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ON THE DIMENSIONS OF THE AUTOMORPHISM GROUPS OF HYPERBOLIC REINHARDT DOMAINS

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ABSTRACT. We study the possible dimensions of the groups of holomorphic automorphisms of hyperbolic Reinhardt domains. We are particularly interested in the problem of characterizing Reinhardt domains with automorphism group of prescribed dimension.

Introduction

Let D be a domain (a connected open set) in $\mathbb{C}^n$, $n \geq 2$. Denote by $\mathrm{Aut}(D)$ the group of holomorphic automorphisms of D; that is, $\mathrm{Aut}(D)$ is the group under composition of all biholomorphic self-maps of D. If D is bounded or, more generally, Kobayashi-hyperbolic, then the group $\mathrm{Aut}(D)$ with the topology of uniform convergence on compact subsets of D is in fact a finite-dimensional Lie group (see [Ko]). We note that this Lie group is always a *real* group but never a complex Lie group (except for the case of zero-dimensional groups). Thus, when we specify the dimension of this group, we shall always be speaking of its real dimension. By contrast, when we speak of the dimension of the domain on which it acts, we shall be referring to complex dimension.

We are interested in characterizing a domain by its automorphism group. Much work has been done on classifying domains with non-compact automorphism group (see [IK3] for a detailed exposition). In this paper we concentrate on the dimension of $\mathrm{Aut}(D)$. Namely, we are interested in the following question: to what extent does the dimension of the automorphism group determine the domain?

In this work we consider only Reinhardt domains, i.e., domains invariant under the (coordinate) rotations

$$z_j \mapsto e^{i\phi_j} z_j, \qquad \phi_j \in \mathbb{R},\ j = 1, \ldots, n.$$

As we shall see below, even this special case leads to difficult problems.

The paper is based on the structure theorem by Kruzhilin [Kr] (see also [Sh] for the case of bounded domains) that allows us to list all possible dimensions that the automorphism groups of hyperbolic Reinhardt domains can have. It turns out that

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all dimensions (except for the case of domains that, up to dilations of coordinates, are the unit ball in $\mathbb{C}^n$—see Corollary 1.2) lie between n and n^2+2 inclusive; the dimensions are even if n is even and odd if n is odd.

We classify all domains whose automorphism groups have dimensions n^2 and n^2+2 (see Theorem 1.9 and Corollary 1.5). The remaining dimensions (i.e., those that lie between n and n^2-2 inclusive) split into two sets: the "bad" and "good" ones (the latter corresponds to the case of domains with non-compact automorphism group). These will be defined in the sequel.

While it will turn out that there is no hope to obtain any reasonable classification of domains whose automorphism groups have dimensions that belong to the "bad" set, one *can* hope to obtain some description for the "good" dimensions. For C^1-smoothly bounded domains such a description has been already found in [IK2].

In this paper we study the structure of the sets of "bad" and "good" dimensions. The main question we are interested in is: what is the asymptotic behavior (as a function of n) of the numbers of "bad" and "good" dimensions as the spatial dimension $n \to \infty$? We have been able to prove that the number of "bad" dimensions behaves asymptotically as $n^2/2$ (see Theorem 1.10) which means that these dimensions asymptotically fill the whole list of all possible automorphism group dimensions. On the other hand, we show that the lim inf of the number of "good" dimensions behaves asymptotically at least as n (see Theorem 1.12). This last result implies that the probability of randomly choosing a "good" dimension from the list of all possible automorphism group dimensions is asymptotically at least of order $1/n$; this information is encouraging compared with the fact that almost any randomly chosen domain in $\mathbb{C}^n$ does not belong to any reasonable classification list.

We have also made numerical computations for the numbers of "bad" and "good" dimensions for up to $n = 36000$ and present some of the results in Section 2. These results were obtained by C programming. The source code of the C program is available on the World Wide Web at

`http://wwwmaths.anu.edu.au/~james/reinhardt`

The complete list of our computational results is available from

`http://wwwmaths.anu.edu.au/~james/reinhardt_domains/table1.html`

As we shall see below, finding the numbers of "bad" and "good" dimensions is also related to determining certain characteristics of partitions by way of their Young diagrams.

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1. Results

Let $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. We denote by $\mathrm{Aut}_{alg}((\mathbb{C}^*)^n)$ the group of algebraic automorphisms of $(\mathbb{C}^*)^n$, i.e., the group of mappings of the form

$$z_i \mapsto \lambda_i z_1^{a_{i1}} \dots z_n^{a_{in}}, \quad i = 1, \dots n, \tag{1.1}$$

where $\lambda_i \in \mathbb{C}^*$, $a_{ij} \in \mathbb{Z}$, and $\det(a_{ij}) = \pm 1$.

For a hyperbolic Reinhardt domain $D \subset \mathbb{C}^n$, denote by $\mathrm{Aut}_{alg}(D)$ the subgroup of $\mathrm{Aut}(D)$ that consists of algebraic automorphisms of D, i.e., automorphisms induced by elements of $\mathrm{Aut}_{\mathrm{alg}}((\mathbb{C}^*)^n)$. Let $\mathrm{Aut}_0(D)$ be the connected component of the identity in $\mathrm{Aut}(D)$, and the dot (the symbol $\cdot$) denote the composition operation in $\mathrm{Aut}(D)$. It is shown in [Kr] that $\mathrm{Aut}(D) = \mathrm{Aut}_0(D) \cdot \mathrm{Aut}_{\mathrm{alg}}(D)$.

By [Kr] any hyperbolic Reinhardt domain in $\mathbb{C}^n$ can—by a biholomorphic mapping of the form (1.1)—be put into a normalized form G written as follows. There exist integers $0 \le s \le t \le p \le n$ and $n_i \ge 1$, $i = 1, \dots, p$, with $\sum_{i=1}^p n_i = n$, and real numbers α_i^j, $i = 1, \dots, s$, $j = t+1, \dots, p$, β_j^k, $j = s+1, \dots, t$, $k = t+1, \dots, p$, such that if we set $z^i := (z_{n_1+\dots+n_{i-1}+1}, \dots, z_{n_1+\dots+n_i})$, $i = 1, \dots, p$, then G can be written in the form

$$G = \left\{ |z^1| < 1, \dots, |z^s| < 1, \right.$$
$$\left(\frac{z^{t+1}}{\prod_{i=1}^s (1 - |z^i|^2)^{\alpha_i^{t+1}} \prod_{j=s+1}^t \exp(-\beta_j^{t+1} |z^j|^2)}, \dots, \right.$$
$$\left. \left. \frac{z^p}{\prod_{i=1}^s (1 - |z^i|^2)^{\alpha_i^p} \prod_{j=s+1}^t \exp(-\beta_j^p |z^j|^2)} \right) \in \widetilde{G} \right\}, \tag{1.2}$$

where $\widetilde{G} := G \bigcap \{z^i = 0,\ i = 1, \dots, t\}$ is some hyperbolic Reinhardt domain in $\mathbb{C}^{n_{t+1}} \times \dots \times \mathbb{C}^{n_p}$. It should be noted that any given domain will have many different normalized forms of type (1.2).

A normalized form can be chosen so that $\mathrm{Aut}_0(G)$ is given by the following formulas:

$$\begin{aligned} z^i &\mapsto \frac{A^i z^i + b^i}{c^i z^i + d^i}, \quad i = 1, \dots, s, \\ z^j &\mapsto B^j z^j + e^j, \quad j = s+1, \dots, t, \\ z^k &\mapsto C^k \frac{\prod_{j=s+1}^t \exp(-\beta_j^k (2 \overline{e^j}^T B^j z^j + |e^j|^2)) z^k}{\prod_{i=1}^s (c^i z^i + d^i)^{2\alpha_i^k}}, \quad k = t+1, \dots, p, \end{aligned} \tag{1.3}$$

where

$$\begin{aligned}
&\begin{pmatrix} A^i & b^i \\ c^i & d^i \end{pmatrix} \in SU(n_i, 1), \quad i = 1, \dots, s, \\
&B^j \in U(n_j), \quad e^j \in \mathbb{C}^{n_j}, \quad j = s+1, \dots, t, \\
&C^k \in U(n_k), \quad k = t+1, \dots, p.
\end{aligned} \tag{1.4}$$

It follows from (1.3), (1.4) that the dimension of the automorphism group of any hyperbolic Reinhardt domain in $\mathbb{C}^n$ is a number of the form

$$\sum_{i=1}^{k} n_i^2 + 2\sum_{j=1}^{m} n_j, \qquad 0 \le m \le k, \tag{1.5}$$

for some partition $(n_1, \dots, n_k)$ of n, $1 \le k \le n$. We will be interested in the structure of the set $\Omega(n)$ of all numbers (1.5). Let $\Omega(n, \ell)$ be the set of all numbers of the form (1.5) with $k = \ell$ and $\Omega(n, \ell, q)$ the set of all numbers of the form (1.5) with $k = \ell$, $m = q$. Clearly, $\Omega(n, \ell) = \cup_{q=0}^{\ell} \Omega(n, \ell, q)$, $\Omega(n) = \cup_{\ell=1}^{n} \Omega(n, \ell)$. First, prove the following:

PROPOSITION 1.1. *Let $N \in \Omega(n)$. Then*:

(i) *N is even (odd) if n is even (odd).*
(ii) *$N \ge n$.*
(iii) *If $N \in \Omega(n, \ell)$ for $\ell \ge 2$ then $N \le n^2 + 2$.*

Proof. Statements (i) and (ii) are obvious. We prove (iii) by induction. It is obvious for $n = 2$, so we assume that $n \ge 3$. Let $N = \sum_{i=1}^{\ell} n_i^2 + 2\sum_{j=1}^{m} n_j$, for some $2 \le \ell \le n$, $0 \le m \le \ell$.

Suppose first that $\ell = 2$, $m \le 1$. Then we have

$$\begin{aligned}
N &= \left(n_1^2 + 2\sum_{j=1}^{m} n_j\right) + n_2^2 \\
&\le (n - n_2)^2 + 2(n - n_2) + n_2^2 = n^2 + 2(n_2^2 - n_2(n+1) + n) < n^2 + 2.
\end{aligned}$$

Let $\ell = 2$, $m = 2$. Then

$$N = (n - n_2)^2 + n_2^2 + 2n = n^2 + 2(n_2^2 - n_2 n + n) \le n^2 + 2.$$

Now suppose that $\ell \ge 3$ and assume first that $m \le \ell - 1$. Then by induction we have

$$\begin{aligned}
N &= \left(\sum_{i=1}^{\ell-1} n_i^2 + 2\sum_{j=1}^{m} n_j\right) + n_\ell^2 \\
&\le (n - n_\ell)^2 + 2 + n_\ell^2 = n^2 + 2 + 2n_\ell(n_\ell - n) < n^2 + 2.
\end{aligned}$$

Assume finally that $\ell \geq 3$ and $m = \ell$. Then by induction we get

$$\begin{aligned} N &= \left(\sum_{i=1}^{\ell-1} n_i^2 + 2\sum_{j=1}^{\ell-1} n_j\right) + n_\ell^2 + 2n_\ell \\ &\leq (n - n_\ell)^2 + 2 + n_\ell^2 + 2n_\ell = n^2 + 2 + 2n_\ell(n_\ell - n + 1) \leq n^2 + 2. \end{aligned}$$

The proposition is proved. □

It follows from Proposition 1.1 that the value n^2+2n can only be taken by $\Omega(n, 1, 1)$ (which is clearly a one-point set) corresponding to the case $s = p = 1$ in formula (1.2). Thus we obtain the following characterization of the unit ball in the class of hyperbolic Reinhardt domains.

COROLLARY 1.2. *Let $D \subset \mathbb{C}^n$ be a hyperbolic Reinhardt domain such that* $\dim \operatorname{Aut}(D) > n^2 + 2$. *Then, up to dilations of coordinates, D is the unit ball B^n.*

Remark. It turns out that any connected hyperbolic n-dimensional complex manifold M with $\dim \operatorname{Aut}(M) > n^2+2$, is biholomorphically equivalent to the unit ball; see [IK1]. For comparison, we also recall here an earlier result from [Ka], [Ko]: if M is a connected hyperbolic n-dimensional complex manifold, then $\dim \operatorname{Aut}(M) \leq n^2+2n$, and if $\dim \operatorname{Aut}(M) = n^2 + 2n$, then M is biholomorphically equivalent to the unit ball.

In the following proposition we establish upper bounds for $\Omega(n, \ell, 0)$, $\ell \geq 2$.

PROPOSITION 1.3. *Let $N \in \Omega(n, \ell, 0)$. Then $N \leq (n - \ell + 1)^2 + \ell - 1$.*

Proof. The inequality is obvious for $\ell = 1$, so we assume that $\ell \geq 2$. The proof for $\ell \geq 2$ proceeds by induction. The inequality is clearly correct for $n = 2$ and we suppose that $n \geq 3$. Let $N = \sum_{i=1}^{\ell} n_i^2$. If $\ell = 2$, we have

$$N = n_1^2 + n_2^2 = (n - n_2)^2 + n_2^2 = (n-1)^2 + 1 + 2(n_2^2 - n_2 n + n - 1) \leq (n-1)^2 + 1.$$

Now assume that $\ell \geq 3$. Then by induction we have

$$\begin{aligned} N &= \sum_{i=1}^{\ell-1} n_i^2 + n_\ell^2 \\ &\leq (n - n_\ell - \ell + 2)^2 + \ell - 2 + n_\ell^2 \\ &= (n - \ell + 1)^2 + \ell - 1 \\ &\quad + 2(n_\ell^2 - n_\ell(n - \ell + 2) + n - \ell + 1) \\ &\leq (n - \ell + 1)^2 + \ell - 1. \end{aligned}$$

The proposition is proved. □

COROLLARY 1.4. *Let $N \in \Omega(n, \ell)$. Then $N \leq (n-\ell+1)^2+\ell-1+2n$.*

It follows from Corollary 1.4 that the value n^2+2 can only be taken by the elements of $\Omega(n, 2, 2)$. The only partition (n_1, n_2) (assuming $n_1 \leq n_2$) that can realize this value is clearly $(1, n-1)$ which corresponds to the case $s = p = 2$, $n_1 = 1$, $n_2 = n-1$ in formula (1.2). Thus, we obtain the following:

COROLLARY 1.5. *Let $D \subset \mathbb{C}^n$ be a hyperbolic Reinhardt domain such that $\dim \operatorname{Aut}(D) = n^2+2$. Then, up to dilations and permutations of coordinates, D is the product $B^{n-1} \times \Delta$ of the unit ball $B^{n-1} \subset \mathbb{C}^{n-1}$ and the unit disc $\Delta \subset \mathbb{C}$.*

Remark. In [IK1] we showed that any connected hyperbolic n-dimensional complex manifold M with $\dim \operatorname{Aut}(M) = n^2+2$, is biholomorphically equivalent to $B^{n-1} \times \Delta$. In [IK5] we also proved that if $n \neq 3$, then $\dim \operatorname{Aut}(M) \neq n^2+1$, and, if $n = 3$ and $\dim \operatorname{Aut}(M) = 10 = 3^2+1$, then M is biholomorphically equivalent to the 3-dimensional Siegel space (the classical domain of type $(III)_2$).

Now we shall deal with dimension n^2 and classify all hyperbolic Reinhardt domains D such that $\dim \operatorname{Aut}(D) = n^2$. For this result, we need to understand what numbers from $\Omega(n)$ can equal n^2. First of all, we clearly have $\Omega(n, 1, 0) = \{n^2\}$. Next, the following holds.

PROPOSITION 1.6. *If $n \geq 4$, then for any $N \in \Omega(n, \ell)$ with $\ell \geq 3$, one has $N < n^2$.*

Proof. It follows from Corollary 1.4 that

$$N \leq n^2+\ell^2-\ell(2n+1)+4n \leq n^2-2(n-3) < n^2.$$

The proposition is proved. □

It follows from Proposition 1.6 that, for $n \geq 4$, if $n^2 \in \Omega(n, \ell)$ and $\ell \geq 2$, then $\ell = 2$.

We now take a closer look at the set $\Omega(n, 2)$. To cover all the elements from $\Omega(n, 2, 0)$ we clearly only need to consider partitions (n_1, n_2) of n with $n_1 \geq n_2$, i.e., partitions of the form

$$\left(\frac{n}{2}+\mu, \frac{n}{2}-\mu\right), \ \mu = 0, \dots, \frac{n}{2}-1, \ \text{if } n \text{ is even},$$

$$\left(\frac{n+1}{2}+\mu, \frac{n-1}{2}-\mu\right), \ \mu = 0, \dots, \frac{n-3}{2}, \ \text{if } n \text{ is odd}.$$

Therefore, the following proposition is obvious.

PROPOSITION 1.7. (i) *If n is even,*

$$\begin{aligned}\Omega(n,2) &= \left\{\frac{n^2}{2}+2\mu^2\colon\ \mu=0,\dots,\frac{n}{2}-1\right\}\\ &\cup\left\{\frac{n^2}{2}+2\mu(\mu-1)+n\colon\ \mu=0,\dots,\frac{n}{2}-1\right\}\\ &\cup\left\{\frac{n^2}{2}+2\mu(\mu+1)+n\colon\ \mu=0,\dots,\frac{n}{2}-1\right\}\\ &\cup\left\{\frac{n^2}{2}+2\mu^2+2n\colon\ \mu=0,\dots,\frac{n}{2}-1\right\}.\end{aligned}$$

(ii) *If n is odd,*

$$\begin{aligned}\Omega(n,2) &= \left\{\frac{n^2+1}{2}+2\mu(\mu+1)\colon\ \mu=0,\dots,\frac{n-3}{2}\right\}\\ &\cup\left\{\frac{n^2+1}{2}+2\mu^2+n-1\colon\ \mu=0,\dots,\frac{n-3}{2}\right\}\\ &\cup\left\{\frac{n^2+1}{2}+2\mu(\mu+2)+n+1\colon\ \mu=0,\dots,\frac{n-3}{2}\right\}\\ &\cup\left\{\frac{n^2+1}{2}+2\mu(\mu+1)+2n\colon\ \mu=0,\dots,\frac{n-3}{2}\right\}.\end{aligned}$$

COROLLARY 1.8. (i) *A number $N\in\Omega(n,2)$, $n\neq 4$, is equal to n^2 only if $N\in\Omega(n,2,1)$ and corresponds to the partition $(n-1,1)$.*

(ii) *A number $N\in\Omega(4,2)$ is equal to 16 only if either*

(a) *$N\in\Omega(4,2,2)$ and N corresponds to the partition $(2,2)$, or*
(b) *$N\in\Omega(4,2,1)$ and N corresponds to the partition $(3,1)$.*

We are now ready to prove the following classification result for dimension n^2.

THEOREM 1.9. *Let $D\subset\mathbb{C}^n$ be a hyperbolic Reinhardt domain such that $\dim\operatorname{Aut}(D)=n^2$. Then D is holomorphically equivalent to one of the following domains:*

(i) $\{z\in\mathbb{C}^n\colon\ r<|z|<R\}$, $0\le r<R<\infty$;
(ii) Δ^3 (*here $n=3$*);
(iii) $B^2\times B^2$ (*here $n=4$*);
(iv) $\{(z',z_n)\in\mathbb{C}^n\colon\ |z'|^2+|z_n|^\alpha<1\}$, $\alpha\in\mathbb{R}$, $\alpha\neq 0,2$;

(v) $\{(z', z_n) \in \mathbb{C}^n \colon |z'| < 1, r(1-|z'|^2)^\alpha < |z_n| < R(1-|z'|^2)^\alpha\}$, $\alpha \in \mathbb{R}$, $0 < r < R \leq \infty$;

(vi) $\{(z', z_n) \in \mathbb{C}^n \colon re^{\alpha|z'|^2} < |z_n| < Re^{\alpha|z'|^2}\}$, *where* $0 < r < R \leq \infty$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$ *and, if* $R = \infty$, *then* $\alpha > 0$.

The equivalence is given by a mapping of the form (1.1).

Proof. Let G be a normalized form of D as in (1.2). We first consider the case of $\Omega(n, 1, 0)$. Then $\operatorname{Aut}_0(G) = U(n)$ (see (1.3)). It is clear that any hyperbolic domain with this property has the form (i).

Next, by Proposition 1.6, the case of $\Omega(n, 3)$ is only non-trivial when $n = 3$. Clearly, G then coincides with Δ^3.

We now turn to the case of $\Omega(n, 2)$ and use Corollary 1.8. Assume first that $n = 4$ and consider $N \in \Omega(4, 2)$ corresponding to the partition (2, 2). Then the only possibility for G is to be $B^2 \times B^2$.

Now let n be arbitrary, and consider the case of $N \in \Omega(n, 2, 1)$ corresponding to the partition $(n-1, 1)$. Thus, in (1.2) we have $p = 2$, $n_1 = n - 1$, $n_2 = 1$ and either $s = t = 1$, or $s = 0, t = 1$. Next, there are the following possibilities for a hyperbolic Reinhardt domain $\widetilde{G} \subset \mathbb{C}$:

(a) $\widetilde{G} = \{|z_n| < R\}$, $\quad 0 < R < \infty$;
(b) $\widetilde{G} = \{r < |z_n| < R\}$, $\quad 0 < r < R \leq \infty$;
(c) $\widetilde{G} = \{0 < |z_n| < R\}$, $\quad 0 < R < \infty$.

Substituting (a), (b), (c) into (1.2), and excluding non-hyperbolic domains, we produce (iv)–(vi) (cf. [IK4]).

The theorem is proved. $\square$

In Corollaries 1.2, 1.5 and Theorem 1.9 we have described all hyperbolic Reinhardt domains whose group of holomorphic automorphisms has dimension n^2 or higher. We now turn to the case of dimensions not exceeding n^2-2. To this end, we introduce the sets

$$\widetilde{\Omega}(n) := \{N \in \Omega(n) \colon N \leq n^2 - 2\},$$
$$\widetilde{\Omega}(n, \ell) := \{N \in \Omega(n, \ell) \colon N \leq n^2 - 2\}, \ \ell \geq 2,$$
$$\widetilde{\Omega}(n, \ell, q) := \{N \in \Omega(n, \ell, q) \colon N \leq n^2 - 2\}, \ \ell \geq 2.$$

The numbers belonging to $\cup_{\ell=2}^n \Omega(n, \ell, 0)$ we call *compact* and the numbers belonging to $\widetilde{\Omega}(n) \setminus \cup_{\ell=2}^n \Omega(n, \ell, 0)$ we call *non-compact* (note that $\widetilde{\Omega}(n, \ell, 0) = \Omega(n, \ell, 0)$ for $\ell \geq 2$).

It is clear from (1.3) that compact numbers arise as the dimensions of the automorphism groups of domains G for which $\operatorname{Aut}_0(G) = U(n_1) \times \cdots \times U(n_\ell)$, for some partition $(n_1, \dots, n_\ell)$ of n with $\ell \geq 2$. For any such partition one can construct many pairwise non-equivalent hyperbolic (and even smoothly bounded) Reinhardt domains for which the identity component of the automorphism groups is $U(n_1) \times \cdots \times U(n_\ell)$. The construction is as follows. Choose a set $Q \subset \mathbb{R}^\ell_+$ (here $\mathbb{R}^\ell_+ := \{(x_1, \dots, x_\ell)\colon x_j \geq 0\})$ in such a way that

$$D_Q := \left\{(z^1, \dots, z^\ell) \in \mathbb{C}^n\colon (|z^1|, \dots, |z^\ell|) \in Q\right\}$$

is a smoothly bounded domain in $\mathbb{C}^n$ containing the origin; here

$$z^i := \left(z_{n_1+\cdots+n_{i-1}+1}, \dots, z_{n_1+\cdots+n_i}\right), \qquad i = 1, \dots, \ell.$$

By [Su], two bounded Reinhardt domains containing the origin are holomorphically equivalent if and only if one is obtained from another by dilations and permutations of coordinates. In [FIK1] we listed all smoothly bounded Reinhardt domains with non-compact automorphism group. Since they all contain the origin, it is not difficult to choose D_Q such that it is not holomorphically equivalent to any of the domains from [FIK1] and thus ensure that $\mathrm{Aut}_0(D_Q)$ is compact and therefore, by (1.3), is isomorphic to a product of unitary groups. Further, $U(n_1) \times \cdots \times U(n_\ell) \subset \mathrm{Aut}_0(D_Q)$, and if necessary, one can vary Q slightly to get $U(n_1) \times \cdots \times U(n_\ell) = \mathrm{Aut}_0(D_Q)$. The freedom in choosing Q satisfying the above requirements is very substantial, and one can produce (uncountably) many non-equivalent domains that clearly cannot be classified in any reasonable way. Thus compact dimensions are virtually "unclassifiable" and so are "bad" dimensions. On the other hand, it is clear from (1.3) that if dim $\mathrm{Aut}(D)$ is non-compact, then $\mathrm{Aut}(D)$ is non-compact. In [IK2] we classified all bounded Reinhardt domains with C^1-smooth boundary and non-compact automorphism group. Thus, at least in the C^1-smooth and bounded situation, non-compact dimensions are "classifiable" and so are termed "good" dimensions.

Thus from now on we will restrict our considerations to bounded Reinhardt domains with C^1-smooth boundary. It follows from [FIK1] and the discussion above that only numbers of the form (1.5) with $m = 0, 1$ can be realized as the dimensions of the automorphism groups of such domains. On the other hand, for any such number it is not difficult to construct a smoothly bounded domain whose automorphism group has dimension equal to this number. Indeed, assume that in (1.5) $m = 1$ (the case $m = 0$ has been discussed above) and consider a number N of the form

$$\sum_{i=1}^{k} n_i^2 + 2n_{i_0}, \tag{1.6}$$

for some index i_0. Consider the domain

$$D^{i_0}_{n_1,\dots,n_k}(s) := \left\{(z^1, \dots, z^k) \in \mathbb{C}^n\colon |z^{i_0}|^2 + \sum_{i \neq i_0} |z^i|^{2s_i} < 1\right\},$$

where $s := (s_1, \dots, s_{i_0-1}, s_{i_0+1}, \dots, s_k)$, $s_i \in \mathbb{N}$, $s_i > 1$, $s_i \neq s_j$ $(i \neq j)$. By an argument similar to that in the proof of Theorem 1 in [FIK2] one can now explicitly determine $\mathrm{Aut}_0(D^{i_0}_{n_1,\dots,n_k}(s))$. It then follows from the explicit formulas that $\dim \mathrm{Aut}(D^{i_0}_{n_1,\dots,n_k}(s)) = N$.

Therefore, in the situation of C^1-smooth bounded domains the sets of interest are

$$\widetilde{\Omega}^{\mathrm{sb}}(n, \ell) := \Omega(n, \ell, 0) \cup \widetilde{\Omega}(n, \ell, 1), \quad \ell \geq 2, \quad \text{and} \quad \widetilde{\Omega}^{\mathrm{sb}}(n) := \cup_{\ell=2}^{n} \widetilde{\Omega}^{\mathrm{sb}}(n, \ell).$$

The set $\widetilde{\Omega}^{\mathrm{sb}}(n)$ appears to have an extremely irregular structure.

Let $C(n)$ and $H(n)$ denote the sets of all compact and non-compact dimensions, respectively, from $\widetilde{\Omega}^{\mathrm{sb}}(n)$. Any number from $C(n)$ does not exceed $n^2 - 2$ and has the form

$$\sum_{i=1}^{k} n_i^2,$$

where $(n_1, \dots, n_k)$ is a partition of n. Any number from $H(n)$ does not exceed $n^2 - 2$ and has the form (1.6). A number of the form (1.6) can be written as

$$\sum_{\substack{1 \leq i \leq k \\ i \neq i_0}} n_i^2 + (n_{i_0} + 1)^2 - 1.$$

Therefore,

$$H(n) = (C(n+1) - 1) \setminus (C(n) \cup \{n^2\}). \tag{1.7}$$

Let $c(n)$, $h(n)$ denote the cardinalities of $C(n)$, $H(n)$ respectively. Clearly we have

$$c(n) \leq \frac{n(n-1)}{2} < \frac{n^2}{2}, \tag{1.8}$$

$$h(n) \leq \frac{n(n-1)}{2} < \frac{n^2}{2}. \tag{1.9}$$

Since for any partition $(n_1, \dots, n_k)$ of n, the $k+1$-tuple $(1, n_1, \dots, n_k)$ is a partition of $n+1$, we have

$$C(n) \cup \{n^2\} \subset C(n+1) - 1,$$

and formula (1.7) implies

$$h(n) = c(n+1) - c(n) - 1. \tag{1.10}$$

Therefore, determining $c(n)$ at the same time yields $h(n)$.

Finding $c(n)$, however, has proved to be a difficult task. In Section 2 below we list some of the results of numerical computations for up to $n = 36000$. In general, we have

$$C(n) = \cup_{\ell=2}^{n} \Omega(n, \ell, 0).$$

One can obtain characterizations of the sets $\Omega(n,\ell,0), 2<\ell\le n-1$, similar to that of $\Omega(n,2,0)$ from Proposition 1.7. However, it is still not clear how one can calculate $c(n)$ by using such characterizations since the sets $\Omega(n,\ell,0)$, $\ell=2,\ldots,n-1$, intersect each other in a chaotic manner. Nevertheless, we have been able to determine the principal term in the asymptotic behavior of $c(n)$.

THEOREM 1.10. *We have*

$$\lim_{n\to\infty}\frac{c(n)}{n^2}=\frac{1}{2}. \tag{1.11}$$

Proof. For any integer $n\ge 0$, let

$$\widehat{C}(n):=\left\{\sum_{i=1}^{k}n_i^2\colon \sum_{i=1}^{k}n_i=n\right\}; \tag{1.12}$$

and let $\hat{c}(n)$ denote the cardinality of the set $\widehat{C}(n)$. Clearly, $\hat{c}(n)=c(n)+1$ for $n\ge 2$.

First, we note that

$$\widehat{C}(n)=\bigcup_{0\le i<n}\left(\widehat{C}(i)+(n-i)^2\right). \tag{1.13}$$

Next, we define sequences of integers $f(0), f(1),\ldots, g(0), g(1),\ldots$ and $k(1), k(2),\ldots$ inductively as follows:

1. $f(0)=0$;
2. $k(1)=0$;
3. $f(n)=(n-k(n))^2+f(k(n))$;
4. $g(n)=\frac{1}{2}(f(n)+n+4)$;
5. $k(n)=\sup\{0\le\kappa<n\colon g(\kappa)\le n\}$.

We note that $k(n)$ is a non-decreasing sequence and $k(n)\to\infty$ as $n\to\infty$.

We shall now show by induction on n that $f(n)$ has the same parity as n. This clearly holds for $n=0$. For arbitrary n we have

$$f(n)-n\equiv(n^2-n)+k^2(n)+f(k(n))\ (\mathrm{mod}\ 2)\equiv k^2(n)+f(k(n))\ (\mathrm{mod}\ 2).$$

But, by induction, $f(k(n))\equiv k(n)\ (\mathrm{mod}\ 2)$ and thus $f(n)-n\equiv 0\ (\mathrm{mod}\ 2)$.

We shall now show by induction on n that $\{n,n+2,\ldots,f(n)\}\subset\widehat{C}(n)$ (we will see below that each of the sets $\{n,n+2,\ldots,f(n)\}$ is non-empty). Certainly the claim is true for $n=0$. For the general case, let $j\in\{n,n+2,\ldots,f(n)\}$.

(a) If $j\ge(n-k(n))^2+k(n)$, then clearly $j\in\{k(n),k(n)+2,\ldots,f(k(n))\}+(n-k(n))^2$. By induction, $\{k(n),k(n)+2,\ldots,f(k(n))\}\subset\widehat{C}(k(n))$. Then it follows from (1.13) that $j\in\widehat{C}(n)$.

(b) Now suppose that $j < (n-k(n))^2 + k(n)$. We define $k := \sup\{\kappa\colon k(n) \leq \kappa < n \text{ and } j < (n-\kappa)^2 + \kappa\}$. Since $j \geq n$, we have $k < n-1$. It now follows from the definitions of k and $f(n)$ and the fact that $g(k+1) > n$ that $j \in \{k+1, k+3, \dots, f(k+1)\} + (n-(k+1))^2$; as above, by induction and (1.13) we obtain that $j \in \hat{C}(n)$.

Next, we claim that $f(n) \geq 2n$ for $n \geq 4$. This is verified by explicit calculation for small n; in particular it is true for $4 \leq n \leq 18$. For general n, we have $f(n) \geq (n-k(n))^2$. Since $n \geq g(k(n)) = \frac{1}{2}(f(k(n)+k(n)+4)$ and $k(n) > 4$ for $n \geq 18$ (one can check that $k(18) = 7$ and use the fact that $k(n)$ is non-decreasing), we have by induction that $n \geq \frac{3}{2}k(n)$ and so

$$f(n) \geq (n-k(n))^2 \geq \frac{1}{9}n^2 \geq 2n \qquad \text{for } n \geq 18. \tag{1.14}$$

The inequalities (1.14) also imply that $f(n)/n \to \infty$ as $n \to \infty$. Since $k(n) \to \infty$ as $n \to \infty$ and $n \geq g(k(n)) \geq \frac{1}{2}f(k(n))$, we find that $n/k(n) \to \infty$ as $n \to \infty$, hence

$$\frac{f(n)}{n^2} \geq \left(\frac{n-k(n)}{n}\right)^2 \to 1 \qquad \text{as } n \to \infty. \tag{1.15}$$

Since the set $\{n, n+2, \dots, f(n)\}$ is contained in $\widehat{C}(n)$ for all n, we have

$$c(n) = \hat{c}(n) - 1 \geq \frac{f(n)-n}{2}.$$

Combining this inequality with (1.8) and (1.15) we obtain (1.11).

The theorem is proved. □

We shall now find a lower bound for $\liminf_{n\to\infty} h(n)/n$. First, we need the following technical lemma.

LEMMA 1.11. *Let $\widehat{C}(n)$ be the sets defined in* (1.12), *and suppose that $n \geq 7$. If $N \in \widehat{C}(n)$ and $N > 3n^2/4$, then $N \in \widehat{C}(i) + (n-i)^2$ for some $i < n/2$.*

Proof. First, we shall show that if $\lambda = (n_1, \dots, n_k)$ is a partition of n such that $n_j \leq n/2$ for all j, then

$$\sum_{j=1}^{k} n_j^2 \leq \frac{3n^2}{4}. \tag{1.16}$$

Let $X(n)$ denote the maximal value of the sum $\sum_{j=1}^{k} n_j^2$ over all such partitions λ. Choose a partition $\lambda' = (n'_1, \dots, n'_m)$ such that $X(n) = \sum_{j=1}^{m} n_j'^2$. We claim that $m \leq 3$. Indeed, suppose that $m \geq 4$. Consider the set S of all indices $j \in \{1, \dots, m\}$ such that

$$n'_j + 1 \leq \frac{n}{2}.$$

Clearly, if $n \geq 7$, the set S contains at least two elements. Let $j_1, j_2 \in S$, $j_1 \neq j_2$, and suppose that $n_{j_1} \geq n_{j_2}$. Consider the partition $\tilde{\lambda} = (\tilde{n}_1, \dots, \tilde{n}_m)$ defined as follows:

$$\begin{aligned}
\tilde{n}_j &:= n_j, \qquad \text{if } j \neq j_1, j_2,\\
\tilde{n}_{j_1} &:= n_{j_1} + 1,\\
\tilde{n}_{j_2} &:= n_{j_2} - 1.
\end{aligned}$$

Then

$$\sum_{j=1}^{m} \tilde{n}_j^2 = X(n) + 2(n_{j_1} - n_{j_2}) + 2 > X(n),$$

which contradicts the definition of $X(n)$. Therefore $m \leq 3$ which implies that $X(n) \leq 3n^2/4$, and thus (1.16) holds.

Thus, the number N can only be realized by a partition for which at least one entry is bigger than $n/2$, and the lemma is proved. □

We are now ready to prove the following theorem.

THEOREM 1.12. *We have*

$$\liminf_{n\to\infty} \frac{h(n)}{n} \geq 1.$$

Proof. Let $\widehat{C}(n)$ be the sets defined in (1.12). Define

$$\widehat{H}(n) := (\widehat{C}(n+1) - 1) \setminus \widehat{C}(n),$$

for $n = 0, 1, \dots$ Let $\hat{h}(n)$ denote the cardinality of $\widehat{H}(n)$. Since $\widehat{C}(n) \subset \widehat{C}(n+1) - 1$, we have

$$\hat{h}(n) = \hat{c}(n+1) - \hat{c}(n), \tag{1.17}$$

for all n, where $\hat{c}(n)$ is the cardinality of $\widehat{C}(n)$.

We now fix n and choose a number $k \in \mathbb{N}$ such that

$$\frac{k^2 + 3k + 1}{2} \leq n < \frac{(k+1)^2 + 3(k+1) + 1}{2}. \tag{1.18}$$

It follows from (1.13) that

$$\begin{aligned}
\widehat{H}(n) = {} & \cup_{0 \leq i < n+1} \left(\widehat{C}(i) + (n+1-i)^2 - 1\right)\\
& \setminus \cup_{0 \leq i < n+1} \left(\widehat{C}(i-1) + (n+1-i)^2\right),
\end{aligned} \tag{1.19}$$

where we set $\widehat{C}(-1) := \emptyset$.

We observe that the sets $\widehat{C}(i)+(n+1-i)^2-1$ for $i \leq k$ lie strictly above $3(n+1)^2/4$, if n is sufficiently large. To prove this, we show that $(n+1-i)^2-1 > 3(n+1)^2/4$ for $i \leq k$. Indeed, it follows from the first inequality in (1.18) that for large n one has $i \leq k \leq 3\sqrt{n}/2$. Therefore

$$(n+1-i)^2-1 \geq (n+1)^2 - 3\sqrt{n}(n+1) + \frac{9n}{4} - 1.$$

The expression on the right-hand side is clearly bigger than $3(n+1)^2/4$ for large n.

Claim. The sets $\widehat{C}(i)+(n+1-i)^2-1$ and $\widehat{C}(j)+(n+1-j)^2-1$ do not intersect if $0 \leq i \leq k$, $0 \leq j < (n+1)/2$, $i \neq j$.

To prove the claim we note that $\widehat{C}(\ell)+(n+1-\ell)^2-1 \subset [\ell, \ell^2]+(n+1-\ell)^2-1$ for any ℓ. Let $0 \leq i, i+1 \leq k$. It then follows from the first inequality in (1.18) that $(i+1)^2+(n+1-(i+1))^2-1 < i+(n+1-i)^2-1$. This inequality also holds true if $i = k$. Since $\ell^2+(n+1-\ell)^2-1$ is a decreasing function of ℓ for $\ell < (n+1)/2$, the claim follows.

Further, Lemma 1.11 and formula (1.19) imply

$$\widehat{H}(n) \supset \cup_{0 \leq i \leq k} \left(\widehat{C}(i)+(n+1-i)^2-1 \setminus \widehat{C}(i-1)+(n+1-i)^2\right),$$

and therefore, by (1.17),

$$\hat{h}(n) \geq \sum_{i=1}^{k} \hat{h}(i) = \hat{c}(k). \tag{1.20}$$

Let $0 \leq \beta < 1$ and $n \geq (k_0^2+3k_0+1)/2$ where k_0 is chosen to satisfy

$$\frac{k_0^2}{k_0^2+5k_0+5} \geq \sqrt{\beta},$$

$$\hat{c}(k) \geq \frac{\sqrt{\beta}k^2}{2} \qquad \text{for all } k \geq k_0,$$

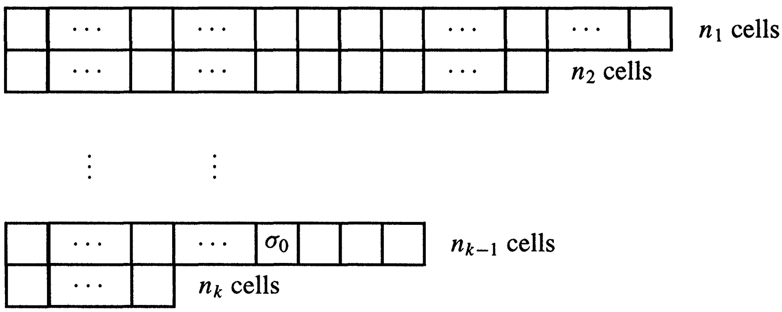
(the second inequality can be satisfied by Theorem 1.10). From (1.20) and the second inequality in (1.18) we now have

$$\frac{\hat{h}(n)}{n} \geq \frac{\sqrt{\beta}k^2}{(k+1)^2+3(k+1)+1} = \frac{\sqrt{\beta}k^2}{k^2+5k+5} \geq \frac{\sqrt{\beta}k_0^2}{k_0^2+5k_0+5} \geq \beta.$$

Since $h(n) = \hat{h}(n) - 1$, the theorem follows. $\square$

Remark. It is plausible—especially in view of our numerical computations (see Section 2)— that in fact one has $\lim_{n\to\infty} h(n)/n = 1$. It is straightforward to show that $\limsup_{n\to\infty} h(n)/(nk(n)) \leq 1$, where $k(n)$ are the numbers defined in the proof of Theorem 1.10. Note that $k(n)$ grows much more slowly than n (cf. (1.9)).

We will now look at the numbers $c(n)$ from the point of view of the theory of partitions (see e.g. [A] for a general exposition of this theory). Let $\lambda = (n_1, \dots, n_k)$, $k \geq 2$, be a partition of n and assume that $n_1 \geq n_2 \geq \cdots \geq n_k$. The partition λ can be pictured by utilizing its *Young diagram* $Y(\lambda)$:



For any cell $\sigma \in Y(\lambda)$ one can define the *arm* of σ (denote it by $a(\sigma)$) as the number of cells to the right of σ. In the diagram above, $a(\sigma_0) = 3$. Clearly, the sum of all arms of the cells in the j^{th} row of $Y(\lambda)$ is equal to

$$\sum_{m=0}^{n_j-1} m = \frac{n_j(n_j-1)}{2},$$

and therefore the total number $A(\lambda)$ of arms in $Y(\lambda)$ is

$$A(\lambda) = \sum_{j=1}^{k} \frac{n_j(n_j-1)}{2} = \frac{1}{2}\left(\sum_{j=1}^{k} n_j^2 - n\right).$$

Therefore, $c(n)$ is equal to the number of distinct values that $A(\lambda)$ takes over all partitions λ of n of length at least 2.

It would be very interesting to know whether the above interpretation of $c(n)$ can be used to get more information on $c(n)$ and $h(n)$ by applying techniques from the theory of partition.

2. Numerical computations for $c(n)$

Equation (1.13) can be used to form an efficient algorithm for the calculation of $\widehat{C}(n) = C(n) \cup \{n^2\}$. Once the sets $\hat{C}(k)$ are known for $1 \leq k \leq n-1$, the set $\hat{C}(n)$ is formed by translating each $\hat{C}(k)$ and taking the union. If $\hat{C}(k)$ is stored as an array of $O(k^2)$ elements, this process will take $O(\sum_{k<n} k^2) = O(n^3)$ time. Therefore to calculate all the sets $\hat{C}(k)$ for $1 \leq k \leq n$ will take $\sum_{k\leq n} O(k^3) = O(n^4)$ time and $O(n^3)$ memory.

A C program implementing this algorithm has been written and used to calculate $\widehat{C}(n)$ for $n \leq 36000$. As previously noted, the source code of this program is available on the World Wide Web. The program stores each element of $\hat{C}(n)$ as a single bit, which makes the program more difficult to understand, but increases its speed and decreases its memory consumption considerably. To give a sense of the constants involved in the $O(n^4)$ time and $O(n^3)$ memory estimates, the program takes 2 seconds to calculate to $n = 1000$ on a Sun Ultra 10, and uses around 0.7 megabytes of memory.

Below we list some results of numerical computations for n up to 36000. We compute $c(n) = \hat{c}(n) - 1$ and then by applying (1.10) calculate the numbers $h(n)$. Further, we compare the growth of $c(n)$ with n^2 and the growth of $h(n)$ with n.

n	$c(n)$	$c(n)/n^2$	$h(n)$	$h(n)/n$
100	3880	0.3880	81	0.8100
1000	464692	0.4646	949	0.9490
5000	12121267	0.4848	4888	0.9776
10000	48947284	0.4894	9840	0.9840
15000	110584806	0.4914	14808	0.9872
20000	197070447	0.4926	19783	0.9891
25000	308425026	0.4934	24768	0.9907
30000	444663933	0.4940	29738	0.99126
35000	605795936	0.4945	34722	0.9920
36000	641010816	0.4946	35707	0.99186

3. Concluding remarks

In this paper we have endeavored to correlate the dimension of the automorphism group of a domain in $\mathbb{C}^n$ with the geometric characteristics of the domain. By restricting attention to Reinhardt domains, we have been able to exploit the structure theorem of Kruzhilin and to come up with (at least in some cases) rather specific conclusions.

We hope that the information obtained here in the Reinhardt case points the way toward what ought to be true for more general classes of domains. In particular, we have identified certain dimensions for the automorphism group that we call "good" and certain others that we call "bad". The former are dimensions in which the automorphism groups are always non-compact; certainly the extant literature (see [IK3]) suggests that the Levi geometry of a boundary accumulation point gives us a chance of classifying such domains. The latter are dimensions for which there exist compact automorphism groups. We are able to determine explicitly that in these cases a holomorphic classification does not exist.

We have been able to find the principal term in the asymptotics of the number of "bad" automorphism group dimensions as well as to bound from below the principle term in the asymptotics of the number of "good" ones. In particular, the "good" dimensions are fairly robust as $n \to \infty$; this is positive information. We also utilize a computer to count these numbers in complex dimension n for values of n up to 36000.

It is clear that further effort is needed to show that $h(n)/n \to \infty$ as well as to determine the forms of the error terms $c(n) - n^2/2$ and $h(n) - n$ that our numerical computations can only suggest. We plan to develop these ideas in future papers. In particular, we wish to extend the work to domains more general than Reinhardt domains.

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