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A SHEAF HOMOLOGY THEORY WITH SUPPORTS

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ABSTRACT. We introduce a homology theory with supports and with coefficients in a sheaf. It has a very explicit description of the chains in terms of a triangulation of an ambient space, making the theory useful for integration purposes. We prove a Poincaré Duality Theorem that states that our homology modules are isomorphic to the classical sheaf cohomology modules with supports. This theorem is a main ingredient in the proof of a criterion on the vanishing of real principal value integrals in terms of cohomology. We briefly explain how real principal value integrals appear as residues of poles of distributions $|f|^s$ and as coefficients of asymptotic expansions of oscillating integrals.

1. Introduction

In this paper we introduce a sheaf homology theory (with supports) on an open subspace W of a topological space X. On W we are given a sheaf $\mathcal{F}$ of L-modules and a family φ of supports. We use a (locally finite) triangulation t of the "big" space X and an orientation o of the simplices of t to build our homology L-modules $H_i^{\varphi}(W, \mathcal{F}, t, o)$. The triangulation t must subtriangulate the complement of W in X. The chains of our homology theory are formal sums of products of a coefficient in the sheaf $\mathcal{F}$ and an intersection of a simplex of t with W. If $W = X$ then we get a more traditional approach to sheaf homology. Note that the triangulation t is finite if X is compact. This fact can be exploited to prove the equality of integrals on homologous cycles of our homology theory; see [D-J].

The main result of this paper is a Poincaré Duality Theorem. We assume that X is a n-dimensional differentiable manifold and that $\mathcal{F}$ is locally constant. Then the theorem states that, under some natural conditions on φ, t and L, our homology L-modules $H_i^{\varphi}(W, \mathcal{F}, t, o)$ are isomorphic to the classical sheaf cohomology modules $H_{\varphi}^{n-i}(W, \mathcal{F})$. Note that this last module is independent of t and o, hence $H_i^{\varphi}(W, \mathcal{F}, t, o)$ is independent of t and o.

If $W = X$ we recover two well-known cases. In the first case we take φ equal to the family of all compact subsets of W. Then our homology modules are isomorphic to the classical singular homology modules. In the second case we take φ equal to the family of all closed subsets of W. Then our homology modules are isomorphic to the classical Borel-Moore homology modules [Bo].

Our version of the Poincaré Duality Theorem forms one of the main ingredients in the proof of the following theorem on the vanishing of principal value integrals. Let X be a non-singular complex projective algebraic variety defined over $\mathbb{R}$ of

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complex dimension m such that $X(\mathbb{R}) \neq \emptyset$. Let ω be a rational differential m-form of degree d on X defined over $\mathbb{R}$. Thus $\omega \in \Gamma(U, \Omega^m_{X/\mathbb{R}}{}^{\otimes d})$ for some Zariski dense open $U \subset X$. Let $|\mathrm{div}(\omega)|$ denote the support of the divisor of ω. Let $\chi\colon X(\mathbb{R}) \to \{1, -1\}$ be a function that is constant on the connected components of $X(\mathbb{R}) - |\mathrm{div}(\omega)|$. Let $\mathrm{div}(\omega) = \sum_i \beta_i D_i$. We formally consider $\omega^{1/d}$ as a multivalued rational differential form on X and define $\mathrm{div}(\omega^{1/d}) = \sum_i \alpha_i D_i$ where $\alpha_i = \beta_i/d$. We assume that $|\mathrm{div}(\omega)|$ has normal crossings over $\mathbb{R}$, meaning that it has normal crossings and that each irreducible component containing an $\mathbb{R}$-rational point is defined over $\mathbb{R}$. Moreover we assume that $\omega^{1/d}$ has no integral poles, meaning that no α_i is a strictly negative integer. Then we can define $PV \int_{X(\mathbb{R})} \chi |\omega^{1/d}|$, the principal value integral of $\chi|\omega^{1/d}|$ over $X(\mathbb{R})$ as in Langlands paper [L]. This is done as follows. Choose a finite number of local coordinates $\{x_p\colon U_p \to \mathbb{R}^m\}_{p\in P}$ for $X(\mathbb{R})$, centered at $p \in X(\mathbb{R})$, such that the U_p cover $X(\mathbb{R})$ and on each U_p: $\omega = \xi_p x_p{}^{\gamma_p}(dx_p)^d$, where ξ_p is a regular function on U_p, defined over $\mathbb{R}$ and nowhere zero on U_p, and $x_p{}^{\gamma_p} = \prod_i x_{p_i}{}^{\gamma_{p_i}}$ with $\gamma_{p_i} \in \mathbb{Z}$. This is possible since $|\mathrm{div}(\omega)|$ has normal crossings over $\mathbb{R}$. Choose a C^∞ partition of unity $\{\varphi_p\}_{p\in P}$ with respect to $\{U_p\}_{p\in P}$. Then for each $p \in P$: $\int_{U_p} \varphi_p \chi |\xi_p|^{1/d} |x_p|^{\gamma_p/d+s} |dx_p|$ converges for $Re(s) >> 0$ and its meromorphic continuation is holomorphic in $s = 0$ because $\omega^{1/d}$ has no integral poles. Then one defines

$$PV \int_{X(\mathbb{R})} \chi |\omega^{1/d}| := \sum\nolimits_{p\in P} \left[\int_{U_p} \varphi_p \chi |\xi_p|^{1/d} |x_p|^{\gamma/d+s} |dx_p| \right]^{mc}_{s=0}$$

where $[-]^{mc}_{s=0}$ means taking the value in $s = 0$ of the meromorphic continuation of the integral. This definition is independent of the choices made. Let $\mathcal{L}(\omega^{1/d})$ be the locally constant sheaf of $\mathbb{C}$-vectorspaces on $X - |\mathrm{div}(\omega)|$ associated to $\omega^{1/d}$, which is locally free of rank 1. A non-zero section of $\mathcal{L}(\omega^{1/d})$ on a connected open U is an analytic branch of $\omega^{1/d}$ on U multiplied with a complex number.

(1.1) THEOREM. *If $H^m(X(\mathbb{C}) - |\mathrm{div}(\omega)|, \mathcal{L}(\omega^{1/d})) = 0$ then*

$$PV \int_{X(\mathbb{R})} \chi |\omega^{1/d}| = 0.$$

For a nice overview of the proof we refer to [D-J]. The whole proof can be found in [J].

Now we briefly explain the connection between coefficients of asymptotic expansions of oscillating integrals and residues of poles of distributions $|f|^s$; see [A-V-G, II,§7], [I1], [I2], [Lae] for more details. The connection of these two with principal value integrals is worked out in detail in [J2]. Let $f\colon \mathbb{R}^m \to \mathbb{R}$ be a non-constant real-analytic function with only isolated singularities. Let $\varphi\colon \mathbb{R}^m \to \mathbb{R}$ be a C^∞-function with compact support. Then one considers the integral

$$I(\tau) := \int_{\mathbb{R}^m} e^{i\tau f(x)} \varphi(x) dx,$$

where τ is a real parameter, $x = (x_1, \dots, x_m)$ and $dx = dx_1 \wedge \cdots \wedge dx_m$. The function $I(\tau)$ has an asymptotic expansion for $\tau \to +\infty$,

$$I(\tau) \approx \sum_{\alpha} \sum_{k=0}^{m-1} a_{k,\alpha}(\varphi)\tau^{\alpha}(\ln \tau)^k,$$

where the coefficients $a_{k,\alpha}$ are distributions of φ and where α runs through a finite set A of numbers in descending arithmetic progressions. One can write these arithmetic progressions in terms of the numerical data (N_i, ν_i), $i \in I$, of an embedded resolution $\pi\colon Y \to \mathbb{R}^m$ of the singularities of f. The set A consists of the numbers in the arithmetic progressions

$$\frac{-\nu_i}{N_i}, \frac{-(\nu_i+1)}{N_i}, \frac{-(\nu_i+2)}{N_i}, \dots \text{ for } i \in I.$$

Now let λ be a complex parameter with real part $\Re(\lambda) >> 0$ and define the functions

$$G_{\pm}(\lambda) := \int_{f>0} f^{\lambda}\varphi dx \pm \int_{f<0} (-f)^{\lambda}\varphi dx.$$

Then

$$G_{+}(\lambda) = \int_{\mathbb{R}^m} |f|^{\lambda}\varphi$$

and

$$G_{-}(\lambda) = \int_{\mathbb{R}^m} \operatorname{sgn}(f)|f|^{\lambda}\varphi dx,$$

where sgn denotes the sign function. So we obtain the classical distribution $|f|^{\lambda}$ and its twisted version $\operatorname{sgn}(f)|f|^{\lambda}$ (twisted by the character sgn). One can show that these functions have meromorphic extensions to the complex plane with poles α in the set A, hereafter called the set of candidate poles. The coefficient $a_{k,\alpha}(\varphi)$ in the asymptotic expansion of $I(\tau)$ can easily be expressed in terms of the coefficients $b^{\pm}_{l,\alpha}(\varphi)$ of $(\lambda - \alpha)^{-(l+1)}$, $l \geq k$, in the Laurent expansions of $G_{\pm}(\lambda)$.

Let $\alpha \in A$ be a candidate pole. Using the resolution π one finds a non-negative integer k_{α} such that $b^{\pm}_{k,\alpha}(\varphi) = 0$ for all φ and all $k > k_{\alpha}$. Then, for most φ, k_{α} is the expected order of α as pole of $G_{\pm}(\lambda)$. Let β be the maximum of A, such that $a_{k_{\beta},\beta}(\varphi)\tau^{\beta}(\ln \tau)^{k_{\beta}}$ is the dominating term in the asymptotic expansion of $I(\tau)$. In [J2] we show that $b^{\pm}_{k_{\beta},\beta}(\varphi)$ and hence $a_{k_{\beta},\beta}(\varphi)$ can be expressed in terms of real principal value integrals (of a meromorphic differential form of higher degree). Thus in the case that f is a polynomial the theorem stated before gives a condition in terms of cohomology for the vanishing of the coefficient $a_{k_{\beta},\beta}(\varphi)$.

This paper is divided into six sections. In the second section we give some definitions and a lemma which we need to build our homology theory in Section 3 and to state our main results in Section 4. Besides the Poincaré Duality Theorem we formulate Propositions A and B, which are useful to check the conditions of the Poincaré Duality Theorem. In the last two sections we prove our results. The proofs of Propositions A and B are given in Section 5. Section 6 is entirely devoted to the proof of the Poincaré Duality Theorem.

The results in this paper can also be found (in a slightly more elaborated way) in my Ph.D. Thesis [J]. At this point I'd like to thank Prof. Jan Denef for the many fruitfull discussions and for his interesting suggestions.

2. Definitions of basic notions

In this section we define the basic notions that we need later on. We introduce simplicial complexes as in [Ka-Sch, VII,§1, p. 321–322]. We also explain the notions of triangulation, orientation and family of support.

2.1 Simplicial complexes

(2.1) *Definition.* A *simplicial complex* $\mathcal{S} = (S, \Delta)$ consists of a set S and a set Δ of subsets of S, satisfying the following axioms.

1. Any σ in Δ is a finite and non-empty subset of S.
2. If τ is a non-empty subset of an element σ of Δ, then τ belongs to Δ.
3. $\{p\}$ belongs to Δ for any p in S.
4. The set $\{\sigma \in \Delta | p \in \sigma\}$ is finite for any p in S.

An element of S is called a *vertex* of $\mathcal{S}$ and an element of Δ is called a *simplex* of $\mathcal{S}$. We equip $\mathbb{R}^S$ with the product topology and for σ in Δ we define *the relative interior of* σ by

$$|\sigma| := \left\{ x \in \mathbb{R}^S | x(p) = 0 \text{ for } p \notin \sigma; x(p) > 0 \text{ for } p \in \sigma \text{ and } \sum_p x(p) = 1 \right\}.$$

We also define $|\mathcal{S}| := \cup_{\sigma \in \Delta} |\sigma|$, called *the space of* $\mathcal{S}$, $\overline{\sigma}$, the *closure of* $|\sigma|$ *in* $|\mathcal{S}|$ and $\dim \sigma$, the *dimension of* σ, which is one less than the cardinality of σ. We denote the set of i-dimensional simplices of $\mathcal{S}$ by Δ_i. A non-empty subset of a simplex σ is called a *subsimplex* or *face* of σ. A face of a simplex σ of dimension $(\dim \sigma - 1)$ is called a *facet* of σ.
A simplicial complex $\mathcal{S}' = (S', \Delta')$ is called a *simplicial subcomplex* of the simplicial complex $\mathcal{S} = (S, \Delta)$ if $S' \subset S$ and $\Delta' \subset \Delta$. □

(2.2) *Convention.* If we write $p = \sum_i \lambda_i p_i$, we will always mean that p is a positive barycentric combination of the p_i, so $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$.

2.2 Triangulations

(2.3) *Definition.* Let X be a topological space. A *triangulation t of X* is a homeomorphism $t\colon |\mathcal{S}| \to X$ from the space $|\mathcal{S}|$ of a simplicial complex $\mathcal{S}$ to X. We denote it by $t\colon \mathcal{S} \to X$. For any simplex σ of $\mathcal{S}$ we denote the set $t(|\sigma|)$ by σ_t and for any subspace W of X we denote the set $t(\bar{\sigma}) \cap W$ by σ_W. If X is a differentiable manifold then we will always assume that the sets σ_t are locally closed differentiable submanifolds of X.

We say that *the triangulation $t'\colon \mathcal{C} \to C$ is a subtriangulation of t* if C is a subspace of X, $\mathcal{C}$ is a simplicial subcomplex of $\mathcal{S}$ and t' equals the restriction of t to the space $|\mathcal{C}|$ of $\mathcal{C}$. If such a triangulation t' exists we also say that *t subtriangulates C*. Note that for any simplex σ of $\mathcal{S}$ with $\sigma_t \subset X - C$ the closure of σ_t in $X - C$ is equal to $\sigma_X - C$. □

(2.4) LEMMA. *Let W be a subset of a topological space X. Let $t\colon (S, \Delta) \to X$ be a triangulation of X which subtriangulates $X - W$. Then:*

1. *W is an open subspace of X.*
2. *$\sigma_t \subset X - W$ or $\sigma_t \subset W$ for every σ in Δ.*
3. *For every $D \subset \Delta$ the subset $\cup_{\sigma \in D} \sigma_W$ of W is a locally finite union and hence closed in W.*
4. *The set σ_W is connected for each σ in Δ.*
5. *For every simplex σ in Δ we have*

$$\sigma_W = \bigcup_{\substack{\tau_t \subset W \\ \tau \text{ face of } \sigma}} \tau_t .$$

The proof of this lemma is elementary. It can be found partially in [Ka-Sch, VII, §1, pp. 321–322].

(2.5) *Definition.* Let $t\colon \mathcal{S} \to X$ be a triangulation of a topological space X and let F be a closed subspace of X. We say that *t is good with respect to F* if t subtriangulates F and for each simplex σ of $\mathcal{S}$ with $\sigma_t \cap F = \emptyset$ there exists a vertex p of σ with $p \notin F$. (Here we identify p with the unique point in the singleton $\{p\}_t$.) □

(2.6) *Example.* Let t be the triangulation of the 2-sphere sketched in Figure 1. Let E be the "equator" and M the "zero meridian". Then t subtriangulates M and E. Moreover t is good with respect to E, but not with respect to M. It is also good with respect to $\{p_0, p_2\}$ but not with respect to $\{p_1, p_2\}$.

(2.7) *Remark.* A triangulation $t\colon \mathcal{S} \to X$ which subtriangulates a closed subspace F of X can always be refined to obtain a triangulation t' of X which is good with respect to F. You can take the first barycentric subdivision of t for example.

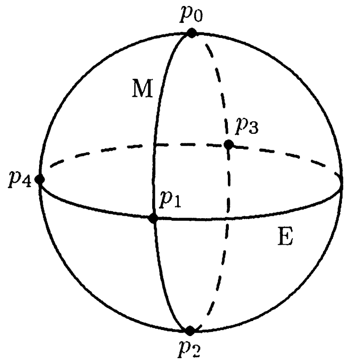

Figure 1

2.3 Orientations

(2.8) *Definition.* Let $\mathcal{S} = (S, \Delta)$ be a simplicial complex. For a simplex $\sigma = \{p_0, \ldots, p_i\}$ we define an *orientation of* σ as a pair $(s, (p_{k_0}, \ldots, p_{k_i}))$ where $s \in \{1, -1\}$ is a sign and $\sigma = \{p_{k_0}, \ldots, p_{k_i}\}$, modulo the following equivalence relation $\sim$. We say that $(s, (p_{k_0}, \ldots, p_{k_i})) \sim (r, (p_{l_0}, \ldots, p_{l_i}))$ if the permutation f of the vertices of σ such that $f(p_{k_j}) = p_{l_j}$ for all $0 \leq j \leq i$, has sign equal to $s.r$. We denote the class of $(s, (p_{k_0}, \ldots, p_{k_i}))$ by $s[p_{k_0}, \ldots, p_{k_i}]$. □

(2.9) *Definition.* Let $\sigma = \{p_0, \ldots, p_i\} \in \Delta$ and let $\tau = \{p_0, \ldots, \hat{p}_j, \ldots, p_i\}$ be a facet of σ (where $\hat{p}_j$ means omitting p_j). Choose an orientation $o(\sigma) = s.[p_0, \ldots, p_i]$ of σ. Then we define the *orientation on* τ *induced by* $o(\sigma)$, denoted by $o(\sigma)|\tau$, as the orientation of τ given by $s(-1)^j[p_0, \ldots, \hat{p}_j, \ldots, p_i]$. One checks that this is well-defined. □

(2.10) *Definition.* An *orientation o of a triangulation t*: $\mathcal{S} \to X$ of a topological space X, is a map which maps each simplex σ of $\mathcal{S}$ to an orientation $o(\sigma)$ of the simplex σ. Note that the orientation $o(\tau)$ of a face τ of a simplex σ is *not* determined by the orientation $o(\sigma)$. Thus the orientations of the simplices need not be 'compatible'. □

2.4 Families of supports

(2.11) *Definition.* Let X be a topological space. A *family of supports on* X is a family φ of closed subsets of X such that:

1. A closed subset of a member of φ belongs to φ.

2. φ is closed under finite unions.

The family φ of supports on X is said to be *paracompactifying* if in addition:

3. Each element of φ is paracompact.
4. Each element of φ has a neighborhood which is in φ.

We write $c(X)$,resp. $cl(X)$, for the family of supports on X consisting of all compact, resp. closed, subsets of X. If W is a subspace of X and φ a family of supports on X, let $\varphi \cap W := \{K \cap W | K \in \varphi\}$ and $\varphi | W := \{K \in \varphi | K \subset W\}$. Both are families of supports on W. □

3. A sheaf homology theory with supports

In this section we build up our homology theory. We assume the following data.

(3.1) *Data.* Let X be a topological space and let W be an open subspace of X. Let L be a ring and $\mathcal{F}$ a sheaf of L-modules on W. Let φ be a family of supports on W. Let t: $(\mathcal{S}, \Delta) \to X$ be a triangulation of X which subtriangulates $X - W$ and let o be an orientation of t.

(3.2) *Notation.* Let $\sigma \in \Delta$. Then we define

$$\mathcal{F}(\sigma_W) := \varinjlim_{\sigma_W \subset V \subset W} \mathcal{F}(V).$$

In this direct limit V runs through all open subsets of W containing σ_W. Note that we have a natural restriction $\mathcal{F}(\sigma_W) \to \mathcal{F}(\tau_W)$: $c \mapsto c|\tau_W$ for a subsimplex τ of σ.

(3.3) *Definition.* We define $C_i(W, \mathcal{F}, t, o)$ as the direct product of L-modules $\prod_{\sigma \in \Delta_i} \mathcal{F}(\sigma_W)$. We denote an element of $\prod_{\sigma \in \Delta_i} \mathcal{F}(\sigma_W)$ by $\prod_{\sigma \in \Delta_i} c(\sigma_W)\sigma_W$, where $c(\sigma_W) \in \mathcal{F}(\sigma_W)$. Note that this notation is justified since for any two different i-dimensional simplices σ and τ of $\mathcal{S}$ we have $\sigma_W \neq \tau_W$ provided that these last two sets are non-empty. If only a finite number of the $c(\sigma_W)$ are non-zero we also write $\sum_{\sigma \in \Delta_i} c(\sigma_W)\sigma_W$. If $\Delta_i = \emptyset$ we agree $C_i(W, \mathcal{F}, t, o) = 0$. □

(3.4) *Definition.* If τ is a facet of a simplex σ we define the sign $\epsilon(\sigma, \tau)$ by the relation $o(\sigma)|\tau = \epsilon(\sigma, \tau)o(\tau)$. □

(3.5) *Remark.* If τ_1 and τ_2 are two different facets of a simplex $\sigma \in \Delta$ with non-empty intersection ρ, one checks that

$$\epsilon(\sigma, \tau_1).\epsilon(\tau_1, \rho) = -\epsilon(\sigma, \tau_2).\epsilon(\tau_2, \rho).$$

(3.6) *Definition.* Now we define boundary morphisms

$$\partial_i\colon\ C_i(W,\mathcal{F},t,o)\to C_{i-1}(W,\mathcal{F},t,o),$$

for $i\geq 0$. If $\Delta_i=\emptyset$ or $i=0$ we put $\partial_i:=0$. Otherwise we define

$$\partial_i(c(\sigma_W)\sigma_W):=\prod_{\tau\subset\sigma,\tau\in\Delta_{i-1}}(\epsilon(\sigma,\tau)c(\sigma_W)|\tau_W)\tau_W$$

for $\sigma\in\Delta_i$, $c(\sigma_W)\in\mathcal{F}(\sigma_W)$. Here $c(\sigma_W)|\tau_W$ is the image of $c(\sigma_W)$ under the natural restriction $\mathcal{F}(\sigma_W)\to\mathcal{F}(\tau_W)$. We extend this definition by linearity to $C_i(W,\mathcal{F},t,o)$.

Using Remark 3.5 one checks that $\partial_{i-1}\partial_i=0$ for all i. Thus we have constructed a chain complex of L-modules $(C_.(W,\mathcal{F},t,o),\partial_.)$. □

(3.7) *Definition.* For $c=\prod_{\sigma\in\Delta_i}c(\sigma_W)\sigma_W\in C_i(W,\mathcal{F},t,o)$ we define the *support of c* by

$$\operatorname{supp}(c):=\bigcup_{\substack{\sigma\in\Delta_i\\ c(\sigma_W)\neq 0}}\sigma_W.$$

Remark that supp(c) is a closed subset of W by lemma 2.4. We also define

$$C_i^{\varphi}(W,\mathcal{F},t,o):=\{c\in C_i(W,\mathcal{F},t,o)|\operatorname{supp}(c)\in\varphi\}\quad\square.$$

(3.8) *Definition.* Since $\operatorname{supp}(\partial_i(c))\subset\operatorname{supp}(c)$ for $c\in C_i(W,\mathcal{F},t,o)$ we have $\partial_i\colon\ C_i^{\varphi}(W,\mathcal{F},t,o)\to C_{i-1}^{\varphi}(W,\mathcal{F},t,o)$. Thus again we have a chain complex of L-modules $(C_.^{\varphi}(W,\mathcal{F},t,o),\partial_.)$. We define $H_.^{\varphi}(W,\mathcal{F},t,o)$ as the homology of the chain complex $(C_.^{\varphi}(W,\mathcal{F},t,o),\partial_.)$. □

(3.9) *Notation.* We denote the module of i-dimensional cycles, resp. boundaries, by $Z_i^{\varphi}(W,\mathcal{F},t,o)$, resp. $B_i^{\varphi}(W,\mathcal{F},t,o)$. We denote $H_.^{cl(W)}(W,\mathcal{F},t,o)$ by $H_.(W,\mathcal{F},t,o)$.

(3.10) *Remark.* The homology modules $H_i^{\varphi}(W,\mathcal{F},t,o)$ depend a priori on the chosen triangulation t and orientation o. However it is easy to check that they don't depend on the chosen orientation. Later on (see 4.3) we will see that in some interesting cases they don't depend on the triangulation neither.

(3.11) *Example.* Let X be the complex projective line and let $W=X-\{0,\infty\}$. We identify W with $\mathbb{C}-\{0\}$ and choose an affine coordinate z on W. Let c,d be relatively prime positive integers with $\alpha:=c/d\notin\mathbb{Z}$. Let ω be the rational differential 1-form $z^c(dz)^d$ on W of degree d, i.e., $\omega\in\Gamma(W,{\Omega^1_{X/\mathbb{R}}}^{\otimes d})$. Let $\mathcal{L}(\omega^{1/d})$ be the locally constant sheaf of $\mathbb{C}$-vectorspaces on W associated to $\omega^{1/d}$, which is locally free of rank 1. A non-zero section of $\mathcal{L}(\omega^{1/d})$ on a connected open U is an analytic branch of $\omega^{1/d}$ on U multiplied with a complex number.

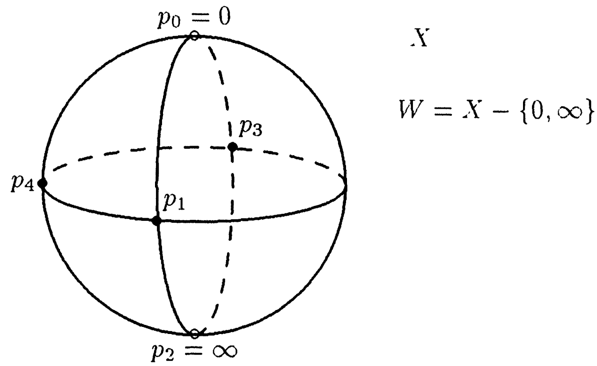

Figure 2

We choose a triangulation t of X which subtriangulates $X - W$, as in Figure 2. We also choose an orientation o of t such that all 2-dimensional simplices have the same orientation, i.e., the orientation induced by one orientation of the Riemann sphere X.

Now we will show that $H_i(W, \mathcal{L}(\omega^{1/d}), t, o) = 0$ for all i. It is clear that all 2-dimensional cycles correspond with global sections of $\mathcal{L}(\omega^{1/d})$ on W. Hence $H_2(W, \mathcal{L}(\omega^{1/d}), t, o)$ vanishes since $\alpha \notin \mathbb{Z}$.

Denote the simplex $\{p_i, p_j\}$ by σ_{ij}. Let $z = \zeta(\sigma_{01})_W$ be a chain. Turning one time around zero we see that $(1 - e^{2\pi i\alpha})z$ is homologous to a chain which contains only simplices with only p_1, p_3 or p_4 as vertices. Using similar arguments one sees that every 1-dimensional cycle is homologous to a cycle of the form $a_{13}(\sigma_{13})_W + a_{34}(\sigma_{34})_W + a_{41}(\sigma_{41})_W$. Such a cycle must vanish since otherwise there would exist an analytic branch of $z^\alpha dz$ on an open neighborhood of $(\sigma_{13})_W \cup (\sigma_{34})_W \cup (\sigma_{41})_W$. This proves that $H_1(W, \mathcal{L}(\omega^{1/d}), t, o) = 0$.

Finally every vertex of t in W is clearly also a boundary, hence also $H_0(W, \mathcal{L}(\omega^{1/d}), t, o) = 0$. Similarly one verifies $H_i^\varphi(W, \mathcal{L}(\omega^{1/d}), t, o) = 0$ for all i and φ equal to $c(W)$, $c(X - \{0\}) \cap W$ or $c(X - \{\infty\}) \cap W$.

4. Statement of the main results

In this section we assume the following data.

(4.1) *Data.* Let X be a real differentiable manifold of dimension d and let W be an open subspace of X. Let L be a ring which is a flat $\mathbb{Z}$-module and let $\mathcal{L}$ be a locally constant sheaf of L-modules on W which is locally free of finite rank r on W. Let φ be a family of supports on W. Let t: $(\mathcal{S}, \Delta) \to X$ be a triangulation of X which is good with respect to $X - W$ (see Definition 2.5). Let o be an orientation of t with respect to which we consider homology. The orientation sheaf of L-modules on W will be denoted by $\mathcal{O}_L$. Thus $\mathcal{O}_L = \mathcal{O} \otimes L$ where $\mathcal{O}$ denotes the orientation sheaf of $\mathbb{Z}$-modules on W.

In this article we prove the following theorem and propositions.

(4.2) POINCARÉ DUALITY THEOREM. *Assume Data* (4.1).
Assume:

1. *φ has the union property, i.e., for every K_0 in φ and for every locally finite family $\mathcal{K}$ of elements in φ the union $\cup\{K \in \mathcal{K} | K \cap K_0 \neq \emptyset\}$ belongs to φ.*
2. $H^j_{\varphi|\sigma_W}(\sigma_W, L) = 0$ *for all* $\sigma \in \Delta$, $j > 0$.

Then for all i, $0 \leq i \leq d$, there is a natural isomorphism

$$H^{\varphi}_{d-i}(W, \mathcal{L}, t, o) \xrightarrow{\simeq} H^i_{\varphi}(W, \mathcal{L} \otimes \mathcal{O}_L).$$

The cohomology modules here are the classical sheaf cohomology modules (e.g., see [Go]). Furthermore these isomorphisms are natural with respect to inclusion of families of supports. More precisely we have commutative diagrams

$$\begin{array}{ccc} H^{\varphi_1}_{d-i}(W, \mathcal{L}, t, o) & \xrightarrow{\simeq} & H^i_{\varphi_1}(W, \mathcal{L} \otimes \mathcal{O}_L) \\ \downarrow & \odot & \downarrow \\ H^{\varphi_2}_{d-i}(W, \mathcal{L}, t, o) & \xrightarrow{\simeq} & H^i_{\varphi_2}(W, \mathcal{L} \otimes \mathcal{O}_L) \end{array}$$

if φ_1, φ_2 are two families of supports as in Data 4.1 *which satisfy conditions* 1 *and* 2 *and for which $\varphi_1 \subset \varphi_2$. Here the vertical maps are the natural maps induced by the inclusion $\varphi_1 \subset \varphi_2$.*

(4.3) COROLLARY. *The homology modules $H^{\varphi}_i(W, \mathcal{L}, t, o)$ don't depend on the triangulation t nor on the orientation o (as long as t and φ satisfy the conditions of the Poincaré Duality Theorem* (4.2)).

(4.4) PROPOSITION A. *Assume Data* (4.1). *Then φ has the union property (see* (4.2)) *in each of the following cases*:

1. *X is compact.*
2. *$\varphi = cl(W)$, the family of all closed subsets of W.*
3. *$\varphi = c(X - F) \cap W$, where F is a closed subset of $X - W$.*

(4.5) *Remark.* If X is a compact Hausdorff space, then a closed subset of W belongs to the family $\varphi = c(X - F) \cap W$ if and only if its closure in X is disjoint from F. For such a family the second condition of the Poincare Duality Theorem is also satisfied, at least if we add the very weak condition that t is good with respect to F.

(4.6) PROPOSITION B. *Assume Data* (4.1). *Let F be a closed subset of $X - W$. Suppose that t is a good triangulation with respect to F. Let $\varphi = c(X - F) \cap W$. Then $H^j_{\varphi|\sigma_W}(\sigma_W, L) = 0$ for all $\sigma \in \Delta$ and $j > 0$.*

(4.7) *Remark.* There are two important cases in which the Poincaré Duality Theorem applies. Take $X = W$.

If $\varphi = c(W)$, the family of all compact subsets of W, then condition 1 is satisfied by Proposition A and condition 2 is satisfied by Proposition B. Hence our homology modules $H_i^{c(W)}(W, \mathcal{L}, t, o)$ are isomorphic to the classical singular homology modules on W.

If $\varphi = cl(W)$, the family of all closed subsets of W, then condition 2 is satisfied since the σ_W are contractible. Also condition 1 is satisfied by Proposition A and our homology modules $H_i(W, \mathcal{L}, t, o)$ are isomorphic to the classical Borel-Moore homology modules on W (e.g., see [Bo, I,2.2]).

(4.8) *Example.* We have $H^i(\mathbb{C} - \{0\}, \mathcal{L}(\omega^{1/d})) = 0$ where ω is the differential form of example 3.11. This follows from the Poincaré Duality Theorem (4.2) and the results of example 3.11. (Note that the triangulation t in that example satisfies the conditions of the Poincaré Duality Theorem. Use proposition B with $F = \emptyset$.)

5. Proofs of Propositions A and B

Proof of Proposition A. The first two cases are evident, so suppose $\varphi = c(X - F) \cap W$. Let $K_0 \in \varphi$. Then there exists a compact subset K_0' of $X - F$ such that $K_0 = K_0' \cap W$. Let $\mathcal{K}$ be a locally finite family of elements in φ. Then there are only finitely many elements of $\mathcal{K}$ that meet K_0'. Hence $\cup\{K \in \mathcal{K} | K \cap K_0 \neq \emptyset\}$ belongs to φ. □

(5.1) LEMMA. *Let t: $\mathcal{S} \to X$ be a triangulation of a space X which is good with respect to a subspace F. Let σ be a simplex of $\mathcal{S}$ such that $\sigma_X \cap F \neq \emptyset$. Then there exists a subsimplex τ of σ such that $\sigma_X \cap F = \tau_X$.*

Proof. We prove this lemma by induction on $\dim \sigma$. If $\dim \sigma = 0$ or if $\sigma_X \subset F$ this is clear. So assume $\sigma_X \not\subset F$ and $\dim \sigma > 0$. Then $\sigma_t \cap F = \emptyset$ by lemma 2.4. Hence there exists a vertex p of σ with $p \notin F$ since t is good with respect to F. Let $\rho = \sigma - \{p\}$. We claim that $\sigma_X \cap F = \rho_X \cap F$. To prove this claim it suffices, by Lemma 2.4, to show that for every face λ of σ with $p \in \lambda$, we have $\lambda_t \cap F = \emptyset$. Choose such a face λ and suppose that $\lambda_t \cap F \neq \emptyset$. Then $\lambda_t \subset F$, by Lemma 2.4 again, and hence $p \in \lambda_X \subset F$ since F is closed. But this is in contradiction with $p \notin F$. By induction on $\dim \sigma$ there exists a subsimplex τ of ρ and hence of σ such that $\tau_X = \rho_X \cap F$. □

Proof of Proposition B. Fix σ in Δ. If $\sigma_X \cap F = \emptyset$ then $\varphi|\sigma_W = cl(\sigma_W)$ and the proposition follows since σ_W is contractible. So we may assume that $\sigma_X \cap F \neq \emptyset$. Since t is good with respect to F it follows from Lemma (5.1) that there exists a subsimplex τ of σ such that $\sigma_X \cap F = \tau_X$. Let $\psi := cl(\sigma_W \cup \tau_X)$, the family of all closed subsets of $\sigma_W \cup \tau_X$. Since $\sigma_W \cup \tau_X$ is paracompact this is a paracompactifying family. Then we have an exact sequence (see [Br, II,10.2]):

$$\cdots \to H^j_{\psi|\sigma_W}(\sigma_W, L) \to H^j_\psi(\sigma_W \cup \tau_X, L) \to H^j_{\psi|\tau_X}(\tau_X, L) \to H^{j+1}_{\psi|\sigma_W}(\sigma_W, L) \to \cdots$$

Since τ_X is closed in $\sigma_W \cup \tau_X$ we see $\psi|\tau_X = cl(\tau_X)$. One checks that

$$\begin{aligned}
\psi|\sigma_W &= cl(\sigma_W \cup \tau_X)|\sigma_W \\
&= (c(\sigma_X) \cap (\sigma_W \cup \tau_X))|\sigma_W \\
&= c(\sigma_X - \tau_X) \cap \sigma_W \\
&= (c(X - F) \cap W)|\sigma_W \\
&= \varphi|\sigma_W
\end{aligned}$$

Since $\sigma_W \cup \tau_X$ and τ_X are both contractible, $H^j_\psi(\sigma_W \cup \tau_X, L)$ and $H^j_{\psi|\tau_X}(\tau_X, L)$ vanish for $j > 0$. Then the sequence implies that $H^j_{\psi|\sigma_W}(\sigma_W, L) = 0$ for $j \geq 2$. Because $\tau_X \neq \emptyset$ the map from $H^0_\psi(\sigma_W \cup \tau_X, L)$ to $H^0_{\psi|\tau_X}(\tau_X, L)$ is an isomorphism, so $H^1_{\psi|\sigma_W}(\sigma_W, L) = 0$. Since $\psi|\sigma_W = \varphi|\sigma_W$ this proves the proposition. □

6. Proof of the Poincaré Duality Theorem

In this section we will prove the Poincaré Duality Theorem (4.2). In the first two paragraphs we give some more results on triangulations and sheaves. The reader who wants to skip the details of the proof may proceed directly to paragraph 3 in which we outline the two main steps of the proof. These steps are proved in the two last paragraphs.

6.1 Triangulations

(6.1) *Definition.* Let $\mathcal{S} = (S, \Delta)$ be a simplicial complex and $p \in |\mathcal{S}|$. Let $\Delta\mathcal{N}_p$ be the set of all simplices σ in Δ for which p belongs to $\overline{\sigma}$ together with all their faces (i.e., non-empty subsets of σ). Let $S\mathcal{N}_p$ be the set of all vertices of simplices in $\Delta\mathcal{N}_p$. Then $\mathcal{N}_p = (S\mathcal{N}_p, \Delta\mathcal{N}_p)$ is a simplicial complex, called the *simplicial neighborhood of p in $\mathcal{S}$*. Let $\Delta\mathcal{L}_p$ be the subset of $\Delta\mathcal{N}_p$ consisting of all simplices σ in $\Delta\mathcal{N}_p$ for which p doesn't belong to $\overline{\sigma}$ and let $S\mathcal{L}_p$ be the set of all vertices of simplices in $\Delta\mathcal{L}_p$. Then $\mathcal{L}_p = (S\mathcal{L}_p, \Delta\mathcal{L}_p)$ is a simplicial complex, called the *simplicial link of p in $\mathcal{S}$*. □

(6.2) LEMMA. *Let $\mathcal{S}$ be a simplicial complex such that $|\mathcal{S}|$ is a d-dimensional topological manifold. Let $p \in |\mathcal{S}|$. Then*:

1. $|\mathcal{N}_p|$ *is contractible.*
2. $|\mathcal{L}_p|$ *is homeomorphic to the* $(d-1)$*-dimensional sphere* S_{d-1}.
3. $H_i(|\mathcal{N}_p|, |\mathcal{L}_p|) = 0$ *for* $0 \leq i \leq d-1$.

Here $H_i(-,-)$ *means the standard singular relative homology.*

Proof. For parts 1 and 2 we refer to [Mau, Prop. 2.4.4, p. 43; Prop. 3.4.3, p. 89]. The third part follows from parts 1 and 2 and the exact homology sequence for the pair $(|\mathcal{N}_p|, |\mathcal{L}_p|)$. □

(6.3) LEMMA. *Let $\mathcal{S}$ be a simplicial complex such that $|\mathcal{S}|$ is a d-dimensional topological manifold. Let τ be a simplex of $\mathcal{S}$ such that $|\tau|$ is a $(d-1)$-dimensional locally closed submanifold of $|\mathcal{S}|$. Then there exist exactly two d-dimensional simplices of $\mathcal{S}$ which contain τ.*

Proof. Take a point p in $|\tau|$ and an open neighborhood U of p in $|\mathcal{S}|$ such that there exists a homeomorphism φ from U to an open subset D of $\mathbb{R}^d$ under which p corresponds to the origin 0 and $|\tau|$ to the set $C = \{x \in D | x_d = 0\}$. Let σ be a simplex of $\mathcal{S}$ of dimension d which contains τ. By making U eventually smaller we may suppose that $|\sigma| \cap U$ is a non-empty connected component of $U - |\tau|$. But $U - |\tau|$ has exactly two connected components whose closure in U contains p. This proves the lemma. □

(6.4) LEMMA. *Let $\mathcal{S} = (S, \Delta)$ be a simplicial complex such that $|\mathcal{S}|$ is a d-dimensional topological manifold. Suppose that $|\sigma|$ is a locally closed submanifold of $|\mathcal{S}|$ for each simplex σ of $\mathcal{S}$. Fix a point p in $|\mathcal{S}|$ and denote the set of j-dimensional simplices σ of $|\mathcal{S}|$ with $p \in \overline{\sigma}$ by $\Delta_{j,p}$. Let $\sigma_1, \sigma_2 \in \Delta_{d,p}$. Then there exists a finite sequence $\sigma_1 = \lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k = \sigma_2$ with $\lambda_i \in \Delta_{d,p}$ and $\lambda_i \cap \lambda_{i+1} \in \Delta_{d-1,p}$ for all i.*

Proof. Define an equivalence relation $\sim$ on $\Delta_{d,p}$ by saying that $\sigma_1 \sim \sigma_2$ if there exists a finite sequence $\sigma_1 = \lambda_1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k = \sigma_2$ with $\lambda_i \in \Delta_{d,p}$ and $\lambda_i \cap \lambda_{i+1} \in \Delta_{d-1,p}$ for all i. Let A be one of the equivalence classes and let B be the union of the other classes. Note that $A \neq \emptyset$. Suppose that $B \neq \emptyset$. Let $\overline{A} := \cup_{\sigma \in A} \overline{\sigma}$ and $\overline{B} := \cup_{\sigma \in B} \overline{\sigma}$ and choose an open connected neighborhood U of p in $|\mathcal{S}|$ contained in $\overline{A} \cup \overline{B}$. Then $\overline{A} \cap \overline{B} \cap U$ is a finite union of closed submanifolds of U of codimension at least 2. Hence $U - (\overline{A} \cap \overline{B})$ is connected. Then we can find a path $\rho\colon [0, 1] \to U - (\overline{A} \cap \overline{B})$ with $\rho(0) \in \overline{A} \cap U$ and $\rho(1) \in \overline{B} \cap U$ (which is non-empty). But $\mathrm{Im}(\rho) \cap \overline{A}$ and $\mathrm{Im}(\rho) \cap \overline{B}$ are closed in $\mathrm{Im}(\rho)$ and they are disjoint. This contradicts the connectedness of $\mathrm{Im}(\rho)$. Hence B must be empty which proves the lemma. □

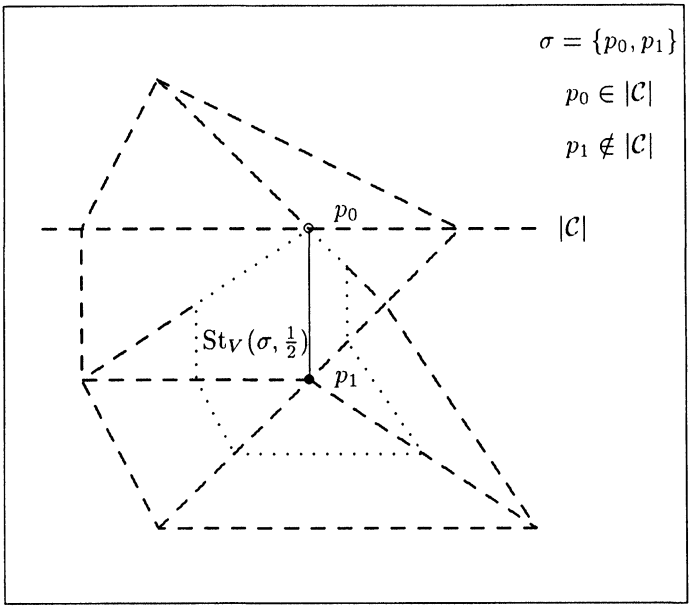

Figure 3

(6.5) *Definition.* Let $\mathcal{S} = (S, \Delta)$ be a simplicial complex and let $\mathcal{C}$ be a simplicial subcomplex of $\mathcal{S}$. Denote $|\mathcal{C}|$ by C and $|\mathcal{S}| - |\mathcal{C}|$ by V. Assume that $\mathcal{S}$ is good with respect to $\mathcal{C}$, which means that for each simplex σ of $\mathcal{S}$ with $|\sigma| \cap |\mathcal{C}| = \emptyset$ there exists a vertex p of σ with $p \in V$. Let $\tau, \sigma \in \Delta$ and let $0 < \epsilon \leq 1$. Then we define

$$\tau_V(\sigma, \epsilon) := \left\{ \sum_{p\in\tau} \lambda_p p \in V \mid \sum_{p\in\tau-\sigma} \lambda_p < \epsilon \text{ and } \sum_{p\in(\tau\cap\sigma)-C} \lambda_p > 0 \right\}.$$

We also define

$$\mathrm{St}_V(\sigma, \epsilon) := \cup_{\tau\in\Delta} \tau_V(\sigma, \epsilon),$$

called the *epsilon star of σ in V*. □

(6.6) LEMMA. *Assume the data of Definition* (6.5). *Then*:

1. *The sets $\tau_V(\sigma, \epsilon)$, $0 < \epsilon \leq 1$, form a base of open neighborhoods of $\tau_V \cap \sigma_V$ in τ_V.*
2. $\tau_V(\sigma, \epsilon) = \emptyset$ *if* $\tau \cap \sigma \subset C$.
3. $\tau_V(\sigma, \epsilon) = \tau_V$ *if* $\tau \subset \sigma$.

4. $\mathrm{St}_V(\sigma, \epsilon) \cap \tau_V = \tau_V(\sigma, \epsilon)$.
5. *The sets* $\mathrm{St}_V(\sigma, \epsilon), 0 < \epsilon \leq 1,$ *form a base of open contractible neighborhoods of* σ_V *in* V.

Proof. If $x \in \tau_V \cap \sigma_V = (\tau \cap \sigma)_V$ then $x = \sum_{p \in \tau \cap \sigma} \lambda_p p$ with $\sum_{p \in (\tau \cap \sigma) - C} \lambda_p > 0$ because of Lemma (5.1). Thus $x \in \tau_V(\sigma, \epsilon)$ for $0 < \epsilon \leq 1$. Then part 1 is clear. If $\tau \cap \sigma \subset C$ then $(\tau \cap \sigma) - C = \emptyset$ from which part 2 follows. If $\tau \subset \sigma$ then $\tau - \sigma = \emptyset$ and $\tau \cap \sigma = \tau$ from which part 3 follows since $\mathcal{S}$ is good with respect to $\mathcal{C}$. Let x belong to $\mathrm{St}_V(\sigma, \epsilon)$. Then $x \in \varphi_V(\sigma, \epsilon)$ for some φ in Δ. Now suppose that $\tau \in \Delta$ such that $x \in \tau_V$. Then $x = \sum_{p \in \varphi \cap \tau} \lambda_p p$ with $\sum_{p \in (\varphi \cap \tau) - \sigma} \lambda_p < \epsilon$ and $\sum_{p \in (\varphi \cap \tau \cap \sigma) - C} \lambda_p > 0$. Thus $x \in (\varphi \cap \tau)_V(\sigma, \epsilon) \subset \tau_V(\sigma, \epsilon)$. This proves part 4. Now we prove that $\mathrm{St}_V(\sigma, \epsilon)$ is open in V. Let x belong to $\mathrm{St}_V(\sigma, \epsilon)$. Since there are only finitely many τ in Δ for which $x \in \tau_V$, there exists an open disk U in V around x such that $U \cap \tau_V \subset \tau_V(\sigma, \epsilon)$ if $x \in \tau_V$ and $U \cap \tau_V = \emptyset$ if $x \notin \tau_V$. But then $U \subset \mathrm{St}_V(\sigma, \epsilon)$ (indeed, suppose $y \in U$ and choose τ in Δ such that $y \in \tau_V$; then $U \cap \tau_V \neq \emptyset$ and hence x must belong to τ_V, so $U \cap \tau_V \subset \tau_V(\sigma, \epsilon) \subset \mathrm{St}_V(\sigma, \epsilon)$; thus $y \in \mathrm{St}_V(\sigma, \epsilon)$).

It is also clear that $\mathrm{St}_V(\sigma, \epsilon)$ contains σ_V since $\sigma_V = \sigma_V(\sigma, \epsilon)$ by part 3.

Now we show that $\mathrm{St}_V(\sigma, \epsilon)$ is contractible. For this purpose we show that σ_V and $\mathrm{St}_V(\sigma, \epsilon)$ are homotopically equivalent. Let $q \in \mathrm{St}_V(\sigma, \epsilon)$ and choose τ in Δ such that $q \in \tau_V(\sigma, \epsilon)$. Then $q = \sum_{p \in \tau} \lambda_p p$ with $\sum_{p \in \tau - \sigma} \lambda_p < \epsilon$ and $\sum_{p \in (\tau \cap \sigma) - C} \lambda_p > 0$. Then we define $\pi(q) := \sum_{p \in \tau \cap \sigma} \frac{\lambda_p}{\sum_{p \in \tau \cap \sigma} \lambda_p} p \in \sigma_V$. This definition is independent of the choice of τ, so we get a well-defined map $\pi\colon \mathrm{St}_V(\sigma, \epsilon) \to \sigma_V$. Moreover $\pi|\sigma_V$ is the identity and π is continuous since it is clearly continuous on the sets $\tau_V(\sigma, \epsilon)$, which are closed in $\mathrm{St}_V(\sigma, \epsilon)$ (by part 4) and cover $\mathrm{St}_V(\sigma, \epsilon)$. Now we define $F\colon [0, 1] \times \mathrm{St}_V(\sigma, \epsilon) \to \mathrm{St}_V(\sigma, \epsilon)\colon (t, q) \mapsto t\pi(q) + (1 - t)q$. Then F is a homotopy, $F(0, -)$ is the identity on $\mathrm{St}_V(\sigma, \epsilon)$ and $F(1, -) = \pi$. Thus σ_V is a strong deformation retract of $\mathrm{St}_V(\sigma, \epsilon)$. Since σ_V is contractible by Lemma (5.1), $\mathrm{St}_V(\sigma, \epsilon)$ is contractible too.

Finally by part 1 and 4 it follows that the sets $\mathrm{St}_V(\sigma, \epsilon)$, $0 < \epsilon \leq 1$, form a base of neighborhoods of σ_V. □

(6.7) *Definition.* Let $t\colon \mathcal{S} \to X$ be a triangulation of a topological space X. Let $t'\colon \mathcal{C} \to C$ be a subtriangulation of t. Denote $|\mathcal{S}| - |\mathcal{C}|$ by V and let $W := X - C$. For σ, τ in Δ and $0 < \epsilon \leq 1$ we define $\tau_W(\sigma, \epsilon) := t(\tau_V(\sigma, \epsilon))$ and $\mathrm{St}_W(\sigma, \epsilon) := t(\mathrm{St}_V(\sigma, \epsilon))$, called the *epsilon star of* σ *in* W. □

(6.8) LEMMA. *Let* X *be a topological space and let* W *be an open subspace of* X. *Let* $t\colon (S, \Delta) \to X$ *be a triangulation of* X *which is good with respect to* $X - W$. *Let* $\tau, \sigma \in \Delta$ *and* $0 < \epsilon \leq 1$. *Then:*

1. *The sets* $\tau_W(\sigma, \epsilon), 0 < \epsilon \leq 1,$ *form a base of open neighborhoods of* $\tau_W \cap \sigma_W$ *in* τ_W.

2. $\tau_W(\sigma, \epsilon) = \emptyset$ *if* $\tau \cap \sigma \cap W = \emptyset$.
3. $\tau_W(\sigma, \epsilon) = \tau_W$ *if* $\tau \subset \sigma$.
4. $\mathrm{St}_W(\sigma, \epsilon) \cap \tau_W = \tau_W(\sigma, \epsilon)$.
5. *The sets* $\mathrm{St}_W(\sigma, \epsilon)$, $0 < \epsilon \leq 1$, *form a base of open contractible neighborhoods of* σ_W *in* W.

(6.9) LEMMA. *Let X be a topological space and let W be an open subspace of X. Let $t\colon \mathcal{S} \to X$ be a triangulation which is good with respect to $X - W$. Let L be a ring and let $\mathcal{L}$ be a locally constant sheaf of L-modules on W. Let σ be a simplex of $\mathcal{S}$. Then σ_W has a base of open contractible neighborhoods in W on which $\mathcal{L}$ is constant.*

Proof. This follows from Lemma (6.8) and the fact that locally constant sheaves are constant on simply connected opens. □

6.2 Sheaves

(6.10) *Definition.* Let $\mathcal{F}$ be a sheaf of L-modules on a topological space W. Let s be a global section of $\mathcal{F}$ on W, i.e. $s \in \mathcal{F}(W)$. Then we define supp(s), *the support of* s, as $\{p \in W | s_p \neq 0\}$. Note that this is a closed subset of W. All global sections with support in a family φ of supports in W form an L-module, denoted by $\Gamma_\varphi(\mathcal{F})$. Moreover $\Gamma_\varphi(-)$ defines a functor of the category of sheaves of L-modules on W to the category of L-modules. Let $(\mathcal{F}^\cdot, \partial^\cdot)$ be a cochain complex of sheaves of L-modules on W. Applying the functor $\Gamma_\varphi(-)$ we get a cochain complex $(\Gamma_\varphi(\mathcal{F}^\cdot), \partial^\cdot)$ of L-modules. The i-th cohomology module of this cochain complex we denote by $H^i(\Gamma_\varphi(\mathcal{F}^\cdot))$. Similarly we get a chain complex $(\Gamma_\varphi(\mathcal{F}_\cdot), \partial_\cdot)$ if $(\mathcal{F}_\cdot, \partial_\cdot)$ is a chain complex of sheaves of L-modules on W. The i-th homology module of this chain complex we denote by $H_i(\Gamma_\varphi(\mathcal{F}_\cdot))$. □

(6.11) *Definition.* Let W be a topological space and $\mathcal{F}$ a sheaf of L-modules on W. Let φ be a family of supports on W. The sheaf $\mathcal{F}$ is said to be *φ-acyclic* if $H^p_\varphi(W, \mathcal{F}) = 0$ for $p > 0$. The sheaf $\mathcal{F}$ is said to be *flabby* if $\mathcal{F}(W) \to \mathcal{F}(U)$ is onto for every open subset U of W. A resolution $\mathcal{C}^\cdot$ of $\mathcal{F}$ is called *φ-acyclic*, resp. *flabby*, if each $\mathcal{C}^i$ is φ-acyclic, resp. flabby. □

(6.12) LEMMA. *Let W be a topological space and $\mathcal{F}$ a flabby sheaf of L-modules on W. Then $\mathcal{F}$ is φ-acyclic for every family of supports φ on W.*

Proof. See [Br, II, 5.2]. □

(6.13) LEMMA. *Let W be a topological space and $\mathcal{C}^\cdot$ a φ_i-acyclic resolution of a sheaf $\mathcal{F}$ of L-modules on W, where φ_i, $i = 1, 2$, are two families of supports on W*

with $\varphi_1 \subset \varphi_2$. Then we have commutative diagrams for all $i \geq 0$:

$$\begin{array}{ccc} H^i(\Gamma_{\varphi_1}(\mathcal{C}^{\cdot})) & \stackrel{\simeq}{\to} & H^i_{\varphi_1}(W, \mathcal{F}) \\ \downarrow & \odot & \downarrow \\ H^i(\Gamma_{\varphi_2}(\mathcal{C}^{\cdot})) & \stackrel{\simeq}{\to} & H^i_{\varphi_2}(W, \mathcal{F}) \end{array}$$

Here the horizontal maps are isomorphisms and the vertical maps are the natural maps coming from the inclusion $\varphi_1 \subset \varphi_2$.

Proof. See [Br, II, 4.1]. □

6.3 Outline of the proof

There are two important steps in the proof.

1. We construct $\mathcal{C}_{\cdot}(\mathcal{L})$, a chain complex of sheaves of L-modules on W and show that
$$H^{\varphi}_{d-i}(W, \mathcal{L}, t, o) \stackrel{\simeq}{\to} H_{d-i}(\Gamma_{\varphi}(\mathcal{C}_{\cdot}(\mathcal{L}))).$$
Then $\mathcal{C}_{d-\cdot}(\mathcal{L})$ is a cochain complex of sheaves of L-modules on W and
$$H_{d-i}(\Gamma_{\varphi}(\mathcal{C}_{\cdot}(\mathcal{L}))) = H^i(\Gamma_{\varphi}(\mathcal{C}_{d-\cdot}(\mathcal{L}))).$$
2. We show that $\mathcal{C}_{d-\cdot}(\mathcal{L})$ is a φ-acyclic resolution of $\mathcal{L} \otimes \mathcal{O}_L$, so
$$H^i(\Gamma_{\varphi}(\mathcal{C}_{d-\cdot}(\mathcal{L}))) \stackrel{\simeq}{\to} H^i_{\varphi}(W, \mathcal{L} \otimes \mathcal{O}_L).$$

At the same time we will check that the isomorphisms in these steps are natural with respect to inclusion of families of supports.

6.4 Proof of step 1

(6.14) *Notation.* We write $\bigotimes$ and $\prod$ for the tensor product and direct product in the category of sheaves. For the tensor product we need only two factors. In the case of the direct product however we need an infinite number of terms. Note that the direct product presheaf of sheaves is again a sheaf. Also note that the tensor product sheaf of two sheaves is isomorphic to the sheafification of the tensor product presheaf of these sheaves.

(6.15) *Definition.* For a simplex σ in Δ let $P\mathcal{L}_{\sigma}$ be the presheaf on W with sections $\varinjlim_{U \cap \sigma_W \subset V \subset W} \mathcal{L}(V)$ above an open subset U of W, where V runs through all open neighborhoods of $U \cap \sigma_W$ in W. (Note that $P\mathcal{L}_{\sigma}(W) = \mathcal{L}(\sigma_W)$.) Let $\mathcal{L}_{\sigma}$ be the sheaf associated to the presheaf $P\mathcal{L}_{\sigma}$. Note that $\mathcal{L}_{\sigma} = (\mu_{\sigma})_*(\mathcal{L}|\sigma_W)$, where μ_{σ} denotes the embedding $\sigma_W \to W$.

Let $\mathcal{C}_i(\mathcal{L}), 0 \leq i \leq d$, be the sheaf $\prod_{\sigma \in \Delta_i} \mathcal{L}_\sigma$. We will define morphisms $\partial_i \colon \mathcal{C}_i(\mathcal{L}) \to \mathcal{C}_{i-1}(\mathcal{L})$ for $1 \leq i \leq d$. Let $\tau \in \Delta_{i-1}$ and let $\sigma \in \Delta_i$ such that τ is a facet of σ. Let $\epsilon(\sigma, \tau)$ be as in Definition 3.4. Let U be an open in W. Let V be an open in W containing $\sigma_W \cap U$. We have a morphism $\mathcal{L}(V) \to \mathcal{L}(V)$: $s \mapsto \epsilon(\sigma, \tau)s$. These morphisms are compatible with restrictions, hence we get a morphism $\mathcal{L}_\sigma \to \mathcal{L}_\tau$. Summing over all $\sigma \in \Delta_i$ which have τ as a facet we get a morphism $\mathcal{C}_i(\mathcal{L}) \to \mathcal{L}_\tau$ for every τ in Δ_{i-1}. These morphisms then give rise vf to a morphism $\partial_i \colon \mathcal{C}_i(\mathcal{L}) \to \mathcal{C}_{i-1}(\mathcal{L})$. □

(6.16) LEMMA. *$(\mathcal{C}.(\mathcal{L}), \partial.)$ is a chain complex of sheaves of L-modules on W.*

Proof. If $\mathcal{L} = \mathbb{Z}$, the constant sheaf of $\mathbb{Z}$-modules with stalks $\mathbb{Z}$, then this lemma follows from Remark 3.5. From this we get the result for general $\mathcal{L}$ by tensoring up. Indeed, it is straightforward to check that

$$\mathcal{C}.(\mathbb{Z}) \otimes \mathcal{L} \cong \prod_{\sigma \in \Delta.} (\mathbb{Z}_\sigma \otimes \mathcal{L}) \cong \mathcal{C}.(\mathcal{L}). \quad \square$$

(6.17) LEMMA. *There are natural isomorphisms $H_i^\varphi(W, \mathcal{L}, t, o) \xrightarrow{\cong} H_i(\Gamma_\varphi(\mathcal{C}.(\mathcal{L})))$. Furthermore we have commutative diagrams*

$$\begin{array}{ccc} H_i^{\varphi_1}(W, \mathcal{L}, t, o) & \xrightarrow{\cong} & H_i(\Gamma_{\varphi_1}(\mathcal{C}.(\mathcal{L}))) \\ \downarrow & \odot & \downarrow \\ H_i^{\varphi_2}(W, \mathcal{L}, t, o) & \xrightarrow{\cong} & H_i(\Gamma_{\varphi_2}(\mathcal{C}.(\mathcal{L}))) \end{array}$$

if φ_1, φ_2 are two families of supports on W with $\varphi_1 \subset \varphi_2$. Here the vertical maps are the natural maps coming from the inclusion $\varphi_1 \subset \varphi_2$.

Proof. Let W^σ be the connected component of W which contains σ_W. Then

$$\begin{aligned} \Gamma(\mathcal{C}_i(\mathcal{L})) &= \prod_{\sigma \in \Delta_i} \Gamma(\mathcal{L}_\sigma) \\ &= \prod_{\sigma \in \Delta_i} \mathcal{L}_\sigma(W^\sigma) \\ &\cong \prod_{\sigma \in \Delta_i} P\mathcal{L}_\sigma(W^\sigma) \\ &= \prod_{\sigma \in \Delta_i} \mathcal{L}(\sigma_W) \\ &= C_i(W, \mathcal{L}). \end{aligned}$$

The isomorphism follows from Lemma (6.18). These isomorphisms $\Gamma(\mathcal{C}_i(\mathcal{L})) \to C_i(W, \mathcal{L})$ clearly commute with ∂ and respect supports. Hence we have natural isomorphisms $H_i(\Gamma_\varphi(\mathcal{C}_.(\mathcal{L}))) \to H_i^\varphi(W, \mathcal{L}, t, o)$. The commutativity of the diagram is evident. □

(6.18) LEMMA. *Let L be a ring and $\mathcal{F}$ a presheaf of L-modules on a topological space W. Let U be an open path-connected subset of W such that for all points $p \in U$ and all open neighborhoods W_p of p in U there exists an open neighborhood $U_p \subset W_p$ of p such that the restriction res_{U,U_p}: $\mathcal{F}(U) \to \mathcal{F}(U_p)$ is an isomorphism. Then the natural map $\theta(U)$: $\mathcal{F}(U) \to \mathcal{S}(\mathcal{F})(U)$ is an isomorphism. Here $\mathcal{S}$ denotes the sheafification functor from the category of presheaves to the category of sheaves (see* [*Ha, II*, §1, *p.* 64]).

Proof. We may assume $U \neq \emptyset$. We will construct an inverse $\psi(U)$ for $\theta(U)$. Let $s \in \mathcal{S}(\mathcal{F})(U)$. Choose a point $p \in U$, an open neighborhood $W_p \subset U$ of p and $t \in \mathcal{F}(W_p)$ such that $t_q = s(q)$ for all $q \in W_p$. By assumption we have an open neighborhood $U_p \subset W_p$ of p such that res_{U,U_p}: $\mathcal{F}(U) \to \mathcal{F}(U_p)$ is an isomorphism. Define $\psi(U)(s)$ in $\mathcal{F}(U)$ by $res_{U,U_p}(\psi(U)(s)) = res_{W_p,U_p}(t)$. Using the condition on U again one sees that $\psi(U)(s)$ is independent of the choices made. One also checks that $\psi(U)$ is the inverse of $\theta(U)$. □

6.5 Proof of step 2

(6.19) LEMMA. *$\mathcal{C}_{d-.}(\mathcal{L})$ is a resolution of $\mathcal{L} \otimes \mathcal{O}$.*

Proof. Since $\mathcal{L} \otimes_{\mathbb{Z}} \mathcal{O} \cong \mathcal{L} \otimes_L \mathcal{O}_L$ and since L is a flat $\mathbb{Z}$-module it is sufficient to prove the lemma in case $\mathcal{L} = \mathbb{Z}$. We denote $\mathcal{C}_i(\mathbb{Z})$ by $\mathcal{C}_i$.

In this proof we introduce some data atached to a d-dimensional simplex σ of $\mathcal{S}$. By Lemma (6.9) we can choose a fixed contractible open neighborhood U_σ of σ_W in W on which $\mathcal{O}$ is constant. The orientation $o(\sigma)$ of σ, represented by $[p_0, \ldots, p_d]$, defines a unique element o_σ of $\mathcal{O}(\sigma_t)$, represented by the chart $\sigma_t \to \mathbb{R}^d$: $t(\sum \lambda_i p_i) \mapsto (\lambda_1, \ldots, \lambda_d)$. Then the restriction $\mathcal{O}(U_\sigma) \to \mathcal{O}(\sigma_t)$ is an isomorphism. Let g_σ be the generator of $\mathcal{O}(U_\sigma)$ which maps to o_σ under this isomorphism.

We start by constructing a morphism of sheaves ϵ: $\mathcal{O} \to \mathcal{C}_d$. It suffices to construct morphisms of presheaves ϵ_σ: $\mathcal{O} \to P\mathbb{Z}_\sigma$ for every σ in Δ_d. Let $\sigma \in \Delta_d$, let U be an open subset of W and $\eta \in \mathcal{O}(U)$. Let $\{U_i\}$ be the set of (non-empty) connected components of $U \cap U_\sigma$. Since the restrictions $\mathcal{O}(U_\sigma) \to \mathcal{O}(U_i)$ are isomorphisms we can find integers z_i such that $\eta|U_i = z_i g_\sigma|U_i$. Now there is a natural morphism f: $\prod_i \mathbb{Z}(U_i) \to P\mathbb{Z}_\sigma(U)$ since $\prod_i \mathbb{Z}(U_i) = \mathbb{Z}(U \cap U_\sigma)$. Then we define $\epsilon_\sigma(U)(\eta) := f(\prod_i z_i)$. One easily verifies that these maps are morphisms which commute with restrictions, giving us the required morphisms $\mathcal{O} \to P\mathbb{Z}_\sigma$.

We have to prove the exactness of the following sequence:

$$0 \to \mathcal{O} \xrightarrow{\epsilon} \mathcal{C}_d \xrightarrow{\partial_d} \mathcal{C}_{d-1} \xrightarrow{\partial_{d-1}} \cdots$$

It suffices to prove exactness at the stalks in points p in W. So let's fix a point p in W. To simplify notations we introduce a new chain complex $(C_.(W_p), \partial_.)$. Let $\Delta_{i,p}(W) := \{\sigma \in \Delta_i | p \in \sigma_W\}$. Define $C_i(W_p) := \bigoplus_{\sigma\in\Delta_{i,p}(W)} \mathbb{Z}$ and denote an element of $C_i(W_p)$ by $\sum_{\sigma\in\Delta_{i,p}(W)} z_\sigma\sigma$. Also define boundary operators

$$\partial_i\colon C_i(W_p) \to C_{i-1}(W_p)\colon \sum_{\sigma\in\Delta_{i,p}(W)} z_\sigma\sigma \mapsto \sum_{\sigma\in\Delta_{i,p}(W)} \sum_{\substack{\tau \text{ facet of } \sigma \\ \tau\in\Delta_{i-1,p}(W)}} z_\sigma\epsilon(\sigma,\tau)\tau.$$

This chain complex is isomorphic to the chain complex $((\mathcal{C}_.)_p, (\partial_.)_p)$. This follows since the stalk $(\mathcal{C}_i)_p$ is clearly isomorphic to $\prod_{\sigma\in\Delta_i}(\mathbb{Z}_\sigma)_p$ and $(\mathbb{Z}_\sigma)_p \cong \mathbb{Z}$ if $p \in \sigma_W$, $(\mathbb{Z}_\sigma)_p \cong 0$ otherwise. We will denote the composition of the morphism $\mathcal{O}_p \to (\mathcal{C}_d)_p$ and the isomorphism $(\mathcal{C}_d)_p \to C_d(W_p)$ also by ϵ. Thus we need to prove the exactness of the following sequence:

$$0 \to \mathcal{O}_p \xrightarrow{\epsilon} C_d(W_p) \xrightarrow{\partial_d} C_{d-1}(W_p) \xrightarrow{\partial_{d-1}} \cdots$$

First we prove exactness at $\mathcal{O}_p$. Choose $\eta \in \mathcal{O}_p$ such that $\epsilon(\eta) = 0$. Now $\epsilon(\eta) = \sum_{\sigma\in\Delta_{p,d}(W)} z_\sigma\sigma$ where the z_σ are integers such that $\eta = z_\sigma(g_\sigma)_p$ in $\mathcal{O}_p$. Since $\Delta_{d,p}(W) \neq \emptyset$ this implies that $\eta = 0$.

Now we prove exactness at $C_d(W_p)$. First we show that $\mathrm{im}(\epsilon) \subset \ker(\partial_d)$. Choose $\eta \in \mathcal{O}_p$ and let z_σ be as above, for σ in $\Delta_{d,p}(W)$. Then $\epsilon(\eta) = \sum_{\sigma\in\Delta_{d,p}(W)} z_\sigma\sigma$. Let $\tau \in \Delta_{d-1,p}(W)$. Since X is a d-dimensional manifold τ is the intersection of exactly two simplices σ_1, σ_2 in $\Delta_{d,p}(W)$ (see Lemma (6.3)). Moreover if $(g_{\sigma_1})_p = s.(g_{\sigma_2})_p$ with $s \in \{1, -1\}$ then $s = -\epsilon(\sigma_1, \tau).\epsilon(\sigma_2, \tau)$ (this can be checked on the level of simplicial complices by using the topological definition of orientation) and $z_{\sigma_1} = s.z_{\sigma_2}$. Hence the coefficient of τ in $\partial_d(\epsilon(\eta))$ is zero. Thus $\epsilon(\eta)$ belongs to $\ker(\partial_d)$.

Now we show that $\ker(\partial_d) \subset \mathrm{im}(\epsilon)$. Let $z = \sum_{\sigma\in\Delta_{d,p}(W)} z_\sigma\sigma \in \ker(\partial_d)$. For $\sigma \in \Delta_{d,p}(W)$ define $\eta_\sigma := z_\sigma(g_\sigma)_p$ in $\mathcal{O}_p$. We check that η_σ is independent of σ. Let $\sigma_1, \sigma_2 \in \Delta_{d,p}(W)$. Since X is a manifold we know that there exists a sequence $\sigma_1 = \lambda_1, \lambda_2, \dots, \lambda_{k-1}, \lambda_k = \sigma_2$ with λ_i in $\Delta_{d,p}(W)$ and $\lambda_i \cap \lambda_{i+1}$ in $\Delta_{d-1,p}(W)$ (see Lemma (6.4)). Thus we may assume that $\tau = \sigma_1 \cap \sigma_2 \in \Delta_{d-1,p}(W)$. Since $z \in \ker(\partial_d)$ we know that $\epsilon(\sigma_1, \tau)z_{\sigma_1} + \epsilon(\sigma_2, \tau)z_{\sigma_2} = 0$. Then

$$\eta_{\sigma_1} = z_{\sigma_1}(g_{\sigma_1})_p = -\epsilon(\sigma_1, \tau).\epsilon(\sigma_2, \tau)z_{\sigma_1}(g_{\sigma_2})_p = z_{\sigma_2}(g_{\sigma_2})_p = \eta_{\sigma_2}.$$

Thus we define $\eta := \eta_\sigma$ where $\sigma \in \Delta_{d,p}(W)$. Then by construction $\epsilon(\eta) = z$. Hence $z \in \mathrm{im}(\epsilon)$.

Finally we prove exactness at $C_i(W_p)$ for $0 \leq i \leq d-1$. Before we proceed we need to build up yet another chain complex. Let $\mathcal{N}_p$ be the simplicial neighborhood of $t^{-1}(p)$ in $\mathcal{S}$ and let $\mathcal{L}_p$ be the simplicial link of $t^{-1}(p)$ in $\mathcal{S}$ (see Definition (6.1)). Let $\Delta_i(\mathcal{N}_p) := \Delta_i \cap \Delta\mathcal{N}_p$ and let $\Delta_i(\mathcal{L}_p) := \Delta_i \cap \Delta\mathcal{L}_p$. Let $C_i(\mathcal{N}_p) := \bigoplus_{\sigma\in\Delta_i(\mathcal{N}_p)} \mathbb{Z}$ and denote an element of $C_i(\mathcal{N}_p)$ by $\sum_{\sigma\in\Delta_i(\mathcal{N}_p)} z_\sigma\sigma$. Define

$$\delta_i\colon C_i(\mathcal{N}_p) \to C_{i-1}(\mathcal{N}_p)\colon \sum_{\sigma\in\Delta_i(\mathcal{N}_p)} z_\sigma\sigma \mapsto \sum_{\sigma\in\Delta_i(\mathcal{N}_p)} \sum_{\tau \text{ facet of } \sigma} \epsilon(\sigma,\tau)z_\sigma\tau.$$

Then $(C.(\mathcal{N}_p), \delta.)$ is a chain complex of $\mathbb{Z}$-modules. Note that $C_i(W_p) \subset C_i(\mathcal{N}_p)$ but $(C.(W_p), \partial.)$ is not a subcomplex of $(C.(\mathcal{N}_p), \delta.)$. In a similar way we define $(C.(\mathcal{L}_p), \delta.)$ which is in fact a subcomplex of $(C.(\mathcal{N}_p), \delta.)$. Hence we can define the relative homology groups $H_i(C.(\mathcal{N}_p, \mathcal{L}_p))$ in the obvious way.

(6.20) CLAIM. $H_i(C.(\mathcal{N}_p, \mathcal{L}_p)) = 0$ *for* $0 \le i \le d-1$.

Before proving this claim, we proceed with the proof of the lemma. Fix an integer i with $0 \le i \le d-1$. Since $C_i(W_p)$ is a chain complex it suffices to prove that $\ker(\partial_i) \subset \operatorname{im}(\partial_{i+1})$. So let $\eta = \sum_{\sigma\in\Delta_{i,p}(W)} \eta_\sigma\sigma \in \ker(\partial_i)$. Then

$$\begin{aligned}
\delta_i(\eta) &= \sum_{\sigma\in\Delta_{i,p}(W)} \eta_\sigma \delta_i(\sigma) \\
&= \sum_{\sigma\in\Delta_{i,p}(W)} \sum_{\tau \text{ facet of } \sigma} \eta_\sigma \epsilon(\sigma,\tau)\tau \\
&= \sum_{\tau\in\Delta_{i-1,p}(W)} \left(\sum_{\substack{\tau \text{ facet of } \sigma \\ \sigma\in\Delta_{i,p}(W)}} \eta_\sigma\epsilon(\sigma,\tau) \right)\tau + \sum_{\tau\in\Delta_{i-1}(\mathcal{L}_p)} \left(\sum_{\substack{\tau \text{ facet of } \sigma \\ \sigma\in\Delta_{i,p}(W)}} \eta_\sigma\epsilon(\sigma,\tau) \right)\tau \\
&= \partial_i(\eta) + \sum_{\tau\in\Delta_{i-1}(\mathcal{L}_p)} \left(\sum_{\substack{\tau \text{ facet of } \sigma \\ \sigma\in\Delta_{i,p}(W)}} \sum \eta_\sigma\epsilon(\sigma,\tau) \right)\tau.
\end{aligned}$$

Thus $\delta_i(\eta) \in C_{i-1}(\mathcal{L}_p)$ since $\partial_i(\eta) = 0$. Hence, by the claim, there exists c in $C_{i+1}(\mathcal{N}_p)$ and u in $C_i(\mathcal{L}_p)$ such that $\eta = \delta_{i+1}(c) + u$. Let $c = \sum_{\lambda\in\Delta_{i+1}(\mathcal{N}_p)} c_\lambda\lambda$ and define $\psi := \sum_{\lambda\in\Delta_{i+1,p}(W_p)} c_\lambda\lambda$. Then $\eta - \delta_{i+1}(\psi) \in C_i(\mathcal{L}_p)$. But by the same argument as above also $\partial_{i+1}(\psi) - \delta_{i+1}(\psi) \in C_i(\mathcal{L}_p)$.Thus $\eta - \partial_{i+1}(\psi) \in C_i(\mathcal{L}_p)$. Since both η and $\partial_{i+1}(\psi)$ belong to $C_i(W_p)$ this means that $\eta = \partial_{i+1}(\psi)$. □

Proof of Claim (6.20). Let $D_i(\mathcal{N}_p)$ be the free abelian group generated by the ordered simplices of $\Delta_i(\mathcal{N}_p)$, divided out by the subgroup generated by the elements $(a_0,\dots,a_i) - sgn(\rho)(a_{\rho(0)},\dots,a_{\rho(i)})$ where $(a_0,\dots,a_i)$ denotes an ordered simplex of $\Delta_i(\mathcal{N}_p)$ and ρ is a permutation of the set $\{0,\dots,i\}$. We let $[a_0,\dots,a_i]$ denote the class of $(a_0,\dots,a_i)$ in $D_i(\mathcal{N}_p)$. Then we define $\partial[a_0,\dots,a_i] := \sum_{j=0}^{i}(-1)^j[a_0,\dots,\hat{a}_j,\dots,a_i]$. This is well-defined on the generators of $D_i(\mathcal{N}_p)$. We extend this definition linearly to $D_i(\mathcal{N}_p)$. Similarly we define $D_i(\mathcal{L}_p)$. Then again we have a chain complex of $\mathbb{Z}$-modules $(D.(\mathcal{N}_p), \partial.)$ with $(D.(\mathcal{L}_p), \partial.)$ as a subcomplex. So we can also consider the relative chain complex $(D.(\mathcal{N}_p, \mathcal{L}_p), \partial.)$. We define morphisms $f_i\colon C_i(\mathcal{N}_p) \to D_i(\mathcal{N}_p)$ and $g_i\colon D_i(\mathcal{N}_p) \to C_i(\mathcal{N}_p)$ by $f_i(\sigma) := o(\sigma)$ and $g_i([a_0,\dots,a_i]) := \operatorname{sgn}[a_0,\dots,a_i]\{a_0,\dots,a_i\}$ where $\operatorname{sgn}[a_0,\dots,a_i][a_0,\dots,a_i] = o(\{a_0,\dots,a_i\})$. One easily checks that these morphisms are well-defined, are each others inverses and commute with ∂. The same is true for there restrictions to $C_i(\mathcal{L}_p)$

and $D_i(\mathcal{L}_p)$. Thus we have isomorphisms $H_i(C_\cdot(\mathcal{N}_p, \mathcal{L}_p)) \cong H_i(D_\cdot(\mathcal{N}_p, \mathcal{L}_p))$. Now by [Mau, Theorem 4.3.9 and Corollary 4.3.5], $H_i(D_\cdot(\mathcal{N}_p, \mathcal{L}_p)) \cong H_i(|\mathcal{N}_p|, |\mathcal{L}_p|)$. (Here the last homology group is the standard singular homology group associated to the pair $(|\mathcal{N}_p|, |\mathcal{L}_p|)$.) Then the claim follows from Lemma (6.2). □

(6.21) LEMMA. *$\mathcal{C}_{d-\cdot}(\mathcal{L})$ is a φ-acyclic resolution of $\mathcal{L} \otimes \mathcal{O}$ if conditions* 1 *and* 2 *of the Poincaré Duality Theorem* (4.2) *hold.*

Proof. Choose $j > 0$ and $i \geq 0$. Since $\mathcal{L}_\sigma$ vanishes outside σ_W we have $H^j_\varphi(W, \mathcal{L}_\sigma) \cong H^j_{\varphi|\sigma_W}(\sigma_W, \mathcal{L}_\sigma|\sigma_W)$ (e.g., see [Br, II,10.1] for general φ or [Go, II,4.10.1] for φ paracompactifying). Thus

$$\begin{aligned}
\prod_{\sigma\in\Delta_i} H^j_\varphi(W, \mathcal{L}_\sigma) &\cong \prod_{\sigma\in\Delta_i} H^j_{\varphi|\sigma_W}(\sigma_W, \mathcal{L}_\sigma|\sigma_W)\\
&\cong \prod_{\sigma\in\Delta_i} H^j_{\varphi|\sigma_W}(\sigma_W, L^r) \quad \text{(by Lemma (6.9))}\\
&= 0 \quad \text{(by condition 2)}
\end{aligned}$$

Then the lemma follows from Lemma (6.22) (since φ has the union property). □

(6.22) LEMMA. *Let Y be a topological space. Let $\{G_i\}_{i\in I}$ be a locally finite family of closed subsets of Y. Let μ_i: $G_i \to Y$ be the inclusions. Let L be a ring. Let $\mathcal{G}_i$ be sheaves of L-modules on G_i and let $\mathcal{F}_i = (\mu_i)_*(\mathcal{G}_i)$. Let φ be a family of supports on Y such that φ has the union property* (*see* (4.2)). *Then for every $j \geq 0$ there is an injection $H^j_\varphi(Y, \prod_i \mathcal{F}_i) \to \prod_i H^j_\varphi(Y, \mathcal{F}_i)$.*

Proof. For every $\mathcal{G}_i$ we choose a flabby resolution $\mathcal{B}_i^\cdot$. Let $\mathcal{A}_i^\cdot = (\mu_i)_*(\mathcal{B}_i^\cdot)$. Then $\mathcal{A}_i^\cdot$ is a flabby resolution of $\mathcal{F}_i$ (see [Br, Corollary 5.6, p. 36]). But then, in addition, $\prod_i \mathcal{A}_i^\cdot$ is a flabby resolution of $\prod_i \mathcal{F}_i$. Thus by Lemmas (6.12) and (6.13) we have isomorphisms

$$\begin{aligned}
H^j_\varphi(Y, \textstyle\prod_i \mathcal{F}_i) &\cong H^j(\Gamma_\varphi(\textstyle\prod_i \mathcal{A}_i^\cdot)),\\
\textstyle\prod_i H^j_\varphi(Y, \mathcal{F}_i) &\cong \textstyle\prod_i H^j(\Gamma_\varphi(\mathcal{A}_i^\cdot)).
\end{aligned}$$

We have a natural map

$$\alpha\colon H^j(\Gamma_\varphi(\textstyle\prod_i \mathcal{A}_i^\cdot)) \to \textstyle\prod_i H^j(\Gamma_\varphi(\mathcal{A}_i^\cdot))\colon [\textstyle\prod_i s_i] \mapsto \textstyle\prod_i [s_i].$$

Here $[-]$ denotes the class of an element in its cohomology module. It suffices to show that α is injective. Let $s = \prod_i s_i \in \ker(\Gamma_\varphi(\prod_i \mathcal{A}_i^j) \xrightarrow{\prod_i \partial_i^j} \Gamma_\varphi(\prod_i \mathcal{A}_i^{j+1}))$ such that $s_i \in \operatorname{im}(\Gamma_\varphi(\mathcal{A}_i^{j-1}) \xrightarrow{\partial_i^{j-1}} \Gamma_\varphi(\mathcal{A}_i^j))$ for every i in I. Thus for every i in I there exists an element t_i in $\Gamma_\varphi(\mathcal{A}_i^{j-1})$ such that $\partial_i^{j-1}(t_i) = s_i$. We choose $t_i = 0$ if $s_i = 0$. Note that $\mathrm{supp}(s_i) \subset \mathrm{supp}(t_i) \cap \mathrm{supp}(s)$. Hence $\mathrm{supp}(t_i) = \emptyset$ if $\mathrm{supp}(t_i) \cap \mathrm{supp}(s) = \emptyset$. The family $\{\mathrm{supp}(t_i)\}_{i\in I}$ is a locally finite family, so

$$\mathrm{supp}(t) = \cup_{i\in I}\mathrm{supp}(t_i) = \cup\{\mathrm{supp}(t_i) | \mathrm{supp}(t_i) \cap \mathrm{supp}(s) \neq \emptyset\} \in \varphi,$$

since φ has the union property. Because $s = (\prod_i \partial_i^{j-1})(t)$ this proves the injectivity of α. □

Proof of the Poincaré Duality Theorem (4.2). Theorem (4.2) now follows immediately from Lemmas (6.17), (6.21) and Lemma (6.13). □

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