CHAIN SEQUENCES AND UNIVALENCE

BY

T. L. HAYDEN[1] AND E. P. MERKES[2]

1. Introduction

This article establishes the radii of univalence and starlikeness of a class of functions Π_f which is defined from the C-fraction expansion of the ratio $zf'(z)/f(z)$.

More precisely let

$$(1.1) \qquad f(z) = \sum_{n=1}^{\infty} c_n z^n, \qquad zf'(z) = \sum_{n=1}^{\infty} nc_n z^n, \qquad c_1 \neq 0,$$

be formal power series. From the one-to-one correspondence between formal power series and C-fractions [5],

$$(1.2) \qquad \frac{zf'(z)}{f(z)} \sim 1 - \frac{a_1 z^{\alpha_1}}{1} - \frac{a_2 z^{\alpha_2}}{1} - \cdots - \frac{a_n z^{\alpha_n}}{1} - \cdots,$$

where $\{\alpha_n\}$ and $\{a_n\}$ are respectively sequences of positive integers and of complex numbers, and the expression on the left is the formal quotient of the series (1.1). The continued fraction (1.2) terminates with k^{th} partial quotient if $a_j \neq 0$ for $j = 1, 2, \cdots, k$ and $a_{k+1} = 0$. In this case, we assume that $a_j = 0$ for $j = k+2, k+3, \cdots$.

For a fixed series $f(z)$ as in (1.1), let Π_f denote the class of formal power series $g(z) = \sum_{n=1}^{\infty} c_n^* z^n$, $c_1^* \neq 0$, such that

$$(1.3) \qquad \frac{zg'(z)}{g(z)} \sim 1 - \frac{a_1^* z^{\alpha_1}}{1} - \frac{a_2^* z^{\alpha_2}}{1} - \cdots - \frac{a_n^* z^{\alpha_n}}{1} - \cdots,$$

where $| a_n^* | \leqq | a_n |$, $n = 1, 2, \cdots$, and the sequences $\{\alpha_n\}$, $\{a_n\}$ are given in the correspondence (1.2). Let $U(\Pi_f)$ denote the radius of univalence of the class Π_f, i.e., $U(\Pi_f)$ is the supremum of the $r \geqq 0$ for which each member of Π_f is an analytic univalent function in $| z | < r$. It is agreed to put $U(\Pi_f) = 0$ in case there is a member of Π_f which is not analytic at $z = 0$. The radius of starlikeness with respect to the origin $S(\Pi_f)$ is defined in a similar manner. Evidently, $U(\Pi_f) \geqq S(\Pi_f) \geqq 0$. Moreover, if $g \in \Pi_f$, then $U(\Pi_g) \geqq U(\Pi_f)$ and $S(\Pi_g) \geqq S(\Pi_f)$.

A sequence of real numbers $k = \{k_n\}_{n=1}^{\infty}$ for which there exist g_{n-1}, $0 \leqq g_{n-1} \leqq 1$, such that $k_n = g_n(1 - g_{n-1})$ for $n = 1, 2, \cdots$ is called a chain sequence and the numbers g_{n-1} are the parameters of the sequence. In general, a chain sequence does not uniquely determine its parameters. How-

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ever, Wall [8, p. 80] proves the existence of minimal and maximal parameter sequences, $\{m_n\}_{n=0}^{\infty}$ and $\{M_n\}_{n=0}^{\infty}$ respectively, such that $m_n \leqq g_n \leqq M_n$, $n = 0, 1, 2, \cdots$, for every parameter sequence $\{g_n\}_{n=0}^{\infty}$ of k. Throughout this paper, the maximal parameter sequence is a judicious choice, although not a necessary one unless so stated, in the application of the results.

THEOREM A. *For a fixed power series $f(z)$ in (1.1), the correspondence (1.2) holds. Let r_0 be the supremum of the $r \geqq 0$ for which $\{|a_n| r^{\alpha n}\}_{n=1}^{\infty}$ is a chain sequence. Then $r_0 \leqq S(\Pi_f) \leqq U(\Pi_f)$. Moreover, if the sequence $\{|a_n| r_0^{\alpha n}\}_{n=1}^{\infty}$ is a chain sequence with uniquely determined parameters, then $S(\Pi_f) = U(\Pi_f) = r_0$.*

THEOREM B. *Let $f(z)$ be a power series (1.1) and let*

$$\frac{zf'(z)}{f(z)} \sim 1 - \frac{a_1 z^\alpha}{1} - \frac{a_2 z^\alpha}{1} - \cdots - \frac{a_n z^\alpha}{1} - \cdots, \tag{1.4}$$

where α is a positive integer. Then $U(\Pi_f) = S(\Pi_f) = r_0$, where r_0 is the supremum of the $r \geqq 0$ such that $\{|a_n| r^\alpha\}$ is a chain sequence. If $f_0(z)$ is a function such that

$$\frac{zf_0'(z)}{f_0(z)} \sim 1 - \frac{|a_1| z^\alpha}{1} - \frac{|a_2| z^\alpha}{1} - \cdots - \frac{|a_n| z^\alpha}{1} - \cdots, \tag{1.5}$$

then r_0 is the smallest nonnegative zero or singularity of $f_0'(z)$.

These results provide a simple numerical and theoretical method to estimate the radii of univalence and starlikeness of the class.

A study of the univalence of the function $F_\nu(z) = z^{1-\nu}J_\nu(z)$, where $J_\nu(z)$ is a Bessel function of order ν, was recently initiated by Kreyszig and Todd [3] for $\nu > -1$ and by Brown [1], [2] for some complex values of ν. Wilf [10] has simplified the proof of the main result in [3] and has replaced some of the inequalities for the radius of univalence of $F_\nu(z)$ with large ν by asymptotic equalities. These results and some extensions of them are shown in §4 to be corollaries of Theorems A and B.

2. Two lemmas from the problem of moments

Before proving Theorems A and B it is helpful to have some elementary consequences of the Stieltjes and the Hausdorff moment problems. For this purpose, let $\{k_n\}_{n=1}^{\infty}$ be a sequence of positive numbers and let $F(z)$ denote the formal power series which corresponds to the S-fraction

$$\frac{1}{1} + \frac{k_1 z}{1} + \frac{k_2 z}{1} + \cdots + \frac{k_n z}{1} + \cdots. \tag{2.1}$$

LEMMA 2.1. *Let the sequence $\{k_n r\}_{n=1}^{\infty}$ be a chain sequence if and only if $0 \leqq r \leqq 1$. Then the formal power series $F(z)$ corresponding to (2.1) converges in the disk $|z| < 1$ and represents a function which is analytic in the complex* *plane cut from* $-\infty$ *to* -1 *along the negative real axis and which has a singularity at* $z = -1$.

Proof. Since $\{k_n\}$ is a chain sequence, the S-fraction (2.1) converges and represents an analytic function in the z-plane cut from $-\infty$ to -1 along the negative real axis [8, p. 116]. Hence the power series $F(z)$ is convergent for $|z| < 1$ and there is a bounded nondecreasing function $\alpha(t)$ on $0 \leqq t \leqq 1$ such that $\alpha(1) - \alpha(0) = 1$ and

$$F(z) = \int_0^1 \frac{d\alpha(t)}{1 + zt}$$

[8, p. 263]. Suppose now $\alpha(t)$ has no point of increase at $t = 1$. This implies that there is an ε, $0 < \varepsilon < 1$, such that

$$F(z) = \int_0^{1-\varepsilon} \frac{d\alpha(t)}{1 + zt} = \int_0^1 \frac{d\alpha[(1-\varepsilon)s]}{1 + \zeta s}$$

$$= \frac{1}{1+} \frac{k_1\zeta/(1-\varepsilon)}{1} \, + \, \frac{k_2\zeta/(1-\varepsilon)}{1} \, + \cdots,$$

where $\zeta = (1 - \varepsilon)z$. Results on the Hausdorff moment problem [8, p. 263] and the last integral representation now imply that $\{k_n/(1-\varepsilon)\}_{n=1}^{\infty}$ is a chain sequence. This is contrary to the hypothesis that $\{k_n r\}_{n=1}^{\infty}$ is not a chain sequence for $r > 1$. Hence $\alpha(t)$ has a point of increase at $t = 1$. Define

$$\beta(s) = \int_{1/(1+s)}^1 \frac{d\alpha(t)}{t}, \qquad 0 \leqq s < \infty.$$

Clearly $\beta(s)$ is nondecreasing and, since $\beta(0) = 0$,

$$\beta(s) \geqq \alpha(1) - \alpha\left(\frac{1}{1+s}\right) > 0, \qquad s > 0.$$

This function has a point of increase at $s = 0$. Since

$$\int_0^{\infty} \frac{d\beta(s)}{s + 1 + z} = \int_0^1 \frac{d\alpha(t)}{1 + zt} = F(z),$$

it follows from well-known results on Stieltjes transforms [9, p. 337] that $F(z)$ has a singularity at $z = -1$.

LEMMA 2.2. *Suppose that for each* $r > 0$, *the sequence* $\{k_n r\}_{n=1}^{\infty}$ *is not a chain sequence. Then the power series* $F(z)$ *corresponding to* (2.1) *diverges in each neighborhood of zero.*

The proof is similar to that of Lemma 2.1 and is, therefore, omitted.

If (2.1) terminates with n^{th} partial quotient, then this continued fraction represents a rational function whose poles are negative real, simple, and have positive residue. Therefore it is found that Lemma 2.1 remains valid when the sequence $\{k_p\}_{p=1}^{\infty}$ is such that $k_p > 0$ for $p = 1, 2, \cdots. \ n - 1$; $k_p = 0$

for $p = n, n+1, \cdots$. Moreover, for each such sequence, $\{k_p r\}_{p=0}^{\infty}$ is a chain sequence for some $r > 0$. Hence the hypothesis of Lemma 2.2 is not fulfilled in this case.

3. Proof of Theorems A and B

Proof of Theorem A. First, it is evident from results on chain sequences [8, p. 86] that $\{| a_n | r_0^{\alpha_n}\}_{n=1}^{\infty}$ is itself a chain sequence. Now if $r_0 = 0$, there is nothing to prove. If $r_0 > 0$, for each $g(z) \in \Pi_f$, the C-fraction expansion (1.3) converges in the disk $|z| < r_0$ (cf. [4, Theorem 3.1]). It follows that the power series $zg'(z)/g(z)$ converges in this disk [5] and, hence, that $g(z)$ is analytic in $|z| < r_0$. Since the moduli of the partial numerators of the continued fraction (1.3) form a chain sequence for $|z| \leqq r_0$, an easy extension of the arguments in [8, p. 46] shows that

$$\operatorname{Re} \frac{zg'(z)}{g(z)} \geqq 0, \qquad |z| \leqq r_0.$$

Since $g(0) = 0$, $g'(0) \neq 0$, this implies that $g(z)$ is univalent and starlike with respect to the origin [6] for $|z| < r_0$. Thus $r_0 \leqq S(\Pi_f) \leqq U(\Pi_f)$.

Let $f_0(z)$ denote the formal series for which

$$\frac{zf_0'(z)}{f_0(z)} \sim 1 - \frac{|a_1| z^{\alpha_1}}{1} - \frac{|a_2| z^{\alpha_2}}{1} - \cdots - \frac{|a_n| z^{\alpha_n}}{1} - \cdots.$$

Then $f_0(z) \in \Pi_f$. If $\{M_j\}_{j=0}^{\infty}$ denotes the maximal parameter sequence of the chain sequence $\{| a_n | r_0^{\alpha_n}\}_{n=1}^{\infty}$, it is known [8, p. 81] that $M_0 = r_0 f_0'(r_0)/f_0(r_0)$. Since $M_0 = 0$ when the parameters are uniquely determined [8, p. 82], $f'(z)$ has a zero or $f_0(z)$ has a singularity at $z = r_0$. In any case, the function $f_0(z)$ is not analytic and univalent in any disk $|z| < R$ for $R > r_0$. This proves the last statement of the theorem.

Proof of Theorem B. By Theorem A, $r_0 \leqq S(\Pi_f) \leqq U(\Pi_f)$. If $r_0 > 0$, then by Lemma 2.1 the ratio $f_0(z)/zf_0'(z)$ obtained from (1.5) is analytic in $|z| < r_0$ and has a singularity at $z = r_0$. Thus $f_0'(z)$ is analytic and nonzero in $|z| < r_0$ and has a zero or a singularity at $z = r_0$. In any case $f_0(z)$ is not analytic and univalent in $|z| < R$ for any $R > r_0$. Therefore, $r_0 = U(\Pi_f) = S(\Pi_f)$. On the other hand, if $r_0 = 0$, the function $f_0(z)$ is not analytic at $z = 0$ by Lemma 2.2. Hence $U(\Pi_f) = S(\Pi_f) = 0$ in this case and the proof is complete.

4. Univalence of Bessel functions

From the recurrence formulas

$$zJ_{\nu+1}(z) = 2\nu J_\nu(z) - zJ_{\nu-1}(z),$$

$$zJ_{\nu+1}(z) = \nu J_\nu(z) - zJ_\nu'(z),$$

it follows that for $\nu \neq -1, -2, \cdots$

$$(4.1)\qquad \frac{zF_\nu'(z)}{F_\nu(z)} \sim 1 - \frac{\frac{1}{2}z^2/(\nu+1)}{1} - \frac{\frac{1}{4}z^2/(\nu+1)(\nu+2)}{1} - \cdots,$$

where $F_\nu(z) = z^{1-\nu}J_\nu(z)$. The continued fraction converges throughout the z-plane except at the zeros of $J_\nu(z)$ and, therefore, the correspondence symbol in (4.1) can be replaced by an equality [7], [8, pp. 347 ff.].

THEOREM C. *Let* $x = \operatorname{Re} \nu > -1$. *The radius of starlikeness* ρ_ν *of* $F_\nu(z) = z^{1-\nu}J_\nu(z)$ *is not less than the smallest positive zero of* $F_x'(z)$. *Moreover,*

$$(4.2)\qquad \rho_\nu^2 \geqq 2|\nu+1|\left\{1 - \frac{1}{1 + 2|\nu+2|[1 - 1/2|\nu+3|]}\right\}$$

and

$$(4.3)\qquad \lim_{\nu\to\infty} \rho_\nu^2/|\nu| = 2.$$

Proof. Since $|\nu + n| \geqq x + n > 0$ for $n = 1, 2, \cdots$, $F_\nu(z)$ is in the class Π_{F_x}. In view of the fact that $F_x(z)$ is an entire function, the first part of the theorem is now a consequence of Theorem B.

Let $|z| \leqq r$, where r^2 is the quantity on the right-hand side of the inequality (4.2) Put

$$0 < g_1 = \frac{r^2}{2|\nu+1|} = 1 - \frac{1}{2|\nu+2|[1 - 1/2|\nu+3|] + 1} < 1,$$

$$g_{n+1} = \frac{r^2}{4|\nu+n||\nu+n+1|(1-g_n)}, \qquad n = 1, 2, \cdots.$$

Since $|\nu + n + 1| > |\nu + n|$ for $n = 1, 2, \cdots$, the assumption $0 < g_{n-1} < 1$, $0 < g_n \leqq g_{n-2} < 1$, $n > 2$, implies

$$0 < g_{n+1} \leqq \frac{r^2}{4|\nu+n-1||\nu+n-2|(1-g_{n-2})} = g_{n-1} < 1.$$

Now $g_1 > g_3 = |\nu+1|g_1/|\nu+2|$ and $0 < g_2 = 1 - 1/2|\nu+3| < 1$. It follows by induction that $0 < g_n < 1$ for $n = 1, 2, \cdots$ and, therefore, that the sequence $r^2/2|\nu+1|$, $r^2/4|\nu+1||\nu+2|$, $\cdots$ is a chain sequence. Consequently $r \leqq r_0$, where r_0 is defined in Theorem B. Since $\rho_\nu \geqq r_0$, (4.2) is now proved.

Finally, for $|\nu + 3| > 4$ and $\operatorname{Re} \nu > -1$,

$$(4.4)\qquad \rho_\nu^2 \leqq \frac{2|\nu+1||\nu+2|}{|\nu+2|-1}.$$

Indeed, set $z_0^2 = 2(\nu+1)|\nu+2|/(|\nu+2|-1)$. Then, for $n = 2, 3, \cdots$, $|z_0|^2/4|\nu+n||\nu+n+1| \leqq \frac{1}{8}$. By Worpitzky's Theorem [8, p. 42] there is a μ such that $|\mu| \leqq 2$ and

$$z_0\frac{F_\nu'(z_0)}{F_\nu(z_0)} = 1 - \frac{z_0^2/2(\nu+1)}{1 - \mu z_0^2/4(\nu+1)(\nu+2)}.$$

An elementary calculation shows $\operatorname{Re}\{z_0F'_\nu(z_0)/F_\nu(z_0)\} \leqq 0$ which implies $\rho_\nu \leqq |z_0|$. The asymptotic equality now follows from (4.2) and (4.4).

For real $\nu > -1$, the bound in (4.2) is a good estimate of ρ_ν. Indeed, by a modification of the methods used in the preceding paragraph, it can be proved that

$$\operatorname{Re}\{z_0F'_\nu(z_0)/F_\nu(z_0)\} \leqq 0$$

when $\nu > -1$ and $z_0^2 = +2(\nu+1)\{1 - 1/[2(\nu+2)+1]\}$.

Theorem B and (4.1) can also be used to obtain information on the univalence and starlikeness of $F_\nu(z)$ when $\operatorname{Re}\nu \leqq -1$. Moreover, it is possible to obtain from the continued fraction of Gauss [8, p. 347] analogues of the preceding results for the confluent hypergeometric function ${}_1F_1(a, b; z)$.

References

1. R. K. Brown, *Univalence of Bessel functions*, Proc. Amer. Math. Soc., vol. 11 (1960), pp. 278–283.
2. ———, *Univalent solutions of $W'' + pW = 0$*, Canad. J. Math., vol. 14 (1962), pp. 69–78.
3. E. O. Kreyszig and J. Todd, *The radius of univalence of Bessel functions*, Illinois J. Math., vol. 4 (1960), pp. 143–149.
4. R. E. Lane and H. S. Wall, *Continued fractions with absolutely convergent even and odd parts*, Trans. Amer. Math. Soc., vol. 67 (1949), pp. 366–380.
5. W. Leighton and W. T. Scott, *A general continued fraction expansion*, Bull. Amer. Math. Soc., vol. 45 (1939), pp. 596–605.
6. L. Špaček, *Contribution à la théorie des fonctions univalentes*, Časopis pro Pěstování Matematiky a Fysiki, vol. 62 (1932), pp. 12–19.
7. E. V. Van Vleck, *On the convergence of the continued fraction of Gauss and other continued fractions*, Ann. of Math. (2), vol. 3 (1901), pp. 1–18.
8. H. S. Wall, *Analytic theory of continued fractions*, New York, Van Nostrand, 1948.
9. D. V. Widder, *The Laplace transform*, Princeton, N. J., 1941.
10. H. S. Wilf, *The radius of univalence of certain entire functions*, Illinois J. Math., vol. 6 (1962), pp. 242–244.

University of Wisconsin
Madison, Wisconsin
Marquette University
Milwaukee, Wisconsin