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CONVERGENCE IN THE CESÀRO SENSE OF ERGODIC OPERATORS ASSOCIATED WITH A FLOW

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ABSTRACT. We study the a.e. convergence of the Cesàro-$(1+\alpha)$ ergodic averages and the a.e. existence in the Cesàro-α sense of the ergodic Hilbert transform associated with a Cesàro bounded flow and $-1 < \alpha \leq 0$.

1. Introduction

Let $(X, \mathcal{F}, \nu)$ be a finite measure space. By a flow $\{\tau_t \colon t \in \mathbb{R}\}$ we mean a group of measurable transformations $\tau_t \colon X \to X$ with τ_0 the identity and $\tau_{t+s} = \tau_t \circ \tau_s$ $(t, s \in \mathbb{R})$. The flow is said to be measure-preserving if the τ_t are measure-preserving, i.e., if $\nu(\tau_{-t}E) = \nu(E)$ for all $E \in \mathcal{F}$. The flow is said to be nonsingular if $\nu(\tau_{-t}E) = 0$ for all $t \in \mathbb{R}$ and all $E \in \mathcal{F}$ with $\nu(E) = 0$. Finally, the flow is said to be measurable if the map $(x, t) \to \tau_t x$ from $X \times \mathbb{R}$ into X is $\widetilde{\mathcal{F}}$-$\mathcal{F}$-measurable where $\widetilde{\mathcal{F}}$ is the completion of the product-σ-algebra $\mathcal{F} \otimes \mathcal{B}$ of $\mathcal{F}$ with the Borel sets, and the completion is taken with respect to the product measure of ν on $\mathcal{F}$ and the Lebesgue measure on $\mathcal{B}$. Analogously we can define what we mean by a semiflow $\{\tau_t \colon t > 0\}$, a measure-preserving semiflow, a nonsingular semiflow and a measurable semiflow.

Y. Deniel studied in [4] the convergence of the Cesàro-$(1 + \alpha)$ ((C,$1 + \alpha$)) ergodic averages, $-1 < \alpha < 0$, associated with a measure-preserving semiflow on a probability space $(\Omega, \mathcal{F}, \mu)$. More precisely, he proved the following result.

THEOREM A ([4]). *Let $\{\tau_t \colon t > 0\}$ be a measure-preserving semiflow of a probability space $(\Omega, \mathcal{F}, \mu)$. Let $-1 < \alpha < 0$, $\frac{1}{1+\alpha} < p < \infty$ and $f \in L^p(d\mu)$. Then, the (C,$1 + \alpha$) ergodic averages*

$$A^+_{T,1+\alpha} f(x) = \frac{1}{T^{1+\alpha}} \int_0^T f(\tau_t x)(T - t)^\alpha \, dt$$

converge, when $T \to \infty$, almost everywhere and in the $L^p(d\mu)$-norm.

Theorem A does not hold in the limit case $p = \frac{1}{1+\alpha}$ [4]. However a positive result was obtained in [2] in this limit case. Their result is the following.

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THEOREM B ([2]). *Let* $\{\tau_t\colon t > 0\}$, $(\Omega, \mathcal{F}, \mu)$ *and* α *be as in Theorem A. Then,* $\lim_{T\to\infty} A^+_{T,1+\alpha} f(x)$ *exists a.e. for all* f *in the Lorentz space* $L_{\frac{1}{1+\alpha},1}(d\mu) = \{f\colon \|f\|_{\frac{1}{1+\alpha},1;\mu} = \int_0^\infty \left(\lambda_f(t)\right)^{1+\alpha} dt < \infty\}$, *where* $\lambda_f(t) = \mu(\{x\colon |f(x)| > t\})$ *is the distribution function of* f.

On the other hand, Lorente Domínguez and Martín-Reyes studied in [6], the convergence of the ergodic averages

$$A_{T,1} f(x) = \frac{1}{2T} \int_{-T}^{T} f(\tau_t x)\, dt,$$

and the ergodic Hilbert transform $Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x)$, where

$$H_\varepsilon f(x) = \int_{\varepsilon<|t|<1/\varepsilon} \frac{f(\tau_t x)}{t}\, dt,$$

associated with a Cesàro bounded flow on a finite measure space $(X, \mathcal{F}, \nu)$ (notice that the flow does not need to preserve the measure ν). The result proved in [6] is as follows.

THEOREM C ([6]). *Let* $(X, \mathcal{F}, \nu)$ *be a finite measure space,* $1 \le p < \infty$ *and let* $\{\tau_t\colon t \in \mathbb{R}\}$ *be a nonsingular measurable flow on* X *such that for some positive constant* C *and all* $f \in L^p(d\nu)$,

$$\sup_{T>0} ||A_{T,1} f||_{p;\nu} \le C||f||_{p;\nu}.$$

(i) *If* $1 < p < \infty$ *and* $f \in L^p(d\nu)$, *then* $\lim_{T\to\infty} A_{T,1} f(x)$ *and* $\lim_{\varepsilon\to 0} H_\varepsilon f(x)$ *exist a.e. and in the* $L^p(d\nu)$*-norm.*
(ii) *If* $p = 1$ *and* $f \in L^1(d\nu)$, *then* $\lim_{T\to\infty} A_{T,1} f(x)$ *exists a.e. and in the* $L^1(d\nu)$*-norm and* $\lim_{\varepsilon\to 0} H_\varepsilon f(x)$ *exists a.e.*

The aim of this paper is to study, for $-1 < \alpha \le 0$, the convergence of the (C,$1+\alpha$) ergodic averages and the existence in the Cesàro-α ((C,α)) sense of the ergodic Hilbert transform in the setting of Theorem C, i.e., for Cesàro bounded flows. More precisely, for the (C,$1+\alpha$) ergodic averages, we shall prove the following theorem.

THEOREM 1.1. *Let* $(X, \mathcal{F}, \nu)$ *be a finite measure space,* $-1 < \alpha \le 0$ *and* $\frac{1}{1+\alpha} \le p < \infty$. *Let* $\{\tau_t\colon t \in \mathbb{R}\}$ *be a nonsingular measurable flow on* X *such that for some positive constant* C *and all* $f \in L^{p(1+\alpha)}(d\nu)$,

$$\sup_{T>0} ||A^+_{T,1} f||_{p(1+\alpha);\nu} \le C||f||_{p(1+\alpha);\nu}. \tag{1.2}$$

(i) *If* $\frac{1}{1+\alpha} < p < \infty$ *and* $f \in L^p(d\nu)$, *then* $\lim_{T\to\infty} A^+_{T,1+\alpha} f(x)$ *exists a.e. and in the* $L^p(d\nu)$*-norm.*

(ii) *If* $p = \frac{1}{1+\alpha}$ *and* $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$, *then* $\lim_{T\to\infty} A^+_{T,1+\alpha} f(x)$ *exists a.e.*

Now, we make precise what we mean by the existence of the ergodic Hilbert transform in the (C,α) sense. Following Hardy [5, §5.14 and Notes on Chapter V], we wish to study the existence of the limit

$Hf(x) = \lim_{\varepsilon\to 0} H_\varepsilon f(x) = \lim_{t\to\infty} H_{1/t} f(x)$ in the (C,α) sense; in the case $\alpha > 0$ that means that we want to study the limit

$$\lim_{T\to\infty} \frac{\alpha}{T^\alpha} \int_0^T H_{1/t} f(x)(T-t)^{\alpha-1}\, dt.$$

Interchanging the integrals we can easily see that studying the above limit is equivalent to studying the limit of

$$H_{\varepsilon,\alpha} f(x) = \int_{\varepsilon<|t|\le 1} \frac{f(\tau_t x)}{t}\left(1-\frac{\varepsilon}{|t|}\right)^\alpha dt + \int_{1<|t|\le 1/\varepsilon} \frac{f(\tau_t x)}{t}(1-\varepsilon|t|)^\alpha\, dt,$$

when $\varepsilon \to 0$. We shall see that for suitable f the above integrals make sense not only for $\alpha \ge 0$ but also for $\alpha > -1$. Since the convergence of $H_{\varepsilon,0} f(x)$ implies the convergence of $H_{\varepsilon,\alpha} f(x)$ for $\alpha > 0$ (see §4, claim (d)), we are interested in the limit $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f$, for $-1 < \alpha \le 0$. In particular, we shall prove the following theorem.

THEOREM 1.3. *Let* $(X, \mathcal{F}, \nu)$ *be a finite measure space,* $-1 < \alpha \le 0$ *and* $\frac{1}{1+\alpha} \le p < \infty$. *Let* $\{\tau_t\colon t \in \mathbb{R}\}$ *be a nonsingular measurable flow on* X *such that for some positive constant* C *and all* $f \in L^{p(1+\alpha)}(d\nu)$,

$$\sup_{T>0} \|A_{T,1} f\|_{p(1+\alpha);\nu} \le C\|f\|_{p(1+\alpha);\nu}. \tag{1.4}$$

(i) *If* $\frac{1}{1+\alpha} < p < \infty$ *and* $f \in L^p(d\nu)$, *then* $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f(x)$ *exists a.e. and in the* $L^p(d\nu)$*-norm.*

(ii) *If* $p = \frac{1}{1+\alpha}$ *and* $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$, *then* $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f(x)$ *exists a.e.*

On one hand, notice that for $\alpha = 0$, Theorem 1.3 is contained in Theorem C. On the other hand, under the assumptions in Theorem 1.3 one can obtain the convergence of the "two-sided"(C,$1+\alpha$) ergodic averages

$$A_{T,1+\alpha} f(x) = \frac{1}{(2T)^{1+\alpha}} \int_{-T}^T f(\tau_t x)(T-|t|)^\alpha\, dt,$$

but this result is an easy consequence of Theorem 1.1 applied to the flows $\{\tau_t\colon t \in \mathbb{R}\}$ and $\{\tilde\tau_t\colon t \in \mathbb{R}\}$, where $\tilde\tau_t = \tau_{-t}$.

The proofs of Theorems 1.1 and 1.3 are based on the study of the maximal operators $S^+_{1+\alpha} f = \sup_{T>0} A^+_{T,1+\alpha}|f|$ and $H^*_\alpha f = \sup_{\varepsilon>0} |H_{\varepsilon,\alpha} f|$ (Theorems 3.1 and 3.7) and the Banach Principle. The boundedness of these operators will be obtained by using transference arguments. This requires knowledge of the behaviour on weighted spaces of some maximal operators in the real line. These last results appear in §2 while the boundedness of $S^+_{1+\alpha}$ and H^*_α are in §3. Finally, the proofs of Theorems 1.1 and 1.3 are given in §4.

Throughout this paper α will be a number such that $-1 < \alpha \le 0$ and if $1 < p < \infty$ then p' will denote its conjugate exponent, i.e., $1/p + 1/p' = 1$. The letter C will mean a positive constant not necessarily the same at each ocurrence.

2. Preliminary results

As we said above, in order to prove the theorems we will need results about some maximal operators in the real line which were studied in [9] and [1]. First we introduce the following definitions about weights.

Definition 2.1 [10]. Let w be a positive measurable function on the real line. It is said that w satisfies the Muckenhoupt A_p condition, $1 \le p < \infty$, if there exists a constant $C > 0$ such that

$$\sup_{a<b} \left(\frac{1}{b-a}\int_a^b w(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b w^{1-p'}(t)\,dt\right)^{p-1} \le C \qquad \text{if} \quad 1 < p < \infty$$

and

$$\sup_{r>0} \left(\frac{1}{2r}\int_{-r}^{r} w(x-t)\,dt\right) \le Cw(x) \quad \text{a.e.} \qquad \text{if} \quad p = 1.$$

Definition 2.2 ([12], [8], [7]). Let w be a positive measurable function on the real line. It is said that w satisfies A^+_p, $1 \le p < \infty$, if there exists a constant $C > 0$ such that

$$\sup_{a<b<c} \left(\frac{1}{c-a}\int_a^b w(t)\,dt\right)\left(\frac{1}{c-a}\int_b^c w^{1-p'}(t)\,dt\right)^{p-1} \le C \qquad \text{if} \quad 1 < p < \infty$$

and

$$\sup_{r>0} \left(\frac{1}{r}\int_0^r w(x-t)\,dt\right) \le Cw(x) \quad \text{a.e.} \qquad \text{if} \quad p = 1.$$

The A^-_p classes are defined in the obvious way, reversing the orientation in the real line.

The boundedness of the ergodic maximal operator $S^+_{1+\alpha}$ associated with the (C, $1+\alpha$) ergodic averages is based on the corresponding result for the maximal operator in $\mathbb{R}$ defined by

$$M^+_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{T^{1+\alpha}} \int_0^T |f(x+t)|(T-t)^\alpha \, dt.$$

The following result has been proved for this operator (see Theorem 2.5, Theorem 3.5 and Final Remarks in [9]).

THEOREM D ([9]). *Let $-1 < \alpha \leq 0$, $\frac{1}{1+\alpha} \leq p < \infty$ and let w be a positive measurable function on the real line.*

(i) *If $\frac{1}{1+\alpha} < p < \infty$ and $w \in A^+_{p(1+\alpha)}$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}} \left[M^+_{1+\alpha} f(t)\right]^p w(t)\, dt \leq C \int_{\mathbb{R}} |f(t)|^p w(t)\, dt$$

for all $f \in L^p(w(t)dt)$.

(ii) *If $p = \frac{1}{1+\alpha}$ and $w \in A^+_1$, then there exists a constant $C > 0$ such that*

$$w(\{t \in \mathbb{R}\colon M^+_{1+\alpha} f(t) > \lambda\}) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;w}$$

for all $f \in L_{\frac{1}{1+\alpha},1}(w(t)dt)$ and all $\lambda > 0$.

Remark 2.3. Actually, in [9], Theorem D (ii) was proved only for characteristic functions but, for $-1 < \alpha < 0$, applying Theorem 3.13 in [13], p. 195 which also holds for the sublinear operator $M^+_{1+\alpha}$, we easily obtain the result for all $f \in L_{\frac{1}{1+\alpha},1}(w(t)dt)$. On the other hand, if $\alpha = 0$, statement (ii) is the known result that $w \in A^+_1$ implies the weak type (1,1) inequality for the one-sided Hardy-Littlewood maximal function with respect to $w(t)dt$ that was proved by E. Sawyer [12] (see also [8] and [7]).

Obviously, a result analogous to Theorem D holds for the other one-sided maximal operator $M^-_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{T^{1+\alpha}} \int_{-T}^0 |f(x+t)|(T+t)^\alpha \, dt$ and the corresponding $A^-_{p(1+\alpha)}$ classes. Now, taking into account that the maximal operator

$$M_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{-T}^T |f(x+t)|(T-|t|)^\alpha \, dt$$

is pointwise equivalent to the sum of the operators $M^+_{1+\alpha}$ and $M^-_{1+\alpha}$ and that $A_{p(1+\alpha)} = A^+_{p(1+\alpha)} \cap A^-_{p(1+\alpha)}$, we see that Theorem D is valid for $M_{1+\alpha}$ with the $A^+_{p(1+\alpha)}$ classes replaced by the $A_{p(1+\alpha)}$ classes. This result will be used to obtain the boundedness of the ergodic maximal operator

$$R_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(4T)^{1+\alpha}} \int_{-2T}^{2T} |f(\tau_t x)|\, |T-|t||^\alpha \, dt.$$

On the other hand, in the study of the ergodic Hilbert transform in the Cesàro-α sense (see §3) the following maximal operator appears:

$$N_{1+\alpha}f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{T<|t|<2T} |f(x+t)|(|t|-T)^{\alpha}\, dt.$$

This operator was studied in [1, Theorems 2.1 and 2.4], which obtained results analogous to the ones for the operator $M_{1+\alpha}$. In the following theorem we collect these results and the corresponding ones for $M_{1+\alpha}$.

THEOREM E ([9], [1]). *Let $-1 < \alpha \leq 0$, $\frac{1}{1+\alpha} \leq p < \infty$ and let w be a positive measurable function on the real line. Let us denote by $\mathcal{M}$ either $M_{1+\alpha}$ or $N_{1+\alpha}$.*

(i) *If $\frac{1}{1+\alpha} < p < \infty$ and $w \in A_{p(1+\alpha)}$, then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}} [\mathcal{M}f(t)]^p\, w(t)\, dt \leq C \int_{\mathbb{R}} |f(t)|^p w(t)\, dt$$

for all $f \in L^p(w(t)dt)$.

(ii) *If $p = \frac{1}{1+\alpha}$ and $w \in A_1$, then there exists a constant $C > 0$ such that*

$$w(\{t \in \mathbb{R}\colon \mathcal{M}f(t) > \lambda\}) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;w}$$

for all $f \in L_{\frac{1}{1+\alpha},1}(w(t)dt)$ and all $\lambda > 0$.

3. Boundedness for the ergodic maximal operators

This section is devoted to establishing the boundedness of the maximal operators

$$S^{+}_{1+\alpha}f(x) = \sup_{T>0} \frac{1}{T^{1+\alpha}} \int_0^T |f(\tau_t x)|(T-t)^{\alpha}\, dt,$$

associated to the (C,1 + α) ergodic averages $A^{+}_{1+\alpha}f$, and H^{*}_{α}. First, we shall prove the following theorem.

THEOREM 3.1. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t\colon t \in \mathbb{R}\}$ be as in Theorem 1.1.*

(i) *If $\frac{1}{1+\alpha} < p < \infty$, then there exists a constant $C > 0$ such that for all $f \in L^p(d\nu)$,*

$$||S^{+}_{1+\alpha}f||_{p;\nu} \leq C||f||_{p;\nu}.$$

(ii) *If* $p = \frac{1}{1+\alpha}$, *then there exists a constant* $C > 0$ *such that for all* $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ *and all* $\lambda > 0$,

$$\nu(\{x \in X\colon S^+_{1+\alpha} f(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}} \|f\|^{1/1+\alpha}_{\frac{1}{1+\alpha},1;\nu}.$$

In order to prove this theorem, we need two lemmas. The proof of the first one is very similar to the proof of the claim in the proof of Theorem 1 in [6]; therefore we omit it.

LEMMA 3.2. *Let* $(X, \mathcal{F}, \nu)$, α, p *and* $\{\tau_t\colon t \in \mathbb{R}\}$ *be as in Theorem* 1.1 *or in Theorem* 1.3. *Then, there exists a measure* μ *equivalent to* ν *such that the flow* $\{\tau_t\colon t \in \mathbb{R}\}$ *preserves the measure* μ.

In what follows, the measure μ will be fixed and w will be the Radon-Nikodym derivative of ν with respect to μ. It is clear that $0 < w < \infty$ a.e.. Let w^x denote the function $w^x\colon \mathbb{R} \to \mathbb{R}$ such that $w^x(t) = w(\tau_t x)$.

LEMMA 3.3. *Let* $(X, \mathcal{F}, \nu)$ *be a finite measure space*, $-1 < \alpha \le 0$ *and* $\frac{1}{1+\alpha} \le p < \infty$. *Let* $\{\tau_t\colon t \in \mathbb{R}\}$ *be a nonsingular measurable flow on* X.

(i) *If* (1.2) *holds, then* $w^x \in A^+_{p(1+\alpha)}$ *for almost every* $x \in X$ *and with the same constant.*
(ii) *If* (1.4) *holds, then* $w^x \in A_{p(1+\alpha)}$ *for almost every* $x \in X$ *and with the same constant.*

Proof. We only sketch the proof of (i), since the proof of (ii) is similar (notice that (ii) was already used in [6]). First, observe that if $p = \frac{1}{1+\alpha}$, then (i) follows from the fact that the flow preserves the measure μ given in Lemma 3.2.

Now assume that $q = p(1+\alpha) > 1$ and let q' be its conjugated exponent. Taking Lemma 3.2 into account, by (1.2) we get that

$$\int_X |A^+_{T,1} f(x)|^q\, w(x)\, d\mu(x) \le C \int_X |f(x)|^q w(x)\, d\mu(x)$$

for all $T > 0$. Then letting $\sigma = w^{1-q'}$, by duality we can write the above inequality as

$$\int_X |(A^+_{T,1})^* f(x)|^{q'}\, \sigma(x)\, d\mu(x) \le C \int_X |f(x)|^{q'} \sigma(x)\, d\mu(x),$$

where $(A^+_{T,1})^* f(x) = \frac{1}{T} \int_{-T}^0 f(\tau_t x)\, dt$ is the adjoint operator of $A^+_{T,1}$ with respect to the measure μ. Let us define the following operators:

$$P_T g = \left[A^+_{T,1}\left(|g|^{q'} w^{-1/q}\right)\right]^{1/q'} w^{\frac{1}{qq'}},$$

$$Q_T g = \left[\left(A^+_{T,1}\right)^*\left(|g|^{q} \sigma^{-1/q'}\right)\right]^{1/q} \sigma^{\frac{1}{qq'}}.$$

P_T and Q_T are sublinear operators and P_T, Q_T: $L^{qq'}(d\mu) \to L^{qq'}(d\mu)$ with $\|P_T\|, \|Q_T\| \leq C$, where C is the constant in (1.2). Clearly, the same holds for the operator $P_T + Q_T$ and $\|P_T + Q_T\| \leq 2C$. Now, given $f \in L^{qq'}(d\mu)$, $f > 0$, let us define

$$g_T = \sum_{i=0}^{\infty} \frac{(P_T + Q_T)^{(i)} f}{(4C)^i},$$

where $(P_T + Q_T)^{(i)}$ denotes the i-th iteration of $P_T + Q_T$. Clearly $g_T \in L^{qq'}(d\mu)$ and

$$P_T\left(g_T\right)(x) \leq 4C g_T(x) \quad \text{and} \quad Q_T\left(g_T\right)(x) \leq 4C g_T(x).$$

From these inequalities we can see that if $v_T = g_T^{q'} w^{-1/q}$ and $u_T = g_T^{q} \sigma^{-1/q'}$ then

$$A^+_{T,1}(v_T) \leq C v_T \quad \text{and} \quad (A^+_{T,1})^*(u_T) \leq C u_T. \tag{3.4}$$

The lemma follows since $w(x) = u_T(x) v_T^{1-q}(x)$ for almost every $x \in X$ and as a consequence we can prove that $w^x \in A^+_q$. In fact, let a, b and c be real numbers such that $a < b < c$. If $t \in (a, b)$, by the inequality for v_T in (3.4) with $T = c - a$ we get

$$\begin{aligned}
\frac{1}{c-a}\int_b^c v_T(\tau_s x)\, ds &= \frac{1}{c-a}\int_{b-t}^{c-t} v_T(\tau_r \tau_t x)\, dr \\
&\leq \frac{1}{c-a}\int_0^{c-a} v_T(\tau_r \tau_t x)\, dr \\
&\leq C v_T(\tau_t x).
\end{aligned}$$

In the same way, by using the inequality for u_T in (3.4) with $T = c - a$ and for $t \in (c, d)$, we get

$$\frac{1}{c-a}\int_a^b u_T(\tau_s x)\, ds \leq C u_T(\tau_t x).$$

Then, from the last inequalities,

$$\int_a^b u_T(\tau_t x) v_T^{1-q}(\tau_t x)\, dt \left(\int_b^c u_T^{1-q'}(\tau_t x) v_T(\tau_t x)\, dt\right)^{q-1} \leq C(c-a)^q.$$

Proof of Theorem 3.1. We only prove (ii) since (i) follows in a similar way. Assume $\alpha < 0$. As we observe in Remark 2.3, in order to prove (ii) we only need to consider characteristic functions, i.e., we need to prove that

$$\nu(\{x \in X\colon\ S^+_{1+\alpha}(\chi_E)(x) > \lambda\}) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}} \int_X \chi_E(x)\, d\nu(x)$$

for all $\lambda > 0$ and all measurable sets E. We shall use a transference argument. For fixed $L > 0$ we define

$$S^+_{1+\alpha,L} f(x) = \sup_{0<T<L} \frac{1}{T^{1+\alpha}} \int_0^T |f(\tau_t x)|(T-t)^\alpha\, dt.$$

Then, for all $N > 0$ we have

$$\begin{aligned}\nu(\{x \in X\colon\ S^+_{1+\alpha}(\chi_E)(x) > \lambda\}) &= \frac{1}{N}\int_0^N \int_X \chi_{\{x\colon S^+_{1+\alpha}(\chi_E)(x)>\lambda\}}(\tau_t x) w(\tau_t x)\, d\mu(x)\, dt \\ &= \frac{1}{N}\int_0^N \int_{\{x\in X\colon S^+_{1+\alpha}(\chi_E)(\tau_t x)>\lambda\}} w(\tau_t x)\, d\mu(x)\, dt.\end{aligned}$$

Since $S^+_{1+\alpha}(\chi_E)(\tau_t x) \leq M^+_{1+\alpha}(\chi^x_E \chi_{(0,N+L)})(t)$, where $\chi^x_E(t) = \chi_E(\tau_t x)$ and w^x satisfies A^+_1 for almost all x with the same constant (Lemma 3.3 (ii)), Theorem D (ii) implies that

$$\begin{aligned}\nu(\{x \in X\colon\ S^+_{1+\alpha}(\chi_E)(x) > \lambda\}) &\leq \frac{1}{N}\int_X \int_{\{t\colon M^+_{1+\alpha}(\chi^x_E \chi_{(0,N+L)})(t)>\lambda\}} w^x(t)\, dt\, d\mu \\ &\leq \frac{C}{N\lambda^{\frac{1}{1+\alpha}}} \int_X \int_0^{N+L} \chi_E(\tau_t x) w(\tau_t x)\, dt\, d\mu \\ &= \frac{C(N+L)}{N\lambda^{\frac{1}{1+\alpha}}} \int_X \chi_E(x)\, d\nu(x)\end{aligned}$$

because the flow preserves the measure μ. Letting $N \to \infty$ and then $L \to \infty$ we finish the proof for $-1 < \alpha < 0$. The case $\alpha = 0$ is proved in the same way but using general functions $f \in L^1(d\mu)$.

In what follows, we shall establish the boundedness of the ergodic maximal operator $H^*_\alpha = \sup_{\varepsilon>0} |H_{\varepsilon,\alpha}|$. This will follow from the boundedness of the operators H^*_0 and the fact that

$$R_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(4T)^{1+\alpha}} \int_{-2T}^{2T} |f(\tau_t x)||T - |t||^\alpha\, dt.$$

The result for the last operator is in the following theorem.

THEOREM 3.5. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t \colon t \in \mathbb{R}\}$ be as in Theorem* 1.3.

(i) *If $\frac{1}{1+\alpha} < p < \infty$, then there exists a constant $C > 0$ such that for all $f \in L^p(d\nu)$,*

$$||R_{1+\alpha} f||_{p;\nu} \leq C||f||_{p;\nu}.$$

(ii) *If $p = \frac{1}{1+\alpha}$, then there exists a constant $C > 0$ such that for all $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ and all $\lambda > 0$,*

$$\nu(\{x \in X \colon R_{1+\alpha} f(x) > \lambda\}) \leq \frac{C}{\lambda^{\frac{1}{1+\alpha}}} ||f||^{1/1+\alpha}_{\frac{1}{1+\alpha},1;\nu}.$$

Proof of Theorem 3.5. The proof of Theorem 3.5 is completely similar to the proof of Theorem 3.1. We only need to notice that the operator $R_{1+\alpha}$ is pointwise equivalent to the sum of the following two maximal operators:

$$R^1_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{-T}^{T} |f(\tau_t x)|(T - |t|)^\alpha \, dt,$$

$$R^2_{1+\alpha} f(x) = \sup_{T>0} \frac{1}{(2T)^{1+\alpha}} \int_{T<|t|<2T} |f(\tau_t x)|(|t| - T)^\alpha \, dt.$$

Then, when we apply the transference arguments we shall need to use the results of Theorem E for the operators $M_{1+\alpha}$ and $N_{1+\alpha}$.

Now we are ready to establish the boundedness of H^*_α. First, we easily see that the ergodic truncations $H_{\varepsilon,\alpha} f$ are well defined. In fact, by Theorem 3.5, we get

$$\int_{\varepsilon<|t|\leq 1} \frac{|f(\tau_t x)|}{|t|} \left(1 - \frac{\varepsilon}{|t|}\right)^\alpha dt + \int_{1<|t|\leq 1/\varepsilon} \frac{|f(\tau_t x)|}{|t|} (1 - \varepsilon|t|)^\alpha \, dt$$
$$\leq C_\varepsilon \, R_{1+\alpha}(f)(x) < \infty$$

for almost every x and $f \in L^p(d\nu)$ if $\frac{1}{1+\alpha} < p < \infty$ or $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ if $p = \frac{1}{1+\alpha}$. Second, we prove the following key pointwise estimate.

LEMMA 3.6. *Let $(X, \mathcal{F}, \nu)$ be a finite measure space, $-1 < \alpha \leq 0$ and let $\{\tau_t \colon t \in \mathbb{R}\}$ be a nonsingular measurable flow on X. Then, there exists a constant $C > 0$ such that*

$$H^*_\alpha f(x) \leq C\left[R_{1+\alpha} f(x) + H^*_0 f(x)\right].$$

Proof. First, we write

$$\begin{aligned} H_{\varepsilon,\alpha}f(x) &= \int_{\varepsilon<|t|\le 2\varepsilon} \frac{f(\tau_t x)}{t}\left(1-\frac{\varepsilon}{|t|}\right)^{\alpha} dt + \int_{2\varepsilon<|t|\le 1} \frac{f(\tau_t x)}{t}\left[\left(1-\frac{\varepsilon}{|t|}\right)^{\alpha} - 1\right] dt \\ &+ \int_{2\varepsilon<|t|<1/2\varepsilon} \frac{f(\tau_t x)}{t}\, dt + \int_{1<|t|<1/2\varepsilon} \frac{f(\tau_t x)}{t}\left[(1-\varepsilon|t|)^{\alpha} - 1\right] dt \\ &+ \int_{1/2\varepsilon\le|t|\le 1/\varepsilon} \frac{f(\tau_t x)}{t}(1-\varepsilon|t|)^{\alpha}\, dt = I + II + III + IV + V. \end{aligned}$$

Clearly, $|III| \le H_0^* f(x)$. Also, we can easily see that $|I|, |V| \le C R_{1+\alpha} f(x)$. On the other hand, by the Mean Value Theorem and by decomposing the integral in II into the sum of integrals over the sets $\{t\colon 2^k\varepsilon < |t| \le 2^{k+1}\varepsilon\}$, we can see that $|II|$ and $|IV|$ are bounded by a constant times the usual ergodic maximal operator $M_0 f(x)$. Then the lemma follows since $M_0 f(x) \le R_{1+\alpha} f(x)$ for $-1 < \alpha \le 0$.

Now, the boundedness of H_α^* follows from the above lemma, Theorem 3.5 and Theorem 1 in [6]. In this way we obtain the following result for the operator H_α^*.

THEOREM 3.7. *Let $(X, \mathcal{F}, \nu)$, α, p and $\{\tau_t\colon t \in \mathbb{R}\}$ be as in Theorem* 1.3.

(i) *If $\frac{1}{1+\alpha} < p < \infty$, then there exists a constant $C > 0$ such that for all $f \in L^p(d\nu)$,*

$$\|H_\alpha^* f\|_{p;\nu} \le C \|f\|_{p;\nu}.$$

(ii) *If $p = \frac{1}{1+\alpha}$, then there exists a constant $C > 0$ such that for all $f \in L_{\frac{1}{1+\alpha},1}(d\nu)$ and all $\lambda > 0$,*

$$\nu(\{x \in X\colon H_\alpha^* f(x) > \lambda\}) \le \frac{C}{\lambda^{\frac{1}{1+\alpha}}} \|f\|_{\frac{1}{1+\alpha},1;\nu}^{1/1+\alpha}.$$

4. Proofs of Theorems 1.1 and 1.3

From Theorem B and Theorem 3.1 we can easily prove Theorem 1.1.

Proof of Theorem 1.1. We only prove (i) since the proof of (ii) is similar. By Theorem 3.1, the Banach Principle and the Dominated Convergence Theorem it will suffice to prove the a.e. convergence of the averages $A_{1+\alpha}^+ f$ for f in a dense subset of $L^p(d\nu)$. Using Theorem B we have the a.e. convergence of $A_{1+\alpha}^+ f$ for $f \in L^p(d\nu) \cap L_{\frac{1}{1+\alpha},1}(d\mu)$ which is a dense subset of $f \in L^p(d\nu)$ (μ is the measure given in Lemma 3.2). Then, the theorem follows.

Proof of Theorem 1.3. As in the proof of Theorem 1.1 we only prove (i) and we only have to show that the a.e. convergence holds for the functions in a dense subset of $L^p(d\nu)$.

Let us fix β and q such that $p < \frac{1}{1+\beta} < q$ and, as before, let μ be the measure given in Lemma 3.2. On one hand, the set $D = L^p(d\nu) \cap L^q(d\mu)$ is a dense subset of $L^p(d\nu)$. On the other hand, since μ is preserved by the flow, for all $f \in D$ we have the following results:

(a) By the classical result by Cotlar [3] (see also [11]) or by Theorem C, $\lim_{\varepsilon\to 0} H_{\varepsilon,0} f(x) = Hf(x)$ exists for almost every $x \in X$.
(b) By Theorem 3.7, $H^*_\beta f$ is a.e. finite, because $q > \frac{1}{1+\beta}$ and the ergodic averages $A_{T,1}$ are uniformly bounded on $L^{q(1+\beta)}(d\mu)$.

In what follows we will prove that, for all $f \in D$, (a) and (b) imply the a.e. existence of $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f(x)$ and that $\lim_{\varepsilon\to 0} H_{\varepsilon,\alpha} f(x) = Hf(x)$. The proof is an adaptation of Lemma 2.27 in [14].

For fixed $f \in D$, let $x \in X$ such that $\lim_{\varepsilon\to 0} H_{\varepsilon,0} f(x) = Hf(x)$ and $H^*_\beta f(x)$ is finite. We may assume without loss of generality that $Hf(x) = 0$. Applying the formula

$$(x-u)^{\alpha+\delta} = C \int_u^x (t-u)^\alpha (x-t)^{\delta-1}\, dt, \qquad \delta > 0, \tag{4.1}$$

with $\delta = \alpha - \beta$, where C depends only on α and δ (in fact, $C = \frac{\Gamma(\alpha+\delta+1)}{\Gamma(\alpha+1)\Gamma(\delta)}$ where Γ is the Gamma function), we obtain

$$H_{\varepsilon,\alpha} f(x) = C\, \varepsilon^\alpha \int_1^{1/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-1}\, t^\beta\, H_{1/t,\beta} f(x)\, dt. \tag{4.2}$$

Given $\eta > 0$, let us fix θ with $1/2 < \theta < 1$ and $(1-\theta)^{\alpha-\beta} < \eta$. Then,

$$\begin{aligned} H_{\varepsilon,\alpha} f(x) &= C\, \varepsilon^\alpha \int_1^{\theta/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-1}\, t^\beta\, H_{1/t,\beta} f(x)\, dt \\ &\quad + C\, \varepsilon^\alpha \int_{\theta/\varepsilon}^{1/\varepsilon} (1/\varepsilon - t)^{\alpha-\beta-1}\, t^\beta\, H_{1/t,\beta} f(x)\, dt = I + II. \end{aligned}$$

First, we estimate II and obtain

$$|II| \le C\, \varepsilon^\alpha\, H^*_\beta f(x)\, (\theta/\varepsilon)^\beta\, (1/\varepsilon - \theta/\varepsilon)^{\alpha-\beta} \le C\, H^*_\beta f(x)\, \eta.$$

To estimate I, we integrate by parts and use (4.2) with $\alpha = \beta + 1$ to obtain

$$\begin{aligned} I &= C\,\varepsilon^\alpha\,(1/\varepsilon - \theta/\varepsilon)^{\alpha-\beta-1}\int_1^{\theta/\varepsilon} s^\beta H_{1/s,\beta} f(x)\,ds \\ &\quad + C\,\varepsilon^\alpha \int_1^{\theta/\varepsilon} (\alpha-\beta-1)(1/\varepsilon - t)^{\alpha-\beta-2}\int_1^t s^\beta H_{1/s,\beta} f(x)\,ds\,dt \\ &= C\,\varepsilon^\alpha\left(\frac{1-\theta}{\varepsilon}\right)^{\alpha-\beta-1}(\theta/\varepsilon)^{\beta+1}\,H_{\varepsilon/\theta,\beta+1}f(x) \\ &\quad + C\,\varepsilon^\alpha\,(\alpha-\beta-1)\int_1^{\theta/\varepsilon}(1/\varepsilon - t)^{\alpha-\beta-2}t^{\beta+1}H_{1/t,\beta+1}f(x)\,dt = III + IV. \end{aligned}$$

Now, we claim that the following hold.

(c) $H^*_{\beta+\delta} f(x)$ is finite for all $\delta > 0$.
(d) $\lim_{\varepsilon\to 0} H_{\varepsilon,\beta+1} f(x) = Hf(x) = 0$.

The above claims follow from (4.1), (4.2), (a) and (b). Taking into account the claims (c) and (d) we obtain

$$|III| \le C\ (1-\theta)^{\alpha-\beta-1}\,\theta^{\beta+1}|H_{\varepsilon/\theta,\beta+1}f(x)| < \eta$$

for ε small enough.

On the other hand, since $\alpha - \beta - 2 \in (-2,-1)$ and $\beta > -1$, we have $(1/\varepsilon - t)^{\alpha-\beta-2} < (1/\varepsilon - \theta/\varepsilon)^{\alpha-\beta-2}$ and $t^{\beta+1} < (\theta/\varepsilon)^{\beta+1}$ for all $t \in (1, \theta/\varepsilon)$. Then,

$$|IV| \le C\,\varepsilon \int_1^{\theta/\varepsilon} |H_{1/t,\beta+1}f(x)|\,dt,$$

which tends to zero as ε goes to zero because $\lim_{t\to\infty} H_{1/t,\beta+1}f(x) = 0$ and $H^*_{\beta+1}f(x) < \infty$. Therefore we are done.

References

1. A. Bernardis and F.J. Martín-Reyes, *Singular integrals in the Cesáro sense*, J. Fourier Anal. Appl., to appear.
2. M. Broise, Y. Déniel and Y. Derriennic, *Réarrangement, inégalités maximales et Théorémes ergodiques fractionnaires*, Ann. Inst. Fourier, Grenoble **39** (1989), 689–714.
3. M. Cotlar, *A unified theory of Hilbert transforms and ergodic theorems*, Rev. Mat. Cuyana **1** (1955), 105–167.
4. Y. Deniel, *On the a.s. Cesàro-α convergence for stationary or orthogonal random variables*, J. Theoret. Prob. **2** (1989), 475–485.
5. G. H. Hardy, *Divergent series*, Clarendam, Oxford, 1973.
6. M. Lorente Domínguez and F. J. Martín-Reyes, *The ergodic Hilbert transform for Cesàro bounded flows*, Tôhoku Math. J. **46** (1994), 541–556.

7. F. J. Martín-Reyes, *New proofs of weighted inequalities for the one sided Hardy-Littlewood maximal functions*, Proc. Amer. Math. Soc. **117** (1993), 691–698.
8. F. J. Martín-Reyes, P. Ortega, Salvador and A. de la Torre *Weighted inequalities for one-sided maximal functions*, Trans. Amer. Math. Soc. **319** (1990), 517–534.
9. F. J. Martín-Reyes and A. de la Torre, *Some weighted inequalities for general one-sided maximal operators*, Studia Math. **122** (1997), 1–14.
10. B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
11. K. Petersen, *Ergodic theory*, Cambridge Studies in Advanced Math., no. 2, Cambridge Univ. Press, 1983.
12. E. Sawyer, *Weighted inequalities for the one sided Hardy-Littlewood maximal functions*, Trans. Amer. Math. Soc. **297** (1986), 53–61.
13. E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, 1971.
14. A. Zygmund, *Trigonometric series*, vols. I and II, Cambridge University Press, 1959.

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