RESIDUAL PROPERTIES OF POLYCYCLIC GROUPS

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1. Let P be a group property. A group G is *residually-P* if to any non-identity element in G there is a normal subgroup K of G excluding x, and such that G/K has P. K. A. Hirsch [6] proved that a polycyclic group, which is a soluble group with maximal condition, is residually finite. Our aim is to sharpen this result.

Let π be a set of prime numbers. A *π-number* is a positive integer whose prime divisors lie in π. A *π-group* is a finite group of order a π-number. If π contains just one prime p, we write p-number and p-group. Thus a p-group here always means a finite p-group.

Our main result is

THEOREM 1. *A polycyclic group is residually a π-group for a finite set of primes π.*

We give below an explicit method for constructing the set π. It depends only on the group G and the finite factors occurring in a normal series for G. In particular if G is completely infinite (defined below), we can give a definite bound for π depending only on an invariant of G. A corollary to the theorem is a result of K. W. Gruenberg [3] on finitely generated nilpotent groups, which are a special class of polycyclic groups [7, p. 232].

Our notation is as follows:

If S is a subset of a group G, $\mathrm{Gp}(S)$ = subgroup generated by S;
$G^t = \mathrm{Gp}(g^t \mid g \in G)$, where t is a positive integer;
$[H, K] = \mathrm{Gp}([h, k] = h^{-1}k^{-1}hk \mid h \in H, k \in K)$, where H, K are subsets of G;
$C_G(F)$ = centraliser in G of a factor group F in G;
$\varphi(G)$ = Frattini subgroup of G;
Greek letters are used for sets of primes.
As usual π' is the complementary set to π.

2. For any positive integer t, G^t is a characteristic subgroup of finite index in a polycyclic group G. If t is a π-number, then G/G^t has exponent a π-number and so is a π-group. Hence any normal subgroup H of index a π-number m in G, contains the characteristic subgroup G^m, also of index a π-number. If H is residually a π-group, then for the same reason any $x \in H$ is excluded from some H^n where n is a π-number. Now H^n is normal in G and of index a π-number. These remarks make the following lemma obvious.

LEMMA 1. *If a normal subgroup of index a γ-number in G is residually a δ-group, then G is residually a π-group, where π is the union of γ and δ.*

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Any polycyclic group G has a series of subgroups of the following type:

$$\text{(A)} \qquad G = G_0 > G_1 > \cdots > G_k = 1,$$

where each G_i is normal in G and G_{i-1}/G_i is finitely generated *free* abelian or finite abelian. We shall use the term "normal series" only in this special sense. If all the factors in some normal series are infinite, G is called *completely infinite.*[1]

We can refine (A) so that it contains the greatest possible number of infinite factors. It is then called a strong series. By mutually refining two strong series we see that the maximum of the ranks of the free abelian factors in such a series is an invariant of G. We call it the *width* $w(G)$. For example a finitely generated nilpotent group has width $\leqq 1$. In what follows (A) is any normal series, not necessarily a strong series.

3. G is a given polycyclic group and we have chosen a normal series (A). Let τ be a finite set of primes such that all the finite factors in (A) are τ-groups. We say τ is a *suitable* set for G. It is clearly not unique since two different normal series may have different finite factors (cf. [2]), and in any case a set which contains τ is also suitable. We always choose τ nonempty. If G is completely infinite we may take τ to be an arbitrary single prime p.

We now construct the set π of Theorem 1. Let w be the product of all the primes in τ, so that any τ-number divides some power of w. Put $F_i = G_{i-1}/G_i$ $(i = 1, \cdots, k)$. Let C_i be the centraliser in G of (i) F_i if it is a finite group, or (ii) F_i/F_i^w if F_i is free abelian. All the G/C_i are finite groups so that the set μ of all primes dividing their orders is finite. We now define π to be the union of τ and μ. This set π apparently depends on series (A) and on the choice of τ, and so we write $\pi = f(\text{A}, \tau)$. We show later that π is in a sense independent of the choice of (A).

The following lemmas demonstrate why π is defined in the above way.

LEMMA 2. *Let t be any τ-number and $F_i = G_{i-1}/G_i$, some factor in* (A). *There is a π-number r such that*

$$[G_{i-1}, G^r] \leqq G_{i-1}^t G_i .$$

We notice that if F_i is finite, its order is a τ-number and so the lemma states that $[G_{i-1}, G^r] \leqq G_i$ for some π-number r. This is obvious since by definition of π, the centraliser of a finite F_i has index a π-number. Now suppose F_i is infinite. The centraliser of F_i/F_i^w is again by definition of index a π-number. Hence $[G_{i-1}, G^s] \leqq G_{i-1}^w G_i$ for some π-number s. But G_{i-1}/G_i is free abelian and one can now prove by induction that for any positive integer n, $[G_{i-1}, G^r] \leqq G_{i-1}^{w^n} G_i$, where $r = sw^{n-1}$ is a π-number. We finally choose n so that the given τ-number t divides w^n, and the lemma is proved.

[1] This term, used in [2] is preferable to "special polycyclic" in [8].

We now extend this result to any finite normal τ-factor in G. First we need the following observation.

LEMMA 3. *Let F be a finitely generated free abelian group containing a finite τ-factor B/C. Then there is a τ-number t such that $B \cap F^t \leqq C$.*

Proof. We are given $|B/C| = s$, say, is a τ-number. F has a free basis $a_1, \cdots, a_n$ such that B has a basis $b_1, \cdots, b_m$ $(m \leqq n)$ with $b_i = a_i^{u_i}$. Choose t to be the product of s and all the primes in $u_1, \cdots, u_m$ occurring in s. Then s divides each v_i/u_i where $v_i = \operatorname{lcm}(t, u_i)$. But $B \cap F^t$ is freely generated by $a_i^{v_i}$ $(i = 1, \cdots, m)$, and so $B \cap F^t \leqq B^s \leqq C$.

LEMMA 4. *Let L, M be normal subgroups of G and let L/M be a τ-group. Then $C_G(L/M)$ is of index a π-number in G.*

Proof. First assume $G_{i-1} \geqq L > M \geqq G_i$ for some i. The result is immediate by Lemma 2 if $F_i = G_{i-1}/G_i$ is finite. If F_i is free abelian, then by Lemma 3 there is a τ-number t such that $L \cap G_{i-1}^t G_i \leqq M$. But L is normal in G, so using Lemma 2 we have a π-number r with $[L, G^r] \leqq M$. This proves the result in this case.

Now assume L/M is any normal τ-factor, of order r say. Put $L_i = (L \cap G_i)M$. Then L_{i-1}/L_i is isomorphic to a τ-factor group in G_{i-1}/G_i by the Zassenhaus isomorphism. This isomorphism preserves transformation by elements of G and so by the first case above, there is a π-number s with $[L_{i-1}, G^s] \leqq L_i$ $(i = 1, \cdots, k)$. Now if $b \in L = L_0$, $x \in G^s$, then

$$[b, x^r] \equiv [b, x]^r \equiv 1 \pmod{L_2}.$$

Similarly $[b, x^{r^2}] \in L_3$. Since $M = L_k$, we can prove by induction that $[L, G^u] \leqq M$ where $u = sr^{k-1}$ and so is a π-number.

4. Lemma 4 is the main tool in the proof of Theorem 1. However before giving the proof, we note that π can be constructed using *any* normal series all of whose finite factors are τ-groups. Thus suppose

$$\text{(B)} \qquad G = K_0 > K_1 > \cdots > K_n = 1$$

is such a normal series. If K_{i-1}/K_i is finite, its centraliser is of index a π-number by Lemma 4. If K_{i-1}/K_i is infinite, then again by Lemma 4 the centraliser of $K_{i-1}/K_{i-1}^w K_i$ is of index a π-number. Hence the set of primes $f(\mathrm{B}, \tau)$ lies in $\pi = f(\mathrm{A}, \tau)$. Reversing the argument, we have equality and so we can put $\pi = f(G, \tau)$. This means G has some normal series whose finite factors are τ-groups, and that π is formed in the way explained from any such series.

We need the following technical lemma.

LEMMA 5. *Let τ be suitable for G and $\pi = f(G, \tau)$.*

(i) *If $\tau \subseteq \sigma$ then $\pi \subseteq f(G, \sigma)$.*

(ii) *τ is suitable for any subgroup H. If H is normal then $f(H, \tau) \subseteq \pi$.*
(iii) *If τ is suitable for G/H then $f(G/H, \tau) \subseteq \pi$.*

Proof. (i) Any finite normal τ-factor in G is also a σ-factor and so has centraliser of index a $f(G, \sigma)$-number by the previous lemma. (ii) The normal series (A) for G induces a normal series on H with terms $H_i = H \cap G_i$. H_{i-1}/H_i is isomorphic to a subgroup of G_{i-1}/G_i and so is a nontrivial finite group if and only if G_{i-1}/G_i is finite. Hence τ is suitable for H. If H is normal in G and we use this induced series to construct $f(H, \tau)$, then the factors concerned are normal τ-factors in G. By Lemma 4 they are all centralised by a subgroup C of index a π-number in G. Then $H/H \cap C$ is a π-group and so $f(H, \tau) \subseteq \pi$. (iii) We are given that G/H has a normal series for which τ is suitable, and the result follows as in (ii). We must specify that τ is suitable for G/H since the induced series with terms HG_i may have finite factors not τ-groups. For instance if G is free abelian and p any prime, then $f(G, p) = p$. However p does not divide $|G/H|$ if $H = G^q$ and $q \neq p$.

We now turn to the main theorem for which we need the following result of P. Hall and G. Higman [5].

LEMMA. *If F is a finite soluble group containing no nontrivial normal p'-subgroup and P is a maximal normal p-subgroup of F, then $P = C_F(P/\varphi(P))$.*

Proof of Theorem 1. Let (A) be a normal series for G of length k with all finite factors τ-groups and $\pi = f(G, \tau)$. We use induction on k to prove that G is residually a π-group. If $k = 1$, G is finitely generated abelian and either a finite τ-group or free. In both cases G is residually a τ-group, and a fortiori residually a π-group. Assume $k > 1$ and put $H = G_{k-1}$, the last nontrivial term in series (A). Truncating (A) at H gives a normal series for G/H whose finite factors are τ-groups. By Lemma 5, $f(G/H, \tau) \subseteq \pi$ and so by induction hypothesis, G/H is residually a π-group. It is now sufficient to take $x \neq 1$ in H and find a normal subgroup K excluding x and such that G/K is a π-group. Whether H is finite or not, there is a prime $p \in \tau$ and a positive integer n such that $x \notin B = H^{p^n}$. Choose K to be a maximal normal subgroup $\geqq B$ and excluding x. Since G is residually finite (Hirsch's theorem), $G/K = F$ is a finite soluble group. Every normal subgroup of F contains xK of p-power order. Hence if $P = L/K$ is a maximal normal p-subgroup of F we have $P = C_F(P/\varphi(P))$ by the lemma above. This implies that L is the centraliser in G of a normal p-factor L/L_1. Since $p \in \tau$, by Lemma 4, $|G/L|$ is a π-number. But L/K is a p-group and so G/K is a π-group. This completes Theorem 1.

5. By [6, Theorem 3.21] any polycyclic group has a completely infinite subgroup of finite index. Here by convention the trivial group is completely infinite. We can make this more precise.

THEOREM 2. *G has a completely infinite subgroup M of index a π-number. (By a remark in §2 we can choose M to be characteristic in G.)*

Proof. Let G have normal series (A) with finite factors all τ-groups. If G is finite, it is a τ-group and $\tau \subseteq \pi$ and so we put $M = 1$. Assume G is infinite. We use induction on the length k of (A). If $k = 1$, G is already free abelian. If $k > 1$, put $G_{k-1} = H$. The theorem holds for G/H and so there is a π-number t such that $(G/H)^t = G^tH/H$ is completely infinite. If H is free abelian, then G^tH itself is completely infinite and we are finished. Assume H is a finite abelian τ-group of order m say. By Lemma 4, $[H, G^s] = 1$ where s is a π-number. Put $L = G^{ts}H$ so that G/L is a π-group, H is central in L, and L/H is completely infinite being a subgroup of G^tH/H. L/H has a normal series $(\bar{A})$ of length $\leqq (k-1)$ induced by (A) and the finite factors of $(\bar{A})$ are τ-groups. The last factor, K/H say, of $(\bar{A})$ must be free abelian since L/H is completely infinite and so certainly torsion-free. Let $x_1 H, \cdots, x_n H$ be a set of free generators of K/H. Then $[x_i, x_j^m] = [x_i, x_j]^m = 1$ for $i, j = 1, \cdots, n$. Hence $K_1 = \text{Gp}\,(x_1^m, \cdots, x_n^m)$ is free abelian and $|K/K_1| = |K/K_1 H|\,|H|$ is a τ-number. Thus for some τ-number r, K^r is free abelian. Now replace K/H in series $(\bar{A})$ by K/K^r to obtain a normal series of length $\leqq (k-1)$ for L/K^r whose finite factors are still τ-groups. Hence by induction L/K^r has a completely infinite subgroup M/K^r of index a π-number. But K^r is free abelian by construction and so M is completely infinite, which proves Theorem 2.

We could of course use Theorem 2 to give a different proof of Theorem 1. It reduces the problem to completely infinite groups which are easier to handle.

6. *Example* 6.1. Suppose G is a finitely generated nilpotent group. We choose (A) to be a central series so that G itself centralises all the factors used in defining π. Therefore $\pi = \tau =$ the set of primes occurring in the finite factors of (A). In this case the torsion elements of G form a τ-subgroup. If G happens to be torsion-free, we can choose π as an arbitrary single prime. We have thus obtained Theorem 2.1 of [3].

This simple result is not true of polycyclic groups in general even if the periodic elements happen to form a subgroup. For example take

$$G = \text{Gp}\,(x, y \mid y^3 = 1, x^{-1}yx = y^2).$$

Any normal subgroup of G excluding y lies in $K = \text{Gp}\,(x^2)$, and $|G/K| = 6$. Put $A = \text{Gp}\,(y)$, the torsion subgroup of G. The normal series $G > A > 1$ gives $f(G, 3) = (2, 3)$. Hence G is residually a π-group if and only if $\pi \supseteq (2, 3)$.

Example 6.2. Suppose G is completely infinite of width n (see §2 above). We can choose τ to be any single prime p. The normal factors F used in defining $f(G, p)$ are elementary abelian p-groups of dimension at most n. Hence $|G/C_G(F)|$ divides $|GL(n, p)|$. Thus π lies in the set of primes dividing $p|GL(n, p)|$. This set is a specific upper bound. However it is different for each choice of p, and is in general much too large. Consider for example the completely infinite metabelian group

$$G = \text{Gp}\ (x, a, b \mid [a, b] = 1, x^{-1}ax = ab, x^{-1}bx = a^5b^6).$$

If $A = \text{Gp}\ (a, b)$, $G > A > 1$ is a completely infinite series.

$$C_G(A/A^2) = \text{Gp}\ (x^3, A), \quad C_G(A/A^3) = \text{Gp}\ (x^6, A)$$

and

$$C_G(A/A^5) = \text{Gp}\ (x^5, A).$$

Hence choosing $p = 2$ or 3 we see G is residually a $(2, 3)$-group, and choosing $p = 5$, G is also residually a 5-group. Thus π is by no means unique.

Example 6.3. A supersoluble group is a polycyclic group with a normal series all of whose factors are cyclic. We may make the finite factors of prime order, $q_1, \cdots, q_m$ say. The infinite cyclic factors have centralisers of index at most 2, and so π is contained in the union of $(2, q_i)$ and the primes dividing $q_i - 1$ $(i = 1, \cdots, m)$. Thus if G is completely infinite it is residually a 2-group. An example is the dihedral group, $\text{Gp}\ (x, y \mid y^{-1}xy = x^{-1})$. This is residually a π-group if and only if $2 \in \pi$ because all normal subgroups of odd index contain the element x.

7. Given a polycyclic group we have constructed a finite set of primes π such that G has a unit filter, (K_α) say, of normal subgroups with all G/K_α π-groups. We can now complete G to $\bar{G} = \text{inverse lim } G/K_\alpha$ with the usual projections (cf. [4]). $\bar{G}$ is soluble, but of course no longer polycyclic. However $\bar{G}$ has certain interesting properties. One can transfer the definition of Hall subgroups and the Hall theorems from the finite soluble groups G/K_α to $\bar{G}$. The details of this process can be found in [1].

8. It is an open question whether the above theorems possess suitable converses. Given that G is residually a p-group, for example, what can be said about possible normal series for G? Several attractive conjectures can be made which are supported by known examples. The author has been unable yet to obtain satisfactory results in this direction.

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