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SINGULARITIES AND WANDERING DOMAINS IN ITERATION OF MEROMORPHIC FUNCTIONS

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ABSTRACT. Let f be a transcendental meromorphic function and let U be a wandering domain of f. Under some conditions, we prove that a finite limit function of $\{f^n\}$ on U is in the derived set of the forward orbit of the set $\operatorname{sing}(f^{-1})$ of singularities of the inverse function of f. The existence of $\{n_k\}$ such that $f^{n_k}|_U$ tends to ∞ is also considered when f is entire. If $\operatorname{sing}(f^{-1})$ is bounded, however, we show that $\{f^n(z)\}_{n=0}^{\infty}$ in $F(f)$ does not tend to ∞.

1. Introduction and results

Let $f\colon \mathbf{C} \mapsto \hat{\mathbf{C}}$ be a transcendental meromorphic function, and f^n, $n \in N$, denote the nth iterate of f. Then $f^n(z)$ is defined for all $z \in \mathbf{C}$ except for a countable set of the poles of $f, f^2, \ldots, f^{n-1}$. Define the Fatou set of f by

$$F(f) = \{z \in \mathbf{C};\ \{f^n\} \text{ is defined and normal in some neighbourhood of } z\}$$

and the Julia set of f by $J(f) = \hat{\mathbf{C}}\backslash F(f)$. It is well known that $F(f)$ is open and complete invariant under f, i.e., $z \in F(f)$ if and only if $f(z) \in F(f)$. Let U be a connected component of $F(f)$; then $f^n(U) \subseteq U_n$, where U_n also is a component of $F(f)$. If $U_n \neq U_m$ for $n \neq m$ then U is called a wandering domain of f. For a wandering domain U, a basic result is that all the limit functions of $\{f^n|_U\}$ are constants. And Sullivan [17] proved that a rational function has no wandering domains. In this paper, we discuss the connection of the wandering domains with the set of singularities of the inverse function, denoted by $\operatorname{sing}(f^{-1})$, that is, the set of critical and asymptotic values and limit points of these values. Define

$$O^+(\operatorname{sing}(f^{-1})) := \bigcup_{k=0}^{\infty} f^k(\operatorname{sing}(f^{-1})).$$

Let W be a hyperbolic open set in the complex plane, that is, $\mathbf{C}\backslash W$ is closed and contains at least two points. We can define the hyperbolic metric on W. By $\lambda_W(z)$ we denote the hyperbolic density on W. (For details, see Section 2). For $a \notin W$, put

$$C_W(a) := \inf\{\lambda_W(z)\operatorname{dist}(z, a);\ z \in W\},$$

where $\operatorname{dist}(z, a) = |z - a|$. If $C_W(a) > 0$, a is called to be a uniformly perfect point of W. We shall describe the case where $C_W(a) > 0$ in Section 2.

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THEOREM 1. *Let f be a transcendental meromorphic function and let U be a wandering domaim of f. Assume that a is a finite limit function of $\{f^n\}$ on U. If $C_{F(f)}(a) > 0$, then a is in the derived set of $O^+(\mathrm{sing}(f^{-1}))$.*

Remark. By Theorem 1 and some results in Section 2, we can have the following:

(A). *If each $f^n(U) \subseteq U_n$ is simply connected, then all the limit functions of $\{f^n\}$ on U is contained in the derived set of $O^+(\mathrm{sing}(f^{-1}))$.*

The result (A) was proved in Bergweiler et al. [9] for the case of f being an entire function by using the Koebe $\frac{1}{4}$ Theorem and the fact that an isolated singularity must be either critical or logarithmic. In this paper, we shall prove Theorem 1 with the Principle of Hyperbolic Metric, which allows us to avoid the requirement of univalent correspondence in the proof of Theorem 1. However, some ideas used in the proof of Theorem 1 comes from [9].

In the case that f is a holomorphic function of $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ onto $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$, all the limit functions of $\{f^n\}$ on a wandering domain are contained in the derived set of $O^+(\mathrm{sing}(f^{-1}))$, for there is at most one multiply connected component of Fatou set for such a function.

Let B denote the class consisting of all meromorphic functions f such that $\mathrm{sing}(f^{-1})$ is bounded. By the methods used in the proofs of [13, Theorem 1] and [7, Theorem 16], we can easily establish the following theorem, which extends [13, Theorem 1] from entire functions to meromorphic functions.

THEOREM 2. *If $f \in B$, then for $z \in F(f)$, the orbit $\{f^n(z)\}_{n=0}^{\infty}$ does not tend to ∞.*

If f is an entire function in B, then, in fact, for each $p > 0$, the orbit $\{f^{pn}(z)\}_{n=0}^{\infty}$ does not tend to ∞. However, this result is not true for a meromorphic function in B. We observe that the function $1/z - e^z$ is in B and has a cycle of Baker domains of order 2 (see [5]).

In [12] an entire function was constructed with a wandering domain on which $f^n(z)$ has an unbounded infinite limit set and in [4] a meromorphic function was constructed with a multiply-connected wandering domain on which $f^n(z)$ does so. Thus the limit functions of $\{f^n(z)\}$ may not be uniformly perfect points of $F(f)$ and cannot be claimed to be in the derived set of $O^+(\mathrm{sing}(f^{-1}))$ by the arguments in this paper. Under certain additional assumptions, we give the following negative answer to [10, Problem 2.87] which asks whether there exists an entire function f with wandering domain V such that $\bigcup_{n\geq 0} f^n(V)$ is bounded in $\mathbf{C}$.

THEOREM 3. *Let f be a transcendental entire function. If f has a multiply-connected component of normality, then for any wandering domain U of f, there exists a subsequence of $\{f^n\}$ on U tending to ∞.*

THEOREM 4. *Let f be a transcendental entire function such that for an unbounded sequence of r,*

$$m(r, f) := \min\{|f(z)|; |z| = r\} > r. \tag{1}$$

Then for any wandering domain U of f, there exists a subsequence of $\{f^n\}$ on U tending to ∞.

We have an immediate consequence of Theorem 4.

COROLLARY. *Let f be a transcendental entire function of growth not exceeding order $1/2$, minimal type. Then for any wandering domain U of f, there exists a subsequence of $\{f^n\}$ on U tending to ∞.*

This is because we can easily prove that for any positive integer n, the f in Corollary satisfies

$$\limsup_{r\to\infty} \frac{m(r, f)}{r^n} = +\infty.$$

2. Some results on hyperbolic metric

Let us recall the basic facts about the hyperbolic metric on a plane hyperbolic domain, that is, the domain whose boundary has at least two points in $\mathbf{C}$. Let $\lambda_\Omega(z)$ denote the density of the hyperbolic metric on hyperbolic domain Ω, Δ the unit disk and $\mathbf{H}$ the half plane $\{\mathrm{Im} z > 0\}$. It is well known that

$$\lambda_\Delta(z) = \frac{1}{1 - |z|^2}, \quad \lambda_{\mathbf{H}}(z) = \frac{1}{2\mathrm{Im} z}.$$

The density $\lambda_\Omega(w)$ of the hyperbolic metric on other hyperbolic domain Ω is defined as follows. Let $p(z)$ be a holomorphic universal covering mapping of Δ onto Ω. Then the density $\lambda_\Omega(w)$ is determined from

$$\lambda_\Omega(p(z))|p'(z)| = \frac{1}{1 - |z|^2}. \tag{2}$$

Indeed, noting that $p(z)$ is locally homeomorphic, from equation (2), $\lambda_\Omega(w)$ can be determined because of conformal invariance of hyperbolic metric. For a hyperbolic open set W, $\lambda_W(z)$ is defined to be the hyperbolic density on each connected component of W. The main results of the paper will be proved using the following Principle of Hyperbolic Metric (see [15]).

THEOREM A. *Let $f(z)$ be holomorphic in Δ and Ω a hyperbolic domain such that $f(\Delta) \subseteq \Omega$. Then*

$$\lambda_\Omega(f(z))|f'(z)| \le \lambda_\Delta(z), \quad \forall z \in \Delta$$

with equality if and only if f is a covering mapping of Δ onto Ω.

From the Principle of Hyperbolic Metric, we easily prove the Picard Theorem which says that any non-constant entire function can take on as values all finite complex numbers, with possibly at most one exception. Indeed, suppose there are two finite complex numbers not taken as values by a non-constant entire function $F(z)$. Without loss of generality, we may assume they are 0 and 1. Then for any positive number R, $F(z)$: $\Delta_R \mapsto \mathbf{C}\backslash\{0, 1\}$, where Δ_R denotes the disk $\{|z| < R\}$. The Principle of Hyperbolic Metric implies

$$\lambda_{0,1}(F(z))|F'(z)| \leq \frac{R}{R^2 - |z|^2}, \tag{3}$$

where $\lambda_{0,1}(w)$ is the hyperbolic density for $\mathbf{C}\backslash\{0, 1\}$. For any fixed z, let $R \to +\infty$, so that the right side of (3) tends to zero. Then we have $F'(z) = 0$, for $\lambda_{0,1}(F(z)) > 0$, and so $F(z)$ is a constant, which contradicts the assumption. The above proof of Picard theorem is essentially due to Ahlfors, who was the first person to prove the big and small Picard theorems, the Schottky theorem and the Bloch theorem by hyperbolic metric (see [1]).

For all $w_0 \in \Omega$, put $c_{w_0} = \lambda_\Omega(w_0)\text{dist}(w_0, \partial\Omega)$, where $\text{dist}(w_0, \partial\Omega)$ is the euclidean distance from w_0 to $\partial\Omega$. Then

$$\left\{|w - w_0| < \frac{c_{w_0}}{\lambda_\Omega(w_0)}\right\} \subset \Omega.$$

Now we introduce a domain constant

$$C_\Omega := \inf\{c_w;\ w \in \Omega\}.$$

In general, $0 \leq C_\Omega \leq \frac{1}{2}$ (see [14]). When $\infty \notin \Omega$, $\partial\Omega$ is called uniformly perfect, provided $C_\Omega > 0$. It is obvious that $C_\Omega = \inf_{a\in\partial\Omega\backslash\{\infty\}}\{C_\Omega(a)\}$. Hence when $\partial\Omega$ is uniformly perfect, then any finite point on $\partial\Omega$ is a uniformly perfect one of Ω. By the argument used in the proof of Corollary 1 in [6], it is easy to see that for $a \in \partial\Omega\backslash\{\infty\}$, $C_\Omega(a) = 0$ if and only if there exists a sequence of annuli $\{A_n\}$ in Ω such that for each n, A_n separates a and ∞ (that is, the bounded component of $\mathbf{C}\backslash A_n$ contains a), $\text{dist}(a, A_n) \to 0$ or ∞ and $\text{mod}(A_n) \to \infty$ as $n \to \infty$, where $\text{mod}(A_n)$ denotes the modulus of A_n. Then for $a \in \partial\Omega\backslash\{\infty\}$, if $\{a\}$ is not a component of $\partial\Omega$, we have $C_\Omega(a) > 0$; otherwise $\text{dist}(a, A_n) \to \infty$. However, this condition is not necessary for $C_\Omega(a) > 0$. Obviously, by a property of hyperbolic density, for $a \notin \overline{\Omega}$, $C_\Omega(a) > 0$. There exist many mutually equivalent conditions of uniform perfectness of a closed set (see [16] and [14]). If Ω is simply connected, from the Koebe $\frac{1}{4}$ Theorem, we easily prove $\frac{1}{4} \leq C_\Omega$. And $C_\Omega = \frac{1}{2}$ if and only if Ω is convex (see [14]).

THEOREM 5. *Let $f(z)$ be holomorphic in Δ with $f'(0) \neq 0$ and $f(\Delta) \subseteq \Omega$. If $C_\Omega > 0$, then*

$$\{|w - f(0)| < C_\Omega|f'(0)|\} \subset \Omega$$

and for all $a \in \partial\Omega$, we have

$$|f(z) - a| \le |f(0) - a| \left(\frac{1+|z|}{1-|z|}\right)^{1/C_\Omega}. \tag{4}$$

Proof. By the Principle of Hyperbolic Metric, we have

$$\lambda_\Omega(f(z))|f'(z)| \le \frac{1}{1-|z|^2}, \forall z \in \Delta. \tag{5}$$

Since $0 < \frac{C_\Omega}{\mathrm{dist}(f(0), \partial\Omega)} \le \lambda_\Omega(f(0))$, from (5) we obtain

$$C_\Omega |f'(0)| \le \mathrm{dist}(f(0), \partial\Omega).$$

Since $\mathrm{dist}(f(z), \partial\Omega) \le |f(z) - a|$, from (5) we have

$$\frac{C_\Omega |f'(z)|}{|f(z)-a|} \le \frac{C_\Omega |f'(z)|}{\mathrm{dist}(f(z), \partial\Omega)} \le \lambda_\Omega(f(z))|f'(z)| \le \frac{1}{1-|z|^2}.$$

Integrating both the sides along the segment $[0, z]$, we get

$$C_\Omega \log \frac{|f(z)-a|}{|f(0)-a|} \le \log \frac{1+|z|}{1-|z|}.$$

This is the desired inequality (4).

Theorem 5 follows.

We observe Theorem 5. When $f(\Delta)$ is simply connected with $f(0) = 0$ and $f'(0) = 1$, we have

$$\left\{|w| < \frac{1}{4}\right\} \subset f(\Delta).$$

This result generalizes the Koebe $\frac{1}{4}$ Theorem, since we do not make the assumption of univalence on $f(z)$. And if $f(z)$ is univalent in Δ, then $f(\Delta)$ must be simply connected, but if $f(\Delta)$ is simply connected, it does not imply that $f(z)$ is univalent. For this, we observe the following example: Choose $z_0 \in \Delta \backslash \{0\}$; then

$$T_1(z) = \frac{z+z_0}{1+\bar{z}_0 z}: \Delta \mapsto \Delta, \; T_2(z) = \frac{z-z_0^2}{1-\bar{z}_0^2 z}: \Delta \mapsto \Delta$$

and $g(z) := T_2(T_1^2(z))$ maps Δ onto Δ. It is obvious that $g(0) = 0$ and $g'(0) = \frac{2z_0}{1+|z_0|^2} \neq 0$, but $g(z)$ is not univalent and

$$\left\{|w| < \frac{1}{4}|g'(0)|\right\} = \left\{|w| < \frac{|z_0|}{2(1+|z_0|^2)}\right\} \subset g(\Delta) = \Delta.$$

When $f(\Delta)$ is convex with $f(0) = 0$ and $f'(0) = 1$, then

$$\left\{|w| < \frac{1}{2}\right\} \subset f(\Delta).$$

3. Proofs of theorems

Proof of Theorem 1. Suppose that $a \in \mathbf{C}\backslash\left(O^{+}(\operatorname{sing}(f^{-1}))\right)'$ is a limit function of $\{f^n|_U\}$, say $f^{n_k}|_U \to a$. For the convenience, we may assume that $a = 0$. Then we can choose a point $z_0 \in U$, $R > 0$ and $r > 0$ such that $\overline{N(z_0, R)} \subset U$, $f^{n_k}\left(\overline{N(z_0, R)}\right) \subset N(0, r)\backslash\{0\}$ and

$$N(0, r) \cap (O^{+}(\operatorname{sing}(f^{-1}))\backslash\{0\}) = \emptyset, \tag{6}$$

where $N(z_0, R)$ denotes the disk of radius R with center z_0. Put $N_k := f^{n_k}(N(z_0, R))$. Since $f^{n_k}\colon N(z_0, R) \mapsto N_k \subset F(f)$, by the Principle of Hyperbolic Metric, we have

$$\lambda_{F(f)}(f^{n_k}(z_0))|(f^{n_k})'(z_0)| \le \frac{1}{R},$$

and therefore

$$C_{F(f)}(0)|(f^{n_k})'(z_0)| \le \frac{1}{R}\operatorname{dist}(f^{n_k}(z_0), 0) = \frac{1}{R}|f^{n_k}(z_0)|, \tag{7}$$

since f^{n_k} is analytic in $N(z_0, R)$.

On the other hand, define $g_k(z) = \log f^{n_k}(z)$ on $N(z_0, R)$ for some branch of the logarithm such that $g_k\colon N(z_0, R) \mapsto H = \{z;\ \operatorname{Re} z < \log r\}$. Since H is simply connected, by (6) we can continue the inverse function of g_k analytically to a single-valued function h_k on H, such that $h_k(g_k(z)) \equiv z$ for all $z \in N(z_0, R)$. Since f has infinitely many periodic cycles of order 2 (see [7, Theorem 3] or [18]), we choose a periodic cycle, say $\{p, q\}$, of order 2 with $N(0, r) \cap \{p, q\} = \emptyset$. We have $f^{-n_k}(N(0, r)\backslash\{0\}) \cap \{p, q\} = \emptyset$, and therefore $\{p, q\} \cap h_k(H) = \emptyset$, that is,

$$h_k\colon H \mapsto \mathbf{C}\backslash\{p, q\}.$$

By the Principle of Hyperbolic Metric, we have

$$\lambda_{p,q}(h_k(w))|h_k'(w)| \le \lambda_H(w) = \frac{1}{2(\log r - \operatorname{Re} w)}.$$

Define $w_k = g_k(z_0)$, then it follows from $h_k(w_k) = z_0$ that

$$\lambda_{p,q}(z_0)|h_k'(w_k)| \le \frac{1}{2(\log r - \operatorname{Re} w_k)},$$

and therefore by $h_k'(w_k)g_k'(z_0) = 1$, we have

$$2\lambda_{p,q}(z_0)(\log r - \operatorname{Re} w_k) \le |g_k'(z_0)| = \frac{|(f^{n_k})'(z_0)|}{|f^{n_k}(z_0)|}.$$

This contradicts (7), for $\lambda_{p,q}(z_0) > 0$ and $\operatorname{Re} w_k \to -\infty$, as $k \to +\infty$.

Theorem 1 follows.

In the proof of Theorem 2, we need the following lemma, which is a combination of [7, Lemma 8] and [8, Lemma 2].

LEMMA 1. *Suppose* $f \in B$ *and* $0 \notin \bigcup_{k=1}^{\infty} f^{-k}(\infty)$. *Then there exist a positive constant* R *and a curve* Γ *connecting* 0 *and* ∞ *such that* $|f(z)| \leq R$ *on* Γ *and for all* $z \in \mathbf{C}\backslash\{0\}$ *which are not poles of* f,

$$|f'(z)| \geq \frac{|f(z)|}{2\pi|z|}\log\frac{|f(z)|}{R}.$$

Proof of Theorem 2. We may assume without any loss of generalities that $0 \notin \bigcup_{k=1}^{\infty} f^{-k}(\infty)$. Then Lemma 1 holds. Suppose that there exists a point $z \in F(f)$ such that $f^n(z) \to \infty$, as $n \to \infty$. Obviously, there exist $z_0 \in F(f)$ and a positive constant R_0 such that $f^n|_{N(z_0,R_0)} \to \infty$, as $n \to \infty$ and $f^n(N(z_0, R_0)) \subset F(f) \cap \{|z| > R\}$. Lemma 1 implies

$$f^n\colon N(z_0, R_0) \mapsto \mathbf{C}\backslash\Gamma.$$

By Theorem 5, we have

$$|(f^n)'(z_0)| \leq \frac{4}{R_0}\mathrm{dist}(f^n(z_0), \Gamma) \leq \frac{4}{R_0}|f^n(z_0)|, \tag{8}$$

for f^n is analytic on $N(z_0, R_0)$ and $\mathbf{C}\backslash\Gamma$ is simply connected.

On the other hand, put $w_n = f^n(z_0)$. Then by Lemma 1, we have

$$|f'(w_n)| \geq \frac{|f(w_n)|}{2\pi|w_n|}\log\frac{|f(w_n)|}{R} = \frac{|w_{n+1}|}{2\pi|w_n|}\log\frac{|w_{n+1}|}{R},$$

and therefore

$$|(f^n)'(z_0)| = \prod_{k=0}^{n-1}|f'(w_k)| \geq \prod_{k=0}^{n-1}\frac{|w_{k+1}|}{2\pi|w_k|}\log\frac{|w_{k+1}|}{R} = \frac{|w_n|}{|z_0|}\prod_{k=0}^{n-1}\frac{1}{2\pi}\log\frac{|w_{k+1}|}{R}.$$

It is obvious that the above inequality contradicts (8), for $\prod_{k=0}^{n-1}\frac{1}{2\pi}\log\frac{|w_{k+1}|}{R} \to \infty$ $(n \to \infty)$.

Theorem 2 follows.

Remarks. (I) In the proof of Theorem 1, if for each $k \geq 0$, there exists a curve Γ_k connecting a and ∞ such that $N_k \subset \mathbf{C}\backslash\Gamma_k$, then a is in the derived set of $O^+(\mathrm{sing}(f^{-1}))$.

(II) Analysing the proof of Theorem 2 or of [7, Theorem 16], we can deduce that if $f^p(\mathrm{sing}(f^{-1}))$ is bounded, then for $z \in F(f)$, $\{f^{pn}(z)\}$ does not tend to ∞. Obviously, Theorem 16 in [7] easily follows from this result.

Proof of Theorem 3. Since f has a multiply-connected component of normality, all the components of $F(f)$ are bounded, and further, for each $n \geq 0$, $f^n(U)$ is bounded. Suppose that the limit set of $\{f^n|_U\}$ is bounded; then for a real number $M > 0$,

$$\overline{\bigcup_{n\geq 0} f^n(U)} \subset N(0, M).$$

By Theorem 3.1 in Baker [2], we can find a closed Jordan curve γ in a multiply-connected component W of $F(f)$ with $N(0, M) \subset \mathrm{int}\gamma$, where $\mathrm{int}\gamma$ denotes the interior which is bounded by γ, such that there exists a closed Jordan curve $\Gamma \subset f(\gamma)$ with $\overline{\mathrm{int}\gamma} \subset \mathrm{int}\,\Gamma$. Let D be a component of $f^{-1}(\mathrm{int}\,\Gamma)$ intersecting $\mathrm{int}\gamma$; then $f(D) = \mathrm{int}\,\Gamma$. It is easy to see that $\partial D \subset F(f)$, and further, $\partial D \subset W$; consequently, $N(0, M) \subset \overline{D} \subset \mathrm{int}\,\Gamma$. D is simply connected. Suppose not. We draw a simple closed Jordan curve α in D which is not homotopic to a point there. Since $f(\partial D) = \Gamma$, $f(\mathrm{int}\,\alpha) \cap \Gamma \neq \emptyset$; therefore, since $f(\alpha) \subset \mathrm{int}\,\Gamma$, we have $\infty \in f(\mathrm{int}\,\alpha)$, but f is analytic on $\overline{\mathrm{int}\,\alpha}$. We will derive a contradiction. Noting that by Theorem 3.1 in Baker [2], f has no asymptotic values, we have proved that $f\colon D \mapsto \mathrm{int}\,\Gamma$ is a polynomial-like mapping (see [11, p. 99]). It follows from Theorem 1.1 in [11, p.99] that there exist a polynomial P and a quasiconformal mapping φ such that $f = \varphi^{-1} \circ P \circ \varphi$ on D. Obviously,

$$\bigcup_{n\geq 0} f^n(U) \subset \mathrm{int}\, K(f) = \mathrm{int} \bigcap_{n\geq 0} f^{-n}(D) \subset F(f),$$

where $K(f)$ is usually called the filled-in Julia set of polynomial-like mapping $f\colon D \mapsto \mathrm{int}\,\Gamma$, and therefore $\varphi(U)$ is a wandering domain of P. But P has no wandering domains.

Theorem 3 follows.

Proof of Theorem 4. Suppose that the limit set of $\{f^n|_U\}$ is bounded. Since f has no asymptotic values, for each $n \geq 0$, $f^n(U)$ is simply-connected and a component of $F(f)$. Since all the limit functions of $\{f^n|_U\}$ are constants, under the assumption of (1), we have:

CLAIM. *U is bounded, and hence so is $f^n(U)$ for each n.*

The claim will be proved after we complete the proof of Theorem 4. Therefore for some real number $M > 0$,

$$\overline{\bigcup_{n\geq 0} f^n(U)} \subset N(0, M).$$

Take $R > M$ so that

$$|f(z)| \geq m(R, f) > R, \text{ on } \gamma\colon |z| = R. \tag{9}$$

We can assume that $f'(z) \neq 0$, on γ, for we can change R a little bit so that (9) still holds. Then there exists a closed Jordan curve $\Gamma \subset f(\gamma)$ such that $\gamma \subset \operatorname{int}\Gamma$ and $\operatorname{int}\Gamma \cap f(\gamma) = \emptyset$. We may assume that $f^2(z)$ has a fixed point z_0 with $\{z_0, f(z_0)\} \subset \operatorname{int}\gamma$. Let D be the bounded component of $f^{-1}(\operatorname{int}\Gamma)$ containing z_0, and then D is simply connected and $f\colon D \mapsto \operatorname{int}\Gamma$ is proper, for f has no asymptotic values. $D \subset \operatorname{int}\Gamma$. Suppose not. We have a point z_1 in $\gamma \cap D$ sent to Γ under f, which contradicts the fact that f is analytic, for z_1 is an interior point of D. Obviously, we can have $N(0, M) \subset D$ as long as R is sufficiently large. Therefore, $f\colon D \mapsto \operatorname{int}\Gamma$ is a polynomial-like mapping.

A contradiction follows from the same argument as in the proof of Theorem 3. Thus we complete the proof of Theorem 4.

Now we are in position to prove the claim. First of all, we need the following.

LEMMA 2 [3]. *Let $f(z)$ be analytic in $A \le |z| \le B$. If $|f(z)| > R$, on $|z| = B$ and $|f(z)| < R$ on $|z| = A$, then there exists a simple closed curve $\Gamma \subset \{A < |z| < B\}$ with $0 \in \operatorname{int}\Gamma$ and $|f(z)| = R$ on Γ.*

It is obvious that Lemma 2 also holds if "$|z| = A$" and "$|z| = B$" are replaced by two simple closed curves which go around the origin once.

Suppose that U is unbounded. Since the limit set of $\{f^n|_U\}$ is bounded, there exists a point $\alpha \in U$ and a positive number B such that for all $n \ge 0$, $|f^n(\alpha)| < B$.

By (1), we can choose a sufficiently large R such that

$$|f(z)| \ge m(R, f) > R, \text{ on } \Gamma\colon |z| = R,$$

and $R = M(R_1, \alpha, f) > R_1 > |f^n(\alpha)|$, where $M(R_1, \alpha, f) = \max\{|f(z)|;\ |z - \alpha| = R_1\}$. Then there exists a simple closed curve $\Gamma_1 \subset \{R_1 < |z - \alpha|\} \cap \{|z| < R\}$ with $\alpha \in \operatorname{int}\Gamma_1$ such that $|f(z)| = R$, on Γ_1. And furthermore on Γ_1,

$$|f^2(z)| \ge m(R, f) > R.$$

Assume that for $k > 0$, there exists a simple closed curve $\Gamma_{k-1} \subset \{|z| < R\}$ with $\alpha \in \operatorname{int}\Gamma_{k-1}$ such that

$$|f^k(z)| \ge m(R, f) > R, \text{ on } \Gamma_{k-1}. \tag{10}$$

Since $R > |f^k(\alpha)|$, it is easy to see that there exists a $R_k > 0$ such that

$$R = M(R_k, \alpha, f^k) > |f^k(\alpha)|.$$

An application of maximal principle to (10) implies that

$$\{|z - \alpha| \le R_k\} \subset \operatorname{int}\Gamma_{k-1} \text{ and } R > R_k > 0.$$

Then on $|z-\alpha| = R_k$, $|f^k(z)| \leq R$. By Lemma 2, we can find a simple closed curve $\Gamma_k \subset \{R_k < |z-\alpha|\} \cap \operatorname{int}\Gamma_{k-1}$ with $\alpha \in \operatorname{int}\Gamma_k$ on which $|f^k(z)| = R$, and therefore

$$|f^{k+1}(z)| \geq m(R, f) > R, \text{ on } \Gamma_k.$$

Thus by induction, we have proved:

ASSERTION. *For arbitrary positive integer s, there exists a simple closed curve $\Gamma_s \subset \{|z| \leq R\}$ with $\alpha \in \operatorname{int}\Gamma_s$ such that*

$$|f^s(z)| \geq m(R, f) > R \text{ on } \Gamma_s. \tag{11}$$

We draw a Jordan curve γ in U connecting α and a point in $\{|z| = R\} \cap U$. It is easy to see from (11) that for all $n \geq 0$, $f^n(\gamma)$ always has points in $\{|z| < B\}$ and in $\{R < |z|\}$, and hence $f^n(z)$ in U has no constant limits, a contradiction.

The claim follows. The idea used in the proof of the claim comes from [19].

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