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REGULAR SUBDIVISION IN $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$

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ABSTRACT. In the ring $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$, there is a natural subdivision technique analogous to regular subdivision in rational algebraic rings like $\mathbf{Z}[\frac{1}{2}]$. The properties of this subdivision process are developed using the matrix associated to the Fibonacci substitution tiling. These properties are applied to prove some finiteness properties for a discrete group of piecewise-linear homeomorphisms.

1. Introduction

In rings of rational algebraic integers there are obvious regular subdivision processes. For example, in the dyadic rationals $\mathbf{Z}[\frac{1}{2}]$, there is a subdivision process given simply by division of intervals whose endpoints lie in $\mathbf{Z}[\frac{1}{2}]$ into two equal-length pieces. By iterating this process, we can get arbitrary elements of $\mathbf{Z}[\frac{1}{2}]$ as endpoints of intervals in our subdivision. The subdivisions of the unit interval obtained in this way are regular subdivisions in the following sense: the points in the subdivision are obtained by successive halving of intervals already in the subdivision.

In other algebraic rings, though, the analogue of this subdivision process is not so obvious. In this work, we develop the analogue of this process for the ring $\mathbf{Z}[\frac{1+\sqrt{5}}{2}]$, inspired by the Fibonacci substitution tiling $L \to LS$, $S \to L$. This notion of subdivision has essential features in common with regular rational subdivision, including the property that any prescribed point can occur as the endpoint of an interval in a regular subdivision. We obtain bounds on the number of steps required for this process for a given point. These properties are applied to prove a finiteness theorem for a particular class of groups of piecewise-linear homeomorphisms of the interval, which are variations of Thompson's group F. Throughout, we will use τ for the golden ratio: $\tau = \frac{1+\sqrt{5}}{2}$. The author would like to thank the referees for helpful comments which led to a more succinct presentation of the results in Section 2.

2. τ-Regular subdivision

One difficulty is to decide what our fundamental subdivision operation will be. In $\mathbf{Z}[\frac{1}{n}]$, our fundamental subdivision operation was dividing a specified interval into n

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equal pieces. The analogue, which would be subdivision into τ pieces, can be accomplished by taking advantage of the characteristic polynomial of τ, $x^2 - x - 1 = 0$. Thus τ^{-1} is a root of $1 = x^2 + x$. We can thus consider the subdivision specified by $1 = \tau^{-2} + \tau^{-1}$ as replacing an interval of length one with two intervals—a long one of length τ^{-1} and a short one of length τ^{-2}. Note that in $\mathbf{Z}[\tau]$, when subdividing an interval of length Δ into two pieces (the long one of length $\tau^{-1}\Delta$ and the short one of length $\tau^{-2}\Delta$), we have a choice about the order of the appearance of the two new intervals. Either we can choose to have the long interval first or the short interval appear first.

DEFINITION 2.1. A *τ-regular subdivision of size* 2 *of the unit interval* $[0, 1]$ is the sequence $\{0, 1\}$.

Given a τ-regular subdivision $\{0 = c_0, \ldots, c_{k-1} = 1\}$ of size k of $[0, 1]$, we construct *τ-regular subdivision of size $k+1$ of the unit interval* $[0, 1]$ as follows:

For some $0 \leq i < k - 1$, we replace the pair c_i, c_{i+1} with one of the following sets of three points:

(1) $c_i, c_i + \tau^{-2}\Delta, c_i + 1$
(2) $c_i, c_i + \tau^{-1}\Delta, c_i + 1$

where $\Delta = c_{i+1} - c_i$.

We say a sequence $\{0 = c_1, \ldots, c_n = 1\}$ is a *τ-regular subdivision of* $[0, 1]$ if it is a τ-regular subdivision of $[0, 1]$ of size n for some n.

The two possibilities above for subdividing to increase the size of the subdivision correspond to the two ways of dividing an interval of length Δ into pieces of lengths $\Delta\tau^{-1}$ and $\Delta\tau^{-2}$, with either the shorter or the longer interval coming first.

We can understand which points p are obtainable at a particular level by considering relatively uniform subdivisions, where the intervals in the subdivision are all one of two possible lengths, within a factor of τ.

DEFINITION 2.2. A τ-regular subdivision of an interval $[0, 1]$ is called *almost uniform of level* N if the lengths of the intervals in the subdivision are all either τ^{-N} or τ^{-N-1}.

Every τ-regular subdivision of some size s that has a shortest interval of length τ^{-N} can be further subdivided to an almost uniform subdivision of level N by merely subdividing all the intervals longer than τ^{-N} successively until they are all of length either τ^{-N} or τ^{-N-1}, so we will suppose in the rest of this section that our τ-regular subdivisions are almost uniform.

Given S_N, an almost uniform τ-regular subdivision of level N, we can create S_{N+1}, a τ-regular subdivision of level $N + 1$ by some simple replacement rules which are related to the change of scale by a factor of τ. In S_N, the long intervals are of length

τ^{-N} and the short intervals are τ^{-N-1} long. In the resulting S_{N+1}, the long intervals are of length τ^{-N-1} and the short intervals are τ^{-N-2} long. Each of the l_N long intervals in S_N will be subdivided into a long interval and short interval in S_{N+1}, and each of the s_N short intervals in S_N will remain unchanged in length but will now be regarded as a long interval in S_{N+1}. This gives the relationship $l_{N+1} = l_N + s_N$ and $s_{N+1} = l_N$, which is the defining relationship for the Fibonacci sequence, denoted as F_n with $F_1 = 1$, $F_2 = 1$.

We would like to show that the τ-regular subdivision process described above has properties similar to those of regular subdivision in rational algebraic rings. The most important property is described in the following theorem.

THEOREM 1. *For any point p in $\mathbf{Z}[\tau] \cap [0, 1]$, there exists a τ-regular subdivision of $[0, 1]$ with p as an endpoint of at least one of the intervals in the subdivision.*

Before proceeding with the proof of Theorem 1, we develop some of the properties of τ-regular subdivisions. First, we consider the representation of a specified point $p \in \mathbf{Z}[\tau] \cap [0, 1]$ with respect to powers of τ.

LEMMA 1. *For any positive p in $\mathbf{Z}[\tau]$, there exists $N \in \mathbf{Z}$ such that $p = \frac{m}{\tau^N} + \frac{n}{\tau^{N+1}}$ for some $m, n \in \mathbf{Z}^+$. Furthermore, N can be chosen large enough so that the ratio $\frac{m}{n}$ is arbitrarily close to τ.*

Proof. Since $p \in \mathbf{Z}[\tau]$, p has an expression as a polynomial in τ with integer coefficients. Since $\tau^2 = 1 + \tau$, we can easily rewrite this expression as an expression of the form $p = l_0 + \frac{s_0}{\tau}$, where l_0, s_0 are integers, not necessarily positive. We consider the subdivision process described above as a Markov chain, with each successive state representing the number of long and short intervals for progressively smaller-scale representations of p. This process is governed by the matrix equation

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} l_i \\ s_i \end{pmatrix} = \begin{pmatrix} l_{i+1} \\ s_{i+1} \end{pmatrix}$$

Since the transition matrix is the Fibonacci matrix, satisfying

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^N = \begin{pmatrix} F_{N+1} & F_N \\ F_N & F_{N-1} \end{pmatrix},$$

we have

$$p = l_0 + \frac{s_0}{\tau} = \frac{l_0 F_{N+1} + s_0 F_N}{\tau^N} + \frac{l_0 F_N + s_0 F_{N-1}}{\tau^{N+1}} = \frac{l_N}{\tau^N} + \frac{s_N}{\tau^{N+1}}.$$

τ is the dominant eigenvalue for the Fibonacci matrix and noting that the conjugate $\hat{\tau} = -\tau^{-1}$, we have the Binet formula (see [8]) $F_n = \frac{\tau^n - (-\tau)^{-n}}{\sqrt{5}}$. Since the original fixed quantity p is positive, so is $l_0\tau + s_0$. We iterate this subdivision process until both

l_i and s_i are positive. Since $\lim_{k\to\infty}\frac{F_{k+1}}{F_k}=\tau$, for large N we have $l_0F_{N+1}+s_0F_N\simeq(l_0\tau+s_0)F_N$ and $l_0F_N+s_0F_{N-1}\simeq(l_0\tau+s_0)F_{N-1}$, and both are positive.

For the "furthermore" part of the lemma, we note that

$$\lim_{k\to\infty}\frac{l_k}{s_k}=\lim_{k\to\infty}\frac{l_0F_{k+1}+s_0F_k}{l_0F_k+s_0F_{k-1}}=\tau.$$

The ratio of long intervals to short intervals in the subdivision thus approaches τ. □

Thus we can realize an arbitrary point of $\mathbf{Z}[\tau]$ as a combination of long and short intervals on some scale. We would further like to realize this quantity explicitly as a segment of subdivision of m intervals of length τ^{-N} and n intervals of length τ^{-N+1}. We cannot realize an arbitrary expression in such a way as, in particular, there is no way to get 3 short intervals in a row at any scale. However, an expression in which the approximate ratio of long intervals to short intervals is τ can be realized as the initial segment of some τ-regular subdivision which can actually be obtained from an appropriate length interval, such as [0, 1].

DEFINITION 2.3. An expression for a point $p\in\mathbf{Z}[\tau]\cap[0,1]$ of the form $p=\frac{m}{\tau^N}+\frac{n}{\tau^{N+1}}$ for some $m,n,N\in\mathbf{Z}$ is called *obtainable at level N* if there is a τ-regular subdivision S of [0, 1] such that there is an initial segment in the subdivision S which has exactly m intervals of length τ^{-N} and exactly n intervals of length τ^{-N-1} and no other intervals.

A point p is called *obtainable* if there is some expression of the form above for which p is obtainable at level N.

A pair of integers (m,n) is called *obtainable at level N* or *an obtainable long-short pair at level N* if $p=\frac{m}{\tau^N}+\frac{n}{\tau^{N+1}}$ is obtainable at level N, and similarly we define a pair of integers (m,n) to be *an obtainable long-short pair* if it is obtainable at some level N.

We would like to show that any quantity in $\mathbf{Z}[\tau]\cap[0,1]$ has an obtainable expression which we do through the following series of lemmas:

LEMMA 2. *If an expression $p=\frac{m}{\tau^N}+\frac{n}{\tau^{N+1}}$ is obtainable at level N, then the quantity $q=\frac{m}{\tau^M}+\frac{n}{\tau^{M+1}}$ is obtainable at level M for $M>N$.*

Proof. The quantity $\frac{1}{\tau^{M-N}}+\frac{0}{\tau^{M-N+1}}$ is obtainable at level $M-N$ by simply iterating the subdivision rule which always chooses the long interval to be first, starting from the interval [0, 1]. From that subdivision, we can obtain the expression for q by applying the sequence of subdivisions of [0, 1] that yielded p to the leading interval of length τ^{M-N} to get the desired expression for q. □

Now we prove that the complete subdivisions of the unit interval are related to pairs of the Fibonacci numbers F_n.

LEMMA 3. *The quantities given by Fibonacci pairs (F_{n+1}, F_n) for expressions $p = \frac{F_{n+1}}{\tau^N} + \frac{F_n}{\tau^{N+1}}$ are obtainable at all levels $N \geq n \geq 1$.*

Proof. By induction on n.

For $n = 1$, we have $p = \frac{F_2}{\tau^1} + \frac{F_1}{\tau^2} = \frac{1}{\tau^1} + \frac{1}{\tau^2}$ which is obtainable as a subdivision of the entire interval [0, 1] by one application of a subdivision rule, and thus for all higher levels N by Lemma 2.

For $n > 1$, we assume the lemma for is true for $n - 1$, so $p = \frac{F_n}{\tau^M} + \frac{F_{n-1}}{\tau^{M+1}}$ is obtainable at all levels $M \geq n - 1 \geq 1$. If we take the subdivision process from the inductive hypothesis that yielded p at level $n - 1$ and merely subdivide all the F_n intervals of length $\tau^{-(n-1)}$ into F_n intervals of length τ^{-n} and F_n intervals of length $\tau^{-(n+1)}$, then we have

$$p = \frac{F_n}{\tau^{n-1}} + \frac{F_{n-1}}{\tau^n} = \frac{F_n}{\tau^n} + \frac{F_n}{\tau^{n+1}} + \frac{F_{n-1}}{\tau^n} = \frac{F_{n+1}}{\tau^n} + \frac{F_n}{\tau^{n+1}},$$

which is the case for n. Thus, by Lemma 2, we have Lemma 3 for all levels $M \geq N$, and by induction, the lemma is proven. □

If there is room, there are always some obtainable long-short pairs, occurring in consecutive blocks:

LEMMA 4. *If $a \leq F_{N+1}$ then there is some long-short pair (a, a') which is obtainable at level N. If $b \leq F_N$ then there is some long-short pair (b', b) which is obtainable at level N. Furthermore, if $1 < b < F_N - 2$ and (b'', b) is obtainable then (b'', b) is one of a series of at least three contiguous obtainable long-short pairs of the form (b', b) and $(b' \pm 1, b)$.*

Proof. Since $1 = \frac{F_{N+1}}{\tau^N} + \frac{F_N}{\tau^{N+1}}$, the pair (F_{N+1}, F_N) is obtainable with F_{N+1} long intervals. Since $a \leq F_{N+1}$, there is some initial segment of that subdivision which has exactly a long intervals, so some pair (a, a') is obtainable.

For the short intervals, the same argument shows that (b', b) is obtainable for some b'. To show that there is always a series of at least three consecutive obtainable pairs, we consider the initial segment I of the subdivision S_N of level N which realizes the long-short pair (b', b), with b' the smallest number of long intervals which are obtainable at level N for our fixed b short intervals. The last interval in I must be a short interval, else there would be a realizable long-short pair with smaller b'. We consider the subsequent 2 intervals after I in S_N. If they are both long intervals, then $(b' + 1, b)$ and $(b' + 2, b)$ are both obtainable, and we have a string of three consecutive obtainable pairs as desired. If first interval following I is a long interval and the second a short interval, then we consider the subdivision S_{N-1} of level $N - 1$ from which we obtained S_N. The last short in I and the first long after I must have come from the same long interval in S_{N-1} since if they came from different

$$\begin{array}{lccccccc}
\text{Level N-1:} & \cdots\cdots\cdots & & l & & l & & \cdots\cdots\cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Level N:} & \underbrace{\cdots\cdots\cdots s}_{\mathrm{I}} & & l & s & & l & \cdots\cdots\cdots
\end{array}$$

becomes

$$\begin{array}{lccccccc}
\text{Level N-1:} & \cdots\cdots\cdots & & l & & l & & \cdots\cdots\cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Level N:} & \underbrace{\cdots\cdots\cdots s}_{\mathrm{I}} & & l & l & & s & \cdots\cdots\cdots
\end{array}$$

Figure 1. Changing subdivision types to yield nearby pairs for case `sls`

$$\begin{array}{lcccccc}
\text{Level N-1:} & \cdots\cdots\cdots l & & & l & & \cdots\cdots\cdots \\
& \downarrow & & & \downarrow & & \downarrow \\
\text{Level N:} & \underbrace{\cdots\cdots\cdots\ l\ s}_{\mathrm{I}} & s & & & l & \cdots\cdots\cdots
\end{array}$$

becomes

$$\begin{array}{lcccccc}
\text{Level N-1:} & \cdots\cdots\cdots l & & & l & & \cdots\cdots\cdots \\
& \downarrow & & & \downarrow & & \downarrow \\
\text{Level N:} & \underbrace{\cdots\cdots\cdots\ s\ l}_{\mathrm{I}} & l & & & s & \cdots\cdots\cdots
\end{array}$$

Figure 2. Changing subdivision types to yield nearby pairs for case `ssl`

intervals in S_{N-1}, we can change the subdivision type from "long-short" to "short-long" for long interval in S_{N-1} that yielded the final short in I and get a (b, b') with a smaller number of long intervals. Thus, the first short interval following I would have come from a long interval in S_{N-1} and we can change the subdivision type from "short-long" to "long-short", thus yielding two long intervals after I and a string of three consecutive obtainable pairs. This type of subdivision change is pictured in Figure 1.

Since three short intervals in a row are not obtainable with this subdivision process, the remaining case is when the short interval in I is followed by another short interval and then a long interval. In this case, when we consider the adjacent long intervals in S_{N-1} which were subdivided to yield the adjacent short intervals in S_N, we can change the order of both subdivisions as illustrated in Figure 2 to obtain the short interval followed by two long intervals, and thus again we have three consecutive obtainable long-short pairs for a fixed b. $\square$

In these obtainable subdivisions, the ratio of long intervals to short intervals will be close to τ, and in fact we can prove the following, which is essentially a converse to that observation:

LEMMA 5. *If an expression $p = \frac{m}{\tau^N} + \frac{n}{\tau^{N+1}}$ for a quantity $p \in [0,1]$ has the property that $|m - n\tau| < \tau$ then that expression for p is obtainable at level N.*

Proof. Note first that since $p \in [0,1]$ and $1 = \frac{F_N}{\tau^N} + \frac{F_{N-1}}{\tau^{N+1}}$, we necessarily have $m \leq F_N$ and $n \leq F_{N-1}$. Also note that the quantity $|m - n\tau|$ is merely the horizontal Euclidean distance from the integer lattice point (m, n) in the plane to the line L: $y = \frac{x}{\tau}$.

We prove this by contradiction. Out of all possible integer pairs for which (m, n) are within horizontal distance τ of the line L which are NOT obtainable at level N, where N is the smallest N for which $\frac{m}{\tau^N} + \frac{n}{\tau^{N+1}} < 1$ and (m, n) is not obtainable at level N, we consider a pair with the smallest n.

By assumption, this pair (m,n) is not obtainable, and we consider the pair $(n, m-n)$, a potential preimage to (m, n) at level $N-1$, whose horizontal distance to the line L is less than τ^2, due to the fact that the inverse matrix to the subdivision matrix above stretches horizontal distances to L by a factor of τ. If the pair $(n, m-n)$ has horizontal distance to L less than τ then it would be obtainable by the assumption that (m, n) was the first non-obtainable pair that close to L. If $(n, m-n)$ is obtainable at level $N-1$, then the pair (m, n) would be obtainable at level N by subdividing every long into a long and a short to get (m, n), contradicting the assumption that (m, n) was not obtainable at level N.

So $(n, m-n)$ is not obtainable and we consider its horizontal distance to the line L which must be between τ and τ^2. We consider two cases.

Case I. If (m, n) is below the line L, then the pair $(n, m-n)$ will be above the line L. It would lie at distance between τ and τ^2 from L so the pair immediately to the right $(n+1, m-n)$ will be within distance less than τ of L at, since $\tau^2 - 1 = \tau$. So since $m-n$ is less than m, by minimality of our alleged counterexample, it would be obtainable since it is close enough to L. Since $(n+1, m-n)$ is obtainable at level $N-1$, $(m+1, n+1)$ is obtainable at level N by subdivision. We consider the possible last two intervals at the end of the initial segment I of S_N, the subdivision which realizes $(m+1, n+1)$. If the last two intervals in I are a long and short interval, then there is an initial segment realizing (m, n), contradicting our assumption. If the last two intervals of I are both short intervals, then we do the subdivision type-swap similar to that pictured in Figure 2 in S_{N-1} to realize (m, n). If the last two intervals of I are both long intervals, we consider not only the subdivision S_{N-1} but also S_{N-2}. We can change the subdivision order type as in Figure 3 to get that (m, n) is obtainable.

Case II. If (m, n) is above the line L, then the pair $(n, m-n)$ will be below the line L. It would lie at distance between τ and τ^2 from L so the pair immediately to the left $(n-1, m-n)$ would be at distance less than τ from L and thus be obtainable. Since

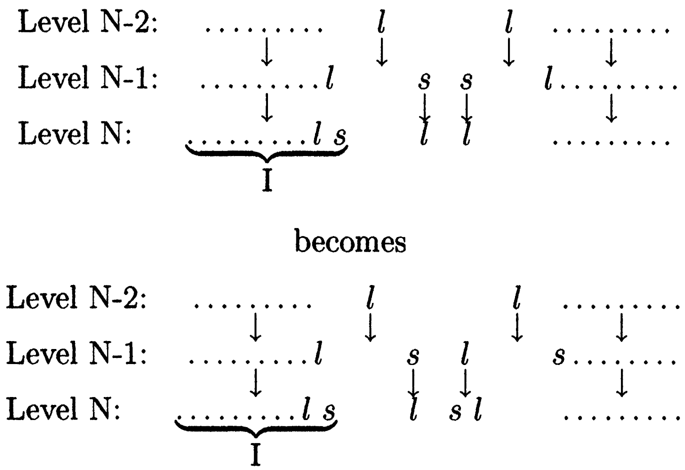

Figure 3. Changing subdivision types to yield nearby pairs

$(n-1, m-n)$ is obtainable at level $N-1$, $(m-1, n-1)$ is obtainable at level N by subdivision. We consider the two intervals immediately following the initial segment I of S_N the subdivision which realizes $(m-1, n-1)$. If the next two intervals in I are a long and short interval, then there would be an initial segment realizing (m, n), contradicting our assumption. If the next two intervals after I are both short intervals, then, similar to the change in Figure 2, we change the subdivision order for the second short interval from "short-long" to "long-short" which will then realize (m, n). If the next two intervals after I are both long intervals, we consider the subdivision type of the second one. If it came from a long interval, then it would have needed to come from a subdivision of type "long-short" from S_{N-1}. We can swap the order of that subdivision to get that (m, n) would be obtainable. If the second long interval came from a short interval in S_{N-1}, we consider the preceding stage S_{N-2}. We can change the subdivision order type as in figure 3 to get that (m, n) is obtainable. Note that it is easy to verify for cases $N \leq 3$ that the needed pairs are obtainable. □

Now we have a very good idea of some long-short pairs which are obtainable; they are the pairs which are close in distance to the line $y = \frac{x}{\tau}$. Also note that these arguments apply to general $p > 1$ subdivided from initial intervals of appropriately long length as well. Lemma 5 is the essential ingredient to proving Theorem 1.

Proof of Theorem 1. Given a quantity p in $\mathbf{Z}[\tau] \cap [0, 1]$, we would like to construct a τ-regular subdivision of $[0, 1]$ containing p.

To construct a τ-regular subdivision of $[0, 1]$ containing p, we use Lemma 1 to find a long-short pair representing p within horizontal distance τ of the line L. By

Lemma 5, that expression for p is obtainable. Thus p occurs as the endpoint of an interval in τ-regular subdivision. □

One consequence from the proof of Theorem 1 is an upper bound of the required number of levels of subdivision that will be required to get one with an interval with endpoint p. We know the original distance of the point (m_0, n_0) from the line $y = \frac{x}{\tau}$ is given by $d = |\frac{m_0 - \tau n_0}{\sqrt{\tau^2+1}}|$, so the number of subdivisions needed to reduce the horizontal distance to be less than τ (and thus the straight line distance to be less than $\frac{\tau}{\sqrt{\tau^2+1}}$) is given by

$$N \leq \log_\tau \left(\frac{d}{\frac{\tau}{\sqrt{\tau^2+1}}} \right)$$

This upper bound on levels of subdivision similar to inspecting of the degree of the denominator in the subdivision for the dyadic rationals $\mathbf{Z}[\frac{1}{2}]$. A further consequence of the theorem is the following corollary:

COROLLARY 1. *For any finite set F in $\mathbf{Z}[\tau] \cap [0, 1]$, there exists a τ-regular subdivision of* $[0, 1]$ *containing intervals whose set of endpoints contain F.*

Proof. We we can successively subdivide a subdivision containing the first point in F into one containing the subsequent ones iteratively by repeated applications of Theorem 1. The adjustments needed to ensure that the point p was an endpoint in the subdivision were all contained in intervals very close to p; at level N, $N-1$ or $N-2$ and involved a change of subdivision order in either a neighboring interval or one beyond that. By subdividing further, we can ensure that the separation between any two points of our finite set is at least, say, 10 intervals and thus the local arguments to prove the theorem above hold. □

3. Substitution and the Fibonacci tiling

The substitutions used in the construction of τ-regular subdivisions above are exactly those used to construct one-dimensional aperiodic tilings of the line, such as those considered in [9] and [1]. The bi-infinite strings of L's and S's that are invariant under the substitution $L \to LS$, $S \to L$ can be thought of as one-dimensional aperiodic tilings of the line where each L is a interval of length τ and each S is an interval of length 1. These aperiodic tilings of the line have a self-similarity which is a scale-invariance under a change of scale by τ.

Though these strings of L's and S's share similar dynamic properties with the patterns in the subdivisions considered here, one essential difference is that in order to have the scale invariance by a factor of τ, we need to always use the same subdivision rule. That is, in the self-similar tilings we decide in advance either to use only the rule $L \to LS$, $S \to L$ or to use only the rule $L \to SL$, $S \to L$. In the consideration of the τ-regular subdivisions above, we were able to use either subdivision rule at any stage. Since we use this choice, we lose the self-similarity properties that these one-dimensional aperiodic tilings possess.

4. An application

We consider the group F_τ of piecewise-linear homeomorphisms from $[0, 1]$ to $[0, 1]$ with the following restrictions:

- Each homeomorphism in the group has only finitely many singularities.
- Each singularity lies in $\mathbf{Z}[\tau]$.
- Away from the singularities, the slopes are powers of τ.

We will show that this group is finitely generated, finitely presented and of type FP_∞. This group is similar to Thompson's group F, a group of piecewise-linear homeomorphisms that has been studied in connection with questions in logic [11], homotopy theory [7], and measure theory of discrete groups [2]. Brown [3], [4] and Stein [10] develop tools to show that the groups with rational slopes and breakpoints are finitely generated, finitely presented and of type FP_∞. The earlier work, with the exception of [6], studied examples of these groups of piecewise-linear homeomorphisms where the singularities and slopes lie in rationally-generated subrings of $\mathbf{Q}$.

The argument to show that F_τ is finitely generated, finitely presented and of type FP_∞ is similar to that presented in [6], so only a brief sketch is described here. A group G is of type FP_∞ if there is a projective $\mathbf{Z}G$ resolution of $\mathbf{Z}$ regarded as a G-module with all terms in the resolution finitely generated. One technique for showing a group F is FP_∞ is to construct a classifying space $K(F, 1)$ with only finitely many n-cells in each dimension, and Brown [3] developed a technique for doing this indirectly, which we will use here.

The connection between τ-regular subdivisions of the unit interval and elements of F_τ comes from the following theorem:

THEOREM 2. *If S and S' are τ-regular subdivisions of the unit interval with the same number of points, then the affine interpolation of S and S' belongs to F_τ. Conversely, if $f \in F_\tau$, then f is the affine interpolation of some S and S', two τ-regular subdivisions of* $[0, 1]$.

Proof. First, suppose that S and S' are τ-regular subdivisions of the same size. Each interval l in an τ-regular subdivision has length τ^i for some i and has endpoints in $\mathbf{Z}[\tau]$. Thus, the singularity set will lie in $\mathbf{Z}[\tau]$. Since the slopes of the interpolation are ratios of lengths of intervals, and both the domain and range intervals have lengths say τ^{-n_i} and τ^{-m_i}, their ratios $\tau^{-m_i+n_i}$ lie in τ^i.

Now suppose that we are given $f \in F_\tau$. We first construct a τ-regular subdivision of $[0, 1]$ containing all the singularities of f. This is accomplished by applying Corollary 1 to the finite collection of breakpoints of f to obtain S_1, a τ-regular subdivision of $[0, 1]$. We consider the set of images of the points of S_1, which include the breakpoints of f, $f(S_1) = S_1'$, which is again a finite subset of $\mathbf{Z}[\tau]$. These form a subdivision of $[0, 1]$ which is not necessarily τ-regular, but this subdivision can be refined to get a subdivision S' which contains S_1' and is τ-regular. Furthermore, we can simultaneously apply the subdivisions used (in the range of f) in making the S_1' into S', a τ-regular subdivision, to the intervals in S_1 (in the domain of f) to obtain S, a τ-regular subdivision of the domain of f. Now we have two τ-regular subdivisions with $f(S) = S'$ and furthermore f is the affine interpolation of S and S', as desired. □

To show the finiteness properties of F_τ, we can use Theorem 2 to regard elements of F_τ as linear interpolations of regular subdivisions. We can control the complexity of τ-regular subdivisions of a particular size by virtue of the fact that there are a limited number of possibilities for such a subdivision of a fixed size. Using methods described in [6], we can build a directed poset on which F_τ acts and construct a classifying space for F_τ with bounds on the complexity of n dimensional cells stemming from the bound on complexity for subdivisions of a particular size. This classifying space, using arguments from [3], is enough to establish that F_τ is finitely generated, finitely presented and of type FP_∞.

5. Other work and results

Some of the theory of regular subdivisions in $\mathbf{Z}[\sqrt{2}]$ and also in $\mathbf{Z}[\sqrt{m^2+1}]$ was developed in [6], but not in as systematic way as the techniques presented here. The techniques developed here generalize readily to subdivisions of general intervals in $\mathbf{Z}[\tau]$ of the form $[a + b\tau, c + d\tau]$, and work is under way to understand subdivision in more general algebraic rings [5].

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