. *Let ω be a solution to* (2.7) *in Ω, $\sigma > 1$ and $0 < s, t < \infty$. Then there exists a constant C, depending only on s, t, a, p, σ and n, such that*

$$\|\omega\|_{s,Q} \le C|Q|^{(t-s)/ts} \|\omega\|_{t,\sigma Q} \tag{2.15}$$

for all cubes Q with $\sigma Q \subset \Omega$.

Proof. It is enough to show that (2.15) holds for ω^+ and ω^-. From the calculations for Theorem 3.34 in [HKM] (with ω^+ in place of u^+ and Q in place of B), using (2.12) and the Moser iteration technique we get

$$\|\omega^+\|_{s,Q} \le C_1 |Q|^{(p-s)/ps} \|\omega^+\|_{p,\sigma Q}.$$

Using Theorem 2 from [IN] we can improve the weak reverse Hölder inequality to get (2.15) for ω^+. The same arguments hold for ω^-.

Next suppose that u is a solution to (2.7) in Ω. At least locally in a ball B, there exists a form $v \in W_q^1(B, \Lambda^{\ell+1})$, $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$A(x, du) = d^\star v.$$

From (2.8) we obtain

$$|du|^{p-1} \leq |d^\star v| \leq a|du|^{p-1}$$

or

$$|du|^p \leq |d^\star v|^q \leq a^q |du|^p.$$

If A is invertible, then v satisfies the following conjugate equation

$$dA^{-1}(x, d^\star v) = 0 \tag{2.16}$$

where $A^{-1}(x, \xi)$ has associated exponent q.

Notice that $w = \star v$ satisfies

$$d^\star A^{-1}(x, dw) = 0.$$

Since the Hodge star operator is an isometry, v satisfies versions of (2.10) and (2.15).

Definition 2.17. When u and v satisfy (1.2) in Ω, and A^{-1} exists in Ω, we call u and v conjugate A-harmonic tensors in Ω.

Of particular interest are p, q-harmonic functions mentioned in the introduction; see (1.1).

We remark that the polar angle θ is p-harmonic for all $1 < p < \infty$ in the domain $\{re^{i\theta} | r > 0, -\pi < \theta < \pi\}$.

The quasi-radial p, q-harmonic tensors in $\mathbb{R}^2$ are described in [A1] and [A2]. These tensors are represented by functions of the form $u = r^k f(\theta)$ and $w = r^\ell g(\theta)$ where $k, \ell \in \mathbb{R}$ and r and θ are the usual polar coordinates in $\mathbb{R}^2$. It is necessary that

$$p(k-1) = q(\ell - 1). \tag{2.18}$$

We will use the following examples later.

Examples 2.19.

(1) $p = q = 2$. Here u and w are conjugate harmonic functions. Notice that $\log r$ and θ are conjugate harmonic functions in $\Omega_1 = \{re^{i\theta} | r > 0 \text{ and } -\pi < \theta < \pi\}$. The only quasi-radial conjugate harmonic functions must have $k = \ell$ and are of the form $r^k \cos k\theta$ and $r^k \sin k\theta$.

(2) If $p, q \neq 2, k = (p-2)/(p-1)$ and $\ell = 0$, then $u = (p-1)r^{(p-2)/(p-1)}/(p-2)$ and $v = - \star \theta$ are conjugate p, q-harmonic tensors in Ω_1.

(3) If $p \neq 2$, $k = 0$ and $\ell = (q-2)/(q-1)$, then $u = \theta$ and $v = - \star (q-1)r^{(q-2)/(q-1)}/(q-2)$ are conjugate p, q-harmonic tensors in Ω_1.

(4) If $q > 2$ and (2.18) holds, then there exists conjugate p, q-harmonic tensors in Ω_1 of the form

$$\begin{aligned} u &= r^k f(\varphi), \\ v &= -\star r^\ell g(\varphi) \end{aligned}$$

where $f(\varphi)$ and $g(\varphi)$ are bounded.

Notice that if u and w are conjugate p, q-harmonic functions in $\mathbb{R}^2$, then they can be embedded as conjugate p, q-harmonic tensors in $\mathbb{R}^n$. There are many ways to do this. One such is as follows:

$$\begin{aligned} u(x_1, x_2, \ldots, x_n) &= u(x_1, x_2), \\ v(x_1, x_2, \ldots, x_n) &= -w(x_1, x_2)\, dx_1 \wedge dx_2. \end{aligned}$$

The following is an interesting example of conjugate harmonic tensors in $\mathbb{R}^3$:

$$\begin{aligned} u(x) &= 3|x|^{-1}, \\ v(x) &= \sum_{i<j}^{3} v_{ij}(x) dx_i \wedge dx_j \end{aligned}$$

where

$$v_{ij}(x) = x_i x_j (x_i^4 - x_j^4) \left[|x| \prod_{k<\ell}^{3} (x_k^2 + x_\ell^2) \right]^{-1}.$$

See [D1] for this and other examples.

The study of equations of the form (2.7) is intimately connected with and partially motivated by the theory of quasiconformal and quasiregular mappings. In this case (2.7) is the Euler-Lagrange equation for a functional defined in terms of the exterior powers of the matrix dilatation of the quasiregular mapping. See [BI], [V], [IM], [I], [HKM] and [N].

Applications of the main results of this paper are given for quasiregular mappings in the last section.

3. The local norm comparison

THEOREM 3.1. *Let u and v be conjugate A-harmonic tensors in $\Omega \subset \mathbb{R}^n$, $\sigma > 1$, and $0 < s, t < \infty$. There exists a constant C, independent of u and v, such that*

$$\|u - u_Q\|_{s,Q} \le C|Q|^\beta \|v - c_1\|_{t,\sigma Q}^{q/p} \tag{3.2}$$

and

$$\|v - v_Q\|_{t,Q} \le C|Q|^{-\beta p/q} \|u - c_2\|_{s,\sigma Q}^{p/q}$$

for all cubes Q with $\sigma Q \subset \Omega$. Here c_1 is any form in $W^1_{p,\mathrm{loc}}(\Omega, \Lambda)$ with $d^\star c_1 = 0$, c_2 is any form in $W^1_{q,\mathrm{loc}}(\Omega, \Lambda)$ with $dc_2 = 0$ and

$$\beta = \frac{1}{s} + \frac{1}{n} - \frac{q}{p}\left(\frac{1}{t} + \frac{1}{n}\right).$$

Proof. Choose ρ so that $\rho^3 = \sigma$. We prove the first inequality in (3.2). The second follows similarly. First we use the weak reverse Hölder inequality (2.15) and the Poincaré inequality (2.2) to get

$$\begin{aligned} \|u - u_{\sigma Q}\|_{s,Q} &\leq C|Q|^{(p-s)/sp}\|u - u_{\sigma Q}\|_{p,\rho Q} \\ &\leq C \text{ diam } Q|Q|^{(p-s)/ps}\|du\|_{p,\rho Q}. \end{aligned}$$

Now using the inequality $|du|^p \leq |\star dv|^q$ we get

$$\|u - u_{\sigma Q}\|_{s,Q} \leq C|Q|^{(pn-sn+sp)/spn}\|d(\star v)\|^{q/p}_{q,\rho Q}.$$

Next, v satisfies (2.16) so applying the Caccioppoli estimate (2.10) and using (2.15) again, we obtain

$$\begin{aligned} \|u - u_{\sigma Q}\|_{s,Q} &\leq C|Q|^{(pn-sn+sp-qs)/spn}\| \star v - \star c\|^{q/p}_{q,\rho^2 Q} \\ &\leq C|Q|^{\beta}\|v - c\|_{t,\rho^3 Q}. \end{aligned}$$

For a weighted version of Theorem 3.1 see [D2].

4. The global result in John domains

In this section we restrict our attention to 0-forms $u \in \mathcal{D}'(\Omega, \Lambda^0)$. As such the conjugate forms v are 2-forms in $\Omega \subset \mathbb{R}^n$.

Definition 4.1. We call a bounded domain Ω δ-John, $\delta > 0$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ so that

$$d(\xi, \partial\Omega) \geq \delta|x - \xi|$$

for each $\xi \in \gamma$. Here $d(\xi, \partial\Omega)$ is the Euclidean distance between ξ and $\partial\Omega$.

Bounded quasiballs and bounded uniform domains are John domains. See [MS] and [M]. In such domains we have the following global result. For given n, p and q we write

$$\Phi(t) = \frac{npt}{nq + t(q - p)}.$$

THEOREM 4.2. *Let $u \in \mathcal{D}'(\Omega, \Lambda^0)$ and $v \in \mathcal{D}'(\Omega, \Lambda^2)$ be conjugate A-harmonic tensors. If Ω is δ-John, $q \leq p$, $v - c \in L^t(\Omega, \Lambda^2)$ and*

$$s = \Phi(t) \tag{4.3}$$

then $u - u_{Q_0} \in L^s(\Omega, \Lambda^0)$ and moreover, there exists a constant C, independent of u and v, such that

$$\|u - u_{Q_0}\|_{s,\Omega} \leq C\|v - c\|_{t,\Omega}^{q/p}. \tag{4.4}$$

Here c is any form in $W^1_{q,\mathrm{loc}}(\Omega, \Lambda)$ with $d^\star c = 0$ and Q_0 is the distinguished cube of Lemma 4.5.

We remark that since $|\Omega| < \infty$ we can increase t or decrease s using Hölder's inequality.

To facilitate the proof of Theorem 4.2, we use the following lemmas.

LEMMA 4.5. *Each Ω has a modified Whitney cover of cubes $W = \{Q_i\}$ which satisfy*

$$\bigcup_i Q_i = \Omega, \qquad \sum_{Q \in W} \chi_{\sqrt{\frac{5}{4}}Q} \leq N\chi_\Omega \tag{4.6}$$

for all $x \in \mathbb{R}^n$ and some $N > 1$ and if $Q_i \cap Q_j \neq \emptyset$, then there exists a cube R ($\notin W$) in $Q_i \cap Q_j$ such that $Q_i \cup Q_j \subset NR$. Moreover if Ω is δ-John, then there is a distinguished cube $Q_0 \in W$ which can be connected with every cube $Q \in W$ by a chain of cubes $Q_0, Q_1, \ldots, Q_k = Q$ from W and such that $Q \subset \rho Q_i$, $i = 0, 1, 2, \ldots, k$, for some $\rho = \rho(n, \delta)$.

Proof. All except the last assertion follows immediately from the properties of a usual Whitney cover $\{W_i\}$ (see [Ste]) if we let $Q_i = \sqrt{5/4}W_i$. If Ω is δ-John, let Q_0 be a member of W containing x_0. Given $Q \in W$ let x be the center of Q. By Definition 4.1 there is a distinguished curve $\gamma \subset \Omega$ joining x_0 to x. The chain $Q_0, Q_1, \ldots, Q_k$ arises as those cubes $Q_i \in W$ such that $\gamma \cap Q_i \neq \emptyset$. It is easy to see that

$$Q \subset \rho Q_i \tag{4.7}$$

for $i = 0, 1, 2, \ldots, k$ with $\rho = 4\sqrt{n}(5 + 1/\delta)$. Also

$$\max(|Q_i|, |Q_{i+1}|) \leq N|Q_i \cap Q_{i+1}| \tag{4.8}$$

for $i = 0, 1, 2, \ldots, k - 1$.

LEMMA 4.9. *Suppose that $0 < s < \infty$, Ω is δ-John, W is the Whitney decomposition of Ω and u is a distribution, $u \in \mathcal{D}'(\Omega, \Lambda^0)$. If for each cube $Q \in W$ there exists a constant b_Q such that*

$$\|u - u_Q\|_{s,Q}^s \le b_Q, \tag{4.10}$$

then there exists a constant C, depending only on s, n and δ, such that

$$\|u - u_{Q_0}\|_{s,\Omega}^s \le C \sum_{Q \in W} b_Q. \tag{4.11}$$

To prove Lemma 4.9 we need the next lemma. A proof appears in [B].

LEMMA 4.12. *If $1 \le s < \infty$, $0 < \rho < \infty$, $\{Q\}$ is an arbitrary collection of cubes in $\mathbb{R}^n$ and $\{a_Q\}$ are nonnegative numbers, then there is a constant C, depending only on s, n and ρ, such that*

$$\|\sum_Q a_Q \chi_{\rho Q}\|_{s,\mathbb{R}^n} \le C \|\sum_Q a_Q \chi_Q\|_{s,\mathbb{R}^n}. \tag{4.13}$$

We now prove Lemma 4.9. Assume that W is a cover of Ω of the form described in Lemma 4.5. Using the properties (4.6) we get

$$\|u - u_{Q_0}\|_{s,\Omega}^s \le 2^s \sum_{Q \in W} \|u - u_Q\|_{s,Q}^s + 2^s \sum_{Q \in W} \|u_{Q_0} - u_Q\|_{s,Q}^s. \tag{4.14}$$

We can estimate the first sum on the right-hand side in (4.14). With (4.6),

$$\sum_{Q \in W} \|u - u_Q\|_{s,Q}^s \le \sum_{Q \in W} b_Q. \tag{4.15}$$

To estimate the second sum in (4.14) we first fix $Q \in W$ and let $Q_0, Q_1, \dots, Q_k = Q$ be the chain from Lemma 4.5. Using (4.8), we have

$$\begin{aligned} |u_{Q_i} - u_{Q_{i+1}}|^s &= |Q_i \cap Q_{i+1}|^{-1} \|u_{Q_i} - u_{Q_{i+1}}\|_{s,Q_i \cap Q_{i+1}}^s \\ &\le N 2^s \sum_{j=i}^{i+1} |Q_j|^{-1} \|u - u_{Q_j}\|_{s,Q_j}^s \\ &\le C \sum_{j=i}^{i+1} |Q_j|^{-1} b_{Q_j} \end{aligned}$$

for $i = 0, 1, \dots, k-1$. Next, by (4.7),

$$|u_{Q_i} - u_{Q_{i+1}}|^s \chi_Q(x) \le C \sum_{j=i}^{i+1} |Q_j|^{-1} b_{Q_j} \chi_{\rho Q_j}(x)$$

for $i = 0, 1, \dots, k-1$. And so, by the triangle inequality,

$$|u_{Q_0} - u_{Q_k}|\chi_{Q_k}(x) \le C \sum_{Q\in W} |Q|^{-1/s} b_Q^{1/s} \chi_{\rho Q}(x)$$

for all $x \in \mathbb{R}^n$. Using Lemma 4.5 we get

$$\sum_{Q\in W} \|u_{Q_0} - u_Q\|_{s,Q}^s \le C \|\sum_{Q\in W} |Q|^{-1/s} b_Q^{1/s} \chi_{\rho Q}\|_{s,\mathbb{R}^n}^s. \tag{4.16}$$

When $0 < s \le 1$, (4.16) becomes

$$\begin{aligned}\sum_{Q\in W} \|u_{Q_0} - u_Q\|_{s,Q}^s &\le C \sum_{Q\in W} |Q|^{-1} |\rho Q| b_Q \\ &= C \sum_{Q\in W} b_Q.\end{aligned}$$

Now (4.11) follows from this and (4.15). On the other hand if $1 \le s < \infty$, then we apply Lemma 4.12 and (4.16) becomes

$$\begin{aligned}\sum_{Q\in W} \|u_{Q_0} - u_Q\|_{s,Q}^s &\le C \left\| \sum_{Q\in W} |Q|^{-1/s} b_Q^{1/s} \chi_Q \right\|_{s,\mathbb{R}^n}^s \\ &\le C \sum_{Q\in W} b_Q.\end{aligned}$$

Again we obtain (4.11).

We are now ready to prove Theorem 4.2. If s and t are related by (4.3), then with $\sigma = \sqrt{5}/2$, (3.2) becomes

$$\|u - u_Q\|_{s,Q}^s \le C \|v - c\|_{t,\sigma Q}^{qs/p}.$$

Choosing $b_Q = \|v - c\|_{t,\sigma Q}^{qs/p}$, we conclude from Lemma 4.9 that

$$\|u - u_{Q_0}\|_{s,\Omega}^s \le C \sum_{Q\in W} b_Q.$$

Now when $s = \Phi(t)$ and $q \le p$ it follows that $qs/pt \ge 1$. In this case we conclude from above that

$$\|u - u_{Q_0}\|_{s,\Omega}^s \le C \|v - c\|_{t,\Omega}^{qs/p}.$$

This completes the proof of Theorem 4.2.

We now show that the condition (4.3) is essentially best possible at least when $n = 2$. Here we use the conjugate p, q-harmonic tensors given by Example 2.19 (4) with $k, \ell < 0$. It is easy to see that $\|u\|_{s,\Omega_1} < \infty$ if and only if $s < -2/k$ and $\|v\|_{t,\Omega_1} < \infty$ if and only if $t < -2/\ell$. Furthermore with the condition $p(k-1) = q(\ell-1)$ we have $\Phi(-2/\ell) = -2/k$.

Remark 4.17. Let $q \leq p$ and $0 < t_0 < 2q/(p-q)$. The above mentioned p, q-harmonic pair of quasi-radial solutions in Ω_1 satisfies

$$\|u\|_{s,\Omega} \leq C\|v\|_{t,\Omega}^{q/p}$$

for all t, $0 < t < t_0$ but only for $s(t) \leq \Phi(t) + \varepsilon(t, t_0)$ where $\varepsilon(t, t_0) \to 0$ as $t \to t_0$. To see this choose $\ell = -2/t_0$ so that $t_0 = -2/\ell$ and define

$$\begin{aligned} \varepsilon(t, t_0) &= -2/k - \Phi(t) \\ &= \Phi(t_0) - \Phi(t). \end{aligned}$$

5. Global results in L^s-averaging domains

Definition 5.1. We call Ω L^s-averaging, $s \geq 1$, if there exists a constant M such that

$$|Q_0|^{-1}\|U - U_{Q_0}\|_{s,\Omega}^s \leq M \sup_{Q \subset \Omega} |Q|^{-1}\|U - U_Q\|_{s,Q}^s \tag{5.2}$$

for some cube $Q_0 \subset \Omega$ and for all $U \in L^s_{\text{loc}}(\Omega, \mathbb{R})$. Here the supremum is over all cubes $Q \subset \Omega$.

These domains were introduced in [St]. See also [H]. It turns out that if Ω is δ-John, then Ω is L^s-averaging for all s. See [St].

In [St], condition (5.2) is characterized by the global L^s-integrability of the quasi-hyperbolic metric $k(x, x_0)$. Theorem 5.3 follows from results in [St].

THEOREM 5.3. *If Ω is L^s-averaging and $\sigma \geq 1$, then there exists a constant N such that*

$$|Q_0|^{-1}\|U - U_{Q_0}\|_{s,\Omega}^s \leq N \sup_{\sigma Q \subset \Omega} |Q|^{-1}\|U - U_Q\|_{s,Q}^s \tag{5.4}$$

for all $U \in L^s_{\text{loc}}(\Omega, \mathbb{R})$. Here the supremum is over all cubes Q with $\sigma Q \subset \Omega$.

THEOREM 5.5. *Suppose that $u \in \mathcal{D}'(\Omega, \Lambda^0)$ and $v \in \mathcal{D}'(\Omega, \Lambda^2)$ are conjugate A-harmonic tensors in Ω and that Ω is L^s-averaging. If $v - c \in L^t(\Omega, \Lambda^2)$, $q < p$ and*

$$t = \frac{nq}{p-q}, \tag{5.6}$$

then $u - u_{Q_0} \in L^s(\Omega, \Lambda^0)$ and there exists a constant C, independent of u and v, such that

$$\|u - u_{Q_0}\|_{s,\Omega} \leq C|Q_0|^{1/s}\|v - c\|_{t,\Omega}^{q/p}. \tag{5.7}$$

If $p = q$, then (5.7) holds with $t = \infty$. Here c is any form in $\mathcal{D}'(\Omega, \Lambda^2)$ with $d^\star c = 0$.

Proof. By using hypothesis (5.6) the result (3.2) becomes

$$|Q|^{-1/s}\|u-u_Q\|_{s,Q} \le C_1\|v-c\|_{t,2Q}^{q/p} \tag{5.8}$$

for all cubes Q with $2Q \subset \Omega$. Since Ω is L^s-averaging we can use (5.4) combined with (5.8) to obtain

$$\begin{aligned} |Q_0|^{-1/s}\|u-u_{Q_0}\|_{s,\Omega} &\le C_2 \sup_{2Q\subset\Omega} \|v-c\|_{t,2Q}^{q/p} \\ &\le C_2\|v-c\|_{t,\Omega}^{q/p}, \end{aligned}$$

This is (5.7) for $q < p$. Next if $p = q$, then we obtain from (3.2), for $2Q \subset \Omega$,

$$\begin{aligned} \|u-u_Q\|_{s,Q} &\le C_3|Q|^{(1/s-1/t)}\|v-c\|_{t,2Q} \\ &\le C_3|Q|^{1/s}\|v-c\|_{\infty,\Omega}. \end{aligned}$$

In particular, it follows that if $f = u + iv$ is analytic, or more generally quasiregular, in an L^s-averaging domain Ω, and if $v \in L^\infty(\Omega,\mathbb{R})$, then $u - u_{Q_0} \in L^s(\Omega,\mathbb{R})$. In an L^s-averaging domain we then get (5.7) with $t = \infty$.

If we invert the relationship between s and t hypothesized in Theorem 4.2, then we get

$$t = \Phi^{-1}(s) = \frac{nqs}{np + s(p-q)}.$$

When $q \le p$ and $s \ge 0$, $\Phi^{-1}(s)$ is increasing. Moreover $\lim_{s\to\infty}\Phi^{-1}(s) = nq/(p-q)$, namely (5.6). Now in a John domain Ω, if $v - c \in L^{nq/(p-q)}(\Omega,\Lambda^2)$, then $u - u_{Q_0} \in L^s(\Omega,\Lambda^0)$ for all $s < \infty$. (Simple examples show that $s = \infty$ is false in general.) This is consistent with the fact that a John domain is L^s-averaging for all s.

Remark 5.9. When $n = 2$, condition (5.6) is sharp. Examples include quasi-radial p, q-harmonic functions in planar domains with cusps.

6. Lipschitz conditions and BMO

Definition 6.1. Assume that $\omega \in L^1_{\text{loc}}(\Omega,\Lambda^\ell)$, $\ell = 0, 1, \ldots, n$. We write $\omega \in \mathsf{BMO}(\Omega,\Lambda^\ell)$ if

$$\sup_{\sigma Q\subset\Omega} |Q|^{-1}\|\omega-\omega_Q\|_{1,Q} < \infty \tag{6.2}$$

for some $\sigma > 1$. Similarly, we write $\omega \in \text{loc Lip}_k(\Omega,\Lambda^\ell)$, $0 < k \le 1$, if

$$\sup_{\sigma Q\subset\Omega} |Q|^{-(n+k)/n}\|\omega-\omega_Q\|_{1,Q} < \infty \tag{6.3}$$

for some $\sigma > 1$. Also we denote the expressions in (6.2) and (6.3) by $\|\omega\|_{\text{loc Lip}_k,\Omega}$ where $k \ge 0$. When ω is a 0-form, (6.2) is the classical definition of $\mathsf{BMO}(\Omega)$. It turns out that these spaces are independent of the expansion factor σ.

A continuous 0-form which satisfies (6.3) is in the usual space loc $\mathrm{Lip}_k(\Omega)$. This result is in [Me]. It, along with the natural connection to the BMO space, inspires (6.3) for forms. We see below that in many ways this definition is natural. The usual local Lipschitz space, $\operatorname{loc}\mathrm{Lip}_k(\Omega, \mathbb{R})$, was introduced in [GM].

Furthermore we write $\mathrm{Lip}_k(\Omega, \Lambda^\ell)$ for those forms whose coefficients are in the usual Lipschitz space with exponent k and write $\|\omega\|_{\mathrm{Lip}_k,\Omega}$ for this norm.

THEOREM 6.4. *Let ω be a solution to* (2.7). *The following are equivalent*:

(*a*) $\omega \in \mathrm{BMO}(\Omega, \Lambda)$;
(*b*) $\sup\{|Q|^{(p-n)/pn}\|d\omega\|_{p,Q} \,|\, \sigma Q \subset \Omega\} < \infty$ *for some* $\sigma > 1$.

Similarly the following are equivalent:

(*c*) $\omega \in \operatorname{loc}\mathrm{Lip}_k(\Omega, \Lambda)$;
(*d*) $\sup\{|Q|^{(p-pk-n)/pn}\|d\omega\|_{p,Q} \,|\, \sigma Q \subset \Omega\} < \infty$ *for some* $\sigma > 1$.

Proof. Assume (a) or (c). Then by (2.10) and (2.15),

$$\begin{aligned}\|d\omega\|_{p,Q} &\le C_1(n,p)|Q|^{-1/n}\|\omega - \omega_{2Q}\|_{p,\sqrt{2}Q}\\ &\le C_2(n,p)|Q|^{(n-p-np)/pn}\|\omega - \omega_{2Q}\|_{1,2Q}.\end{aligned}$$

The results (b) and (d) follow by taking the supremum over all cubes Q with $2\sigma Q \subset \Omega$. Next assume (b) or (d).

By Hölder's inequality and (2.2),

$$\begin{aligned}\|\omega - \omega_Q\|_{1,Q} &\le |Q|^{(p-1)/p}\|\omega - \omega_Q\|_{p,Q}\\ &\le C_3(n,p)|Q|^{(pn+p-n)/pn}\|d\omega\|_{p,Q}.\end{aligned}$$

We now take the supremum over all cubes Q with $2Q \subset \Omega$ to obtain (a) or (c).

In view of Theorem 6.4 we get the following results.

COROLLARY 6.5. *Suppose that ω is a solution to* (2.7) *in* Ω.

If the coefficients of ω, ω_I, are in $\mathrm{BMO}(\Omega)$, *then* $\omega \in \mathrm{BMO}(\Omega, \Lambda)$. *If the coefficients of ω are in* loc $\mathrm{Lip}_k(\Omega)$, *then* $\omega \in \operatorname{loc}\mathrm{Lip}_k(\Omega, \Lambda)$.

Conversely, if $\omega \in \mathrm{BMO}(\Omega, \Lambda)$, *then the coefficients of* $\omega - \omega_Q$ *are in* $\mathrm{BMO}(Q)$, *and if* $\omega \in \operatorname{loc}\mathrm{Lip}_k(\Omega, \Lambda)$, *then* $\omega - \omega_Q \in \mathrm{Lip}_k(Q, \Lambda)$ *for all cubes* $Q \subset \Omega$.

THEOREM 6.6. *Suppose that* $0 \le k, \ell \le 1$ *satisfy* $p(k-1) = q(\ell - 1)$. *There exists a constant C such that*

$$\begin{aligned}\frac{\|u\|^p_{\operatorname{loc}\mathrm{Lip}_k,\Omega}}{C} &\le \|v\|^q_{\operatorname{loc}\mathrm{Lip}_\ell,\Omega}\\ &\le C\|u\|^p_{\operatorname{loc}\mathrm{Lip}_k,\Omega}\end{aligned}\tag{6.7}$$

for all conjugate A-harmonic tensors u and v in Ω.

Proof. We prove the first inequality from the local result (3.2). The second inequality follows similarly. From Definition 6.1,

$$\|u\|_{\mathrm{loc\,Lip}_k,\Omega} \le C \sup_{2Q\subset\Omega} |Q|^{-(n+k)/n}\|u-u_Q\|_{1,Q}. \tag{6.8}$$

Next, using the condition $p(k-1)=q(\ell-1)$, (3.2) becomes

$$|Q|^{-(n+k)/n}\|u-u_Q\|_{1,Q} \le C_1(|Q|^{-(n+\ell)/n}\|v-c\|_{1,2Q})^{q/p} \tag{6.9}$$

for all cubes Q with $2Q\subset\Omega$. Now choose c so that $\star c = (\star v)_{2Q}$. The first inequality follows from (6.8) and (6.9).

Now we have the following global result over a cube.

THEOREM 6.10. *There exists a constant C such that*

$$\begin{aligned}\|u-u_Q\|_{\mathrm{Lip}_k,Q}/C &\le \|\star v - c\|_{\mathrm{Lip}_k,Q}\\ &\le C\|u-u_Q\|_{\mathrm{Lip}_k,Q}\end{aligned}$$

for all conjugate A-harmonic tensors u and v in a cube $Q\subset\mathbb{R}^n$. Here $0<k\le 1$ and $\star c = \star(\star v)_Q$.

For conjugate A-harmonic functions in $\mathbb{R}^2$ we have a global version of Theorem 6.10.

Definition 6.11. We call Ω a $\mathrm{Lip}_{k,k'}$-extension domain, $0<k'\le k\le 1$, if the following equivalent conditions are satisfied. Here $\sigma>1$.
(a) Given (6.13), there exists a constant M, depending only on n, k, σ and N, such that each pair of points $x_1, x_2\in\Omega$ can be joined by a continuous curve $\gamma\subset\Omega$ with

$$\int_\gamma d(\gamma(s),\partial\Omega)^{k-1}\,ds \le M|x_1-x_2|^{k'}. \tag{6.12}$$

(b) Given (6.12), there exists a constant N, depending only on n, k, σ and M, such that

$$\|U\|_{\mathrm{Lip}_{k'},\Omega} \le N \sup_{\sigma B\subset\Omega}\|U\|_{\mathrm{Lip}_k,B} \tag{6.13}$$

for all $U\colon\Omega\to\mathbb{R}$. Here the supremum is over all balls B with $\sigma B\subset\Omega$.

The class of $\mathrm{Lip}_{k,k}$-extension domains is wide including quasiballs and uniform domains. Certain internal cusps however are ruled out; see [GM] and [L].

THEOREM 6.14. *Suppose that $\Omega \subset \mathbb{R}^2$ is a $\mathrm{Lip}_{k,k'}$-extension domain. There is a constant C such that*

$$\|u\|_{\mathrm{Lip}_{k'},\Omega} \le C\|v\|_{\mathrm{loc\,Lip}_{\ell},\Omega}$$

for all conjugate A-harmonic tensors u and v in Ω.

Remark 6.15. The quasi-radial p, q-harmonic functions in Example 2.19 again show that Theorem 6.7 is sharp with respect to k and ℓ in the plane.

7. Quasiregular mappings

We record here some interesting inequalities for conjugate A-harmonic tensors that arise from a quasiregular mapping $f = (f^1, f^2, \ldots, f^n)$.

The following function u and two-form v are conjugate A-harmonic tensors, (see [IM]):

$$\begin{aligned} u &= f^1, \\ v &= \star f^2 df^3 \wedge \cdots \wedge df^n. \end{aligned}$$

We state this special case of Theorem 4.2 over the unit ball $\mathbb{B}^n$ for simplicity.

COROLLARY 7.1. *There is a constant C such that*

$$\|f^1 - f^1(0)\|_{s,\mathbb{B}} \le C\|f^2 df^3 \wedge \cdots \wedge df^n\|^{1/(n-1)}_{sn/[n(n-1)+s(n-2)],\mathbb{B}^n} \tag{7.2}$$

for all K-quasiregular $f = (f^1, f^2, \ldots, f^n)\colon \mathbb{B}^n \to \mathbb{R}^n$. Here C depends only on s, n and K.

When $n = 2$, (7.2) reduces to

$$\|f^1 - f^1(0)\|_{s,\mathbb{B}^2} \le C\|f^2\|_{s,\mathbb{B}^2}.$$

Since a coclosed form may be added inside the norm on the right-hand side, we also have

$$\|f^1 - f^1(0)\|_{s,\mathbb{B}^2} \le C\|f^2 - f^2(0)\|_{s,\mathbb{B}^2}. \tag{7.3}$$

This result appears in [IN] where in fact it is shown to hold in all dimensions. As such we can replace $\|f^1 - f^1(0)\|_{s,\mathbb{B}^n}$ in (7.2) by $\|f - f(0)\|_{s,\mathbb{B}^n}$. In particular, in dimension 3, (7.2) becomes

$$\|f - f(0)\|_{s,\mathbb{B}^3} \le C\sum_{j=1}^{3}\left\|f_2\frac{\partial f_3}{\partial x_j}\right\|^{1/2}_{3s/(s+6),\mathbb{B}^3}. \tag{7.4}$$

COROLLARY 7.5. *If Ω is an L^s-averaging domain, then there is a constant C such that*

$$\|f - f(0)\|_{s,\Omega} \leq C\|f^2 df^3 \wedge \cdots \wedge df^n\|^{1/(n-1)}_{n/(n-2),\Omega} \tag{7.6}$$

for all K-quasiregular mappings $f = (f^1, f^2, \ldots, f^n)\colon \Omega \to \mathbb{R}^n$. Here C depends only on n, s and K.

In 3-dimensions, notice that 7.6 reads

$$\|f - f(0)\|_{s,\Omega} \leq C \sum_{j=1}^{3} \left\| f_2 \frac{\partial f_3}{\partial x_j} \right\|^{1/2}_{3,\Omega}.$$

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