ON THE EXTENSION OF OPERATORS WITH A FINITE-DIMENSIONAL RANGE

BY

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1. Introduction

In this note we continue the study of problems concerning extension of compact operators and the characterization of the spaces having various extension properties. In [11], [12] and [13] our main interest was in characterizing the spaces X for which certain classes of operators having X as range space can be extended. In [14] some further results concerning the *"into"* extension problem are announced. Our aim here is to complement these results by studying the corresponding *"from"* extension properties. The main result of the present note shows that if all the operators defined on X and having a 3-dimensional range can be extended in a norm-preserving manner then the same is true for every compact operator defined on X, and this is the case if and only if X^* is an L_1 space (see Theorem 1 for a precise formulation). It is shown (Theorems 2 and 3) that in the preceding results 3 cannot be replaced by 2. Indeed, all the L_1 spaces have the "from" extension property for operators with a 2-dimensional range. In Section 4 a comparison between the "into" and "from" extension properties for compact operators is given. The paper ends with two theorems concerning the lifting of operators with a 2 or 3-dimensional range. These theorems complement some results of Köthe and Grothendieck [3].

Notations. By "operator" we mean a bounded linear operator. All Banach spaces are assumed to be over the reals. Let I be a set; the Banach space of all bounded real-valued functions on I with the supremum as norm is denoted by $m(I)$. If I is finite and consists of n points we denote $m(I)$ also by l_∞^n. The conjugate of l_∞^n is denoted by l_1^n; it is the n-dimensional L_1 space. The cell $\{x; x \in X, \| x - x_0 \| \leqq r\}$ is denoted by $S_X(x_0, r)$. A Banach space X is called a $\mathcal{P}_\lambda$ space if from every $Y \supset X$ there is a projection of norm $\leqq \lambda$ onto X. For the basic facts concerning $\mathcal{P}_\lambda$ spaces we refer to the book of Day [2, pp. 94–96]. The projection constant $\mathcal{P}(X)$ of a Banach space X is defined by (cf. [4])

$$\mathcal{P}(X) = \inf\{\lambda; X \text{ is a } \mathcal{P}_\lambda \text{ space}\}.$$

For a set K in a Banach space $\mathrm{Cl}(K)$ [resp. $\mathrm{Int}(K)$] denotes the norm closure [resp. interior] of K.

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2. The main result

In the proof of Theorem 1 we shall use three known results which we state here as lemmas. Lemma 1 is due to Nachbin [15] and it plays a decisive role in many extension problems. Grünbaum [5, the proof of Theorem 1] gave a proof of the lemma which is simpler than that of Nachbin. We give here a different simple proof of it.

LEMMA 1. *Let X be a Banach space and let $\{S_X(x_\alpha, r_\alpha)\}_{\alpha\in A}$ be a collection of mutually intersecting cells in X. Then there is a Banach space Z containing X with* $\dim Z/X = 1$ *such that* $\bigcap_{\alpha\in A} S_Z(x_\alpha, r_\alpha) \neq \emptyset$.

Proof. X is (isometric to) a subspace of $m(I)$ for a set I with a sufficiently large cardinality. In $m(I)$ every collection of mutually intersecting cells has a common point (this is an immediate consequence of Helly's theorem [7] in one dimension). Hence $\bigcap_{\alpha\in A} S_{m(I)}(x_\alpha, r_\alpha) \neq \emptyset$. Let z be any point in this intersection. If $z \notin X$ then the subspace Z of $m(I)$ spanned by X and z has the required properties. If $z \in X$ we may take as Z any space containing X and satisfying $\dim Z/X = 1$.

The next lemma is due to Klee [8].

LEMMA 2. *Let $\{C_i\}_{i=1}^{n+1}$ be convex open sets in a Banach space X of dimension $\geqq n$ such that $\bigcap_{i=1}^{n+1} C_i = \emptyset$. Then there is a closed subspace V of X with* $\dim X/V = n$ *such that neither V nor any translate of V intersects all the C_i.*

Klee stated his results in [8] only for finite-dimensional Banach spaces but (as remarked in [8]) the results and their proofs hold also for infinite-dimensional Banach spaces (and even in more general linear spaces). The assertion of Lemma 2 follows from paragraphs (3.1) and (3.2) of [8].

The proof of the next lemma is given in [11] and [12].

LEMMA 3. *Let X be a Banach space. X^* is an L_1 space if and only if for every collection of four mutually intersecting cells $\{S_X(x_i, r_i)\}_{i=1}^4$ in X and every* $\varepsilon > 0$, $\bigcap_{i=1}^4 S_X(x_i, r_i + \varepsilon) \neq \emptyset$.

We are now ready to prove

THEOREM 1. *Let X be a Banach space. The following four statements are equivalent.*

(i) *X^* is an $L_1(\mu)$ space for some measure μ.*

(ii) *For every two Banach spaces Z and Y with $Z \supset X$ and every compact operator T from X into Y there is a compact norm-preserving extension of T from Z into Y.*

(iii) *For every two Banach spaces Z and Y with $Z \supset X$ and every weakly compact operator T from X into Y there is a weakly compact norm-preserving extension of T from Z into Y.*

(iv) *For every two Banach spaces Z and Y with $Z \supset X$,* $\dim Z/X = 1$, dim $Y = 3$ *and every operator* T *from* X *into* Y *there is a norm-preserving extension of* T *from* Z *into* Y.

Proof. That (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) was shown in [10] and [12]. Actually it was shown there that under an additional hypothesis (the "metric approximation property") either of (ii) and (iii) is equivalent to (i) (cf. [13, properties (2), (8) and (9) of Theorem 1]). Thus the only new fact in the present theorem concerning the equivalence of the first three properties is that the requirement concerning the metric approximation property can be discarded. That (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv) is obvious. Hence we have to show only that (iv) $\Rightarrow$ (i). If dim $X \leqq 3$ we may take as T in (iv) the identity operator on X and hence in this case the fact that (iv) $\Rightarrow$ (i) is well known (cf. [15]). We assume therefore that dim $X > 3$.

By Lemma 3 it is sufficient to show that if in a Banach space X there are four mutually intersecting cells $\{S_X(x_i, r_i)\}_{i=1}^4$ and an $\varepsilon > 0$ such that

$$\bigcap_{i=1}^4 S_X(x_i, r_i + \varepsilon) = \emptyset, \tag{1}$$

then X does not satisfy (iv). Let a Banach space X, four mutually intersecting cells in it and an $\varepsilon > 0$ be given such that (1) holds. We apply Lemma 2 to the four sets $C_i = \mathrm{Int}(S_X(x_i, r_i + \varepsilon))$. Let V be a subspace of X having the properties stated in Lemma 2, let $Y = X/V$ and let T be the quotient map from X onto Y. By the choice of V, $\bigcap_{i=1}^4 TC_i = \emptyset$. Since $TC_i \supset \mathrm{Cl}(TS_X(x_i, r_i))$ it follows that

$$\bigcap_{i=1}^4 \mathrm{Cl}(TS_X(x_i, r_i)) = \emptyset. \tag{2}$$

Let now Z be a Banach space containing X with dim $Z/X = 1$ such that there is a $z \in Z$ with $\| z - x_i \| \leqq r_i$, $i = 1, \cdots, 4$ (cf. Lemma 1). Suppose T has an extension $\tilde{T}$ of norm 1 from Z into X. Then $\tilde{T}(z - x_i) \in r_i S_Y(0, 1)$ for every i. But since $S_Y(0, 1)$ is, by the definition of the quotient norm, equal to $\mathrm{Cl}(TS_X(0, 1))$ it would follow that for every i

$$\tilde{T}z \in Tx_i + r_i S_Y(0, 1) = \mathrm{Cl}(TS_X(x_i, r_i)),$$

and this contradicts (2).

This concludes the proof of Theorem 1.

In statement (iv) of Theorem 1 we cannot replace 3 by 2. The spaces having the weaker extension property obtained from (iv) if we replace 3 by 2 are characterized in

THEOREM 2. *Let* X *be a Banach space. The following two statements are equivalent.*

(i) *Every three mutually intersecting cells in* X *have a point in common.*

(ii) *For every two Banach spaces* Z *and* Y *with* $Z \supset X$, dim $Z/X = 1$, dim $Y = 2$ *and every operator* T *from* X *into* Y *there is a norm-preserving extension of* T *from* Z *into* Y.

Proof. That (ii) $\Rightarrow$ (i) follows as in the proof of (iv) $\Rightarrow$ (i) in Theorem 1. We have only to use the fact (proved in [11]) that if for every collection of three mutually intersecting cells $\{S(x_i, r_i)\}_{i=1}^3$ in X and every $\varepsilon > 0$, $\bigcap_{i=1}^3 S(x_i, r_i + \varepsilon) \neq \emptyset$ then X has property (i) of Theorem 2. We show now that (i) $\Rightarrow$ (ii). Let z be any point in Z but not in X and let T be an operator with norm 1 from X into Y. We have to choose $\tilde{T}z \in Y$ so that

$$\| \lambda \tilde{T}z - Tx \| \leqq \| \lambda z - x \|, \qquad x \in X, \lambda \text{ real.} \tag{3}$$

Since (3) holds for $\lambda = 0$ (and every choice of $\tilde{T}z$) we may divide (3) by $|\lambda|$ and hence it is sufficient to show that there is a $u \in Y$ such that $\| u - Tx \| \leqq \| z - x \|$ for every $x \in X$. In other words we have to show that

$$\bigcap_{x \in X} S_Y(Tx, \| z - x \|) \neq \emptyset.$$

(The preceding argument is due to Nachbin [15].) Clearly $S_Y(Tx, \| z - x \|)$ contains $\mathrm{Cl}(TS_X(x, \| z - x \|))$. By the triangle inequality in Z any two of the cells $S_X(x, \| z - x \|)$, $x \in X$, intersect. Since X satisfies (i) any three of these cells have a point in common. Hence any three of the compact convex sets $\mathrm{Cl}(TS_X(x, \| x - z \|))$ have a point in common. Since Y is 2-dimensional the desired result follows from Helly's theorem [7].

Hanner [6] gave an intrinsic characterization of the unit cells of finite-dimensional spaces X satisfying (i) of Theorem 2. Some results concerning infinite-dimensional spaces X satisfying (i) of Theorem 2 are proved in [11, chapter 2]. No functional representation is known (even in the finite-dimensional case) of the spaces having this intersection property. The most important spaces having the properties of Theorem 2 but not those of Theorem 1 are the L_1 spaces of dimension $\geqq 3$ (cf. [11] and Theorem 3 below; if X is an L_1 space then X^* is an L_1 space if and only if $\dim X \leqq 2$).

3. An extension theorem for operators defined on an L_1 space

In this section we prove that L_1 spaces have the extension property obtained from property (ii) in Theorem 2 by discarding the requirement $\dim Z/X = 1$. It seems likely that the same holds for every Banach space which has property (i) of Theorem 2. We shall need the following elementary lemma.

LEMMA 4. *Let K be a convex polygon in the plane, symmetric with respect to the origin, having $4n + 2$ vertices. Let $\{\pm z_i\}_{i=1}^{2n+1}$ be the vertices of K, the indices being chosen so that*

$$\arg z_1 \leqq \arg z_2 \leqq \cdots \leqq \arg z_{2n+1} \leqq \pi + \arg z_1,$$

for a suitable choice of the arguments. Put

$$y = z_1 - z_2 + z_3 - + \cdots - z_{2n} + z_{2n+1}.$$

Then $y \in K$.

Proof. Suppose we renumber the vertices by choosing a different vertex as the first one but still requiring that the vertices are numbered in the counterclockwise direction. Let $\tilde{y}$ denote the alternating sum of the vertices corresponding to this new choice of indices. It is easy to check that either $y = \tilde{y}$ or $y = -\tilde{y}$. Hence in order to prove the lemma we have only to show that if a linear functional f satisfies

$$f(z_1) = 1; \qquad |f(z)| \leqq 1, \qquad z \in K, \tag{4}$$

then $|f(y)| \leqq 1$. Let now f satisfy (4). Then

$$f(z_1) \geqq f(z_2) \geqq \cdots \geqq f(z_{2n+1}). \tag{5}$$

Indeed, if $i > j$ then $z_j = \alpha z_i + \beta z_1$ with $\alpha, \beta \geqq 0, \alpha + \beta \geqq 1$. Hence $f(z_j) = \alpha f(z_i) + \beta \geqq (\alpha + \beta) f(z_i)$. Thus $f(z_j) \geqq f(z_i)$ if at least one of these numbers is positive. If $f(z_j), f(z_i) \leqq 0$ we get that $f(z_j) \geqq f(z_i)$ from the relation $z_i = \gamma z_j + \delta(-z_1)$, $\gamma, \delta \geqq 0$, $\gamma + \delta \geqq 1$. This proves (5). Now

$$f(z_1) \geqq f(z_1) - f(z_2) + - \cdots + -f(z_{2n}) + f(z_{2n+1}) \geqq f(z_{2n+1});$$

thus $|f(y)| \leqq 1$ and this concludes the proof of the lemma.

THEOREM 3. *Let X be an L_1 space. For every two Banach spaces Z and Y with $Z \supset X$, $\dim Y = 2$ and every operator T from X into Y there is a norm-preserving extension of T from Z into Y.*

Proof. We assume that $\| T \| = 1$. We shall first prove the theorem for $X = l_1^n$. This will be done by induction on n. For $n = 1$ the assertion is clear. Denote the extreme points of the unit cell of l_1^n by $\{\pm e_i\}_{i=1}^n$. Suppose first that some Te_j is contained in the convex hull of $\{\pm Te_i\}_{i \neq j}$, say $Te_n = \sum_{i=1}^{n-1} \lambda_i Te_i$ with $\sum |\lambda_i| \leqq 1$. Let l_1^{n-1} be the subspace of l_1^n spanned by $\{e_i\}_{i=1}^{n-1}$ and let P be a projection of norm 1 from l_1^n onto l_1^{n-1} defined by $Pe_i = e_i$, $i < n$, and $Pe_n = \sum_{i=1}^{n-1} \lambda_i e_i$. Further let V be any $\mathcal{P}_1$ space containing l_1^{n-1}. The projection P from l_1^n onto l_1^{n-1} has a norm-preserving extension to an operator $\tilde{P}$ from Z into V (cf. [2, p. 94]). By the induction hypothesis the restriction of T to l_1^{n-1} has a norm-preserving extension T' from V into Y. Put $\tilde{T} = T'\tilde{P}$. $\tilde{T}$ is a norm-preserving extension of T from Z into Y.

Hence we may assume that no Te_j is in the convex hull of $\{\pm Te_i\}_{i \neq j}$. We assume also that

$$\arg Te_1 \leqq \arg Te_2 \leqq \cdots \leqq \arg Te_n \leqq \pi + \arg Te_1 ; \tag{6}$$

this can always be achieved by a suitable naming of the extreme points of l_1^n. Consider the space l_∞^m (where $m = 2^n$) and let $\{u_k\}_{k=1}^m$ denote the basis vectors of this space—all the coordinates of u_k are zero except the k^{th} which is one. It is well known and easy to see (see for example [4, p. 458]) that l_1^n can be embedded isometrically into l_∞^m by taking $e_i = \sum_{k=1}^m \varepsilon_{i,k} u_k$, $i = 1, \cdots, n$, with $\varepsilon_{i,k} = +1$ or -1 for every i and k and such that for every choice of signs $\{\varepsilon_i\}_{i=1}^n$ there is a k for which $\varepsilon_{i,k} = \varepsilon_i$ for every i. We assume that the u_k

are numbered so that $\varepsilon_{i,k} = +1$ for $i \geqq k$ and $\varepsilon_{i,k} = -1$ for $i < k \leqq n$ (the signs of $\varepsilon_{i,k}$ for $k > n$ are of no importance in the sequel). We now extend T to an operator $\tilde{T}$ from l_∞^m into Y. We take $\tilde{T}u_k = 0$ for $k > n$, hence in order that $\tilde{T}$ be an extension of T we must have

$$Te_i = \sum_{k=1}^n \varepsilon_{i,k} \tilde{T}u_k = \sum_{k=1}^i \tilde{T}u_k - \sum_{k=i+1}^n \tilde{T}u_k, \qquad i = 1, 2, \cdots, n,$$

that is

$$\tilde{T}u_1 = (Te_1 + Te_n)/2, \qquad \tilde{T}u_k = (Te_k - Te_{k-1})/2, \qquad k = 2, \cdots, n.$$

We have to show that the $\tilde{T}$ just defined has norm 1, that is that the images by $\tilde{T}$ of the extreme points of the unit cell of l_∞^m are in the unit cell of Y. In other words, we have to show that $\| \sum_{k=1}^n \pm \tilde{T}u_k \| \leqq 1$ for every choice of signs. Any sum $\sum_{k=1}^n \pm \tilde{T}u_k$ is equal to an expression of the form

$$\pm(Te_{i_1} - Te_{i_2} + - \cdots - Te_{i_{2j}} + Te_{i_{2j+1}})$$
$$\text{with } 1 \leqq i_1 < i_2 < \cdots < i_{2j+1} \leqq n.$$

Since $\| T \| = 1$ it follows from Lemma 4 that $\| \sum \pm \tilde{T}u_k \| \leqq 1$.

We show next that T has an extension of norm 1 from Z into Y. Since l_∞^m is a $\mathcal{P}_1$ space there is an operator T_0 of norm 1 from Z into l_∞^m whose restriction to l_1^n is the identity. $\tilde{T}T_0$ is a norm-preserving extension of T from Z into Y. This proves Theorem 3 for $X = l_1^n$.

Finally let X be a general $L_1(\mu)$ space. Denote the measure space on which μ is defined by Ω and consider the set Π of all partitions π of Ω into a finite number of disjoint measurable sets. We say that π_2 is finer than π_1 ($\pi_2 > \pi_1$) if each set in π_1 is a union of sets in π_2. In this ordering Π is a directed set. For each $\pi \in \Pi$ let X_π be the subspace of X spanned by those characteristic functions of the sets of π which are μ-integrable. From what we have already shown it follows that for every π there is an operator $\tilde{T}_\pi$ of norm $\leqq 1$ from Z into Y whose restriction to X_π is equal to the restriction of T to X_π. Let $\tilde{T}$ be any limiting point of the net $\tilde{T}_\pi$ in the weak operator topology (for operators from Z into Y). $\tilde{T}$ is a norm-preserving extension of T from Z into Y (cf. the proof of the lemma in [9]). This concludes the proof of Theorem 3.

COROLLARY 1. *Let the Banach space X contain a subspace isometric to l_1. Then every 2-dimensional Banach space is a quotient space of X.*

This follows immediately from Theorem 3 and the fact that every separable Banach space is a quotient space of l_1 [1].

COROLLARY 2. *Let B be a 2-dimensional Banach space. Then B is (isometric to) a subspace of $L_1(0, 1)$.*

Proof. By Corollary 1, B^* is a quotient space of $C(0, 1)$ and hence B is a subspace of $C(0, 1)^*$. But every separable subspace of $C(0, 1)^*$ is isometric to a subspace of $L_1(0, 1)$ [1].

As pointed out to me by Professor S. Kakutani it is not difficult to verify Corollary 2 directly by constructing a subspace of $L_1(0, 1)$ isometric to B for every given B. In Corollary 2 we cannot replace $L_1(0, 1)$ by l_1 since l_1 has, for example, no smooth subspace of dimension $\geqq 2$ (cf. the proof of Theorem 5).

COROLLARY 3. *If B is a 2-dimensional Banach space then $\mathcal{P}(B) = \mathcal{P}(B^*)$.*

This Corollary follows immediately from Corollary 2 and the following.

LEMMA 5. *Let the Banach space X be a subspace of an L_1 space Z. Then $\mathcal{P}(X) \geqq \mathcal{P}(X^*)$.*

Proof. Let $\lambda > \mathcal{P}(X)$ and let P be a projection from Z onto X with $\| P \| \leqq \lambda$. Let V be a Banach space containing X^*. Since Z^* is a $\mathcal{P}_1$ space the operator P^* from X^* into Z^* has a norm-preserving extension T_0 from V into Z^*. Let T_1 be the canonical quotient map from Z^* into X^* (that is the operator which assigns to each functional on Z its restriction to X). Then $\tilde{P} = T_1 T_0$ is a projection from V onto X^* with norm $\leqq \lambda$.

Corollaries 1, 2 and 3 do not hold if we consider Banach spaces of dimension greater than 2. $\mathcal{P}(l_1^3) > \mathcal{P}(l_\infty^3) = 1$ and hence no 3-dimensional space sufficiently close[2] to l_1^3 is a quotient space of a $C(K)$ space and no 3-dimensional space sufficiently close to l_∞^3 is a subspace of an L_1 space.

4. Comparison with the "into" extension property

We now point out the main differences between the "into" and "from" extension theorems.

1. In the "into" extension problem there is a significant difference between the existence of a norm-preserving extension and the existence of an "almost norm-preserving" extension. We have for example (cf. [3] and [13]) that X^* is an L_1 space if and only if for every two Banach spaces $Z \supset Y$, every compact T from Y into X *and every $\varepsilon > 0$* there is a compact extension $\tilde{T}$ of T from Z into X with $\| \tilde{T} \| \leqq (1 + \varepsilon)\| T \|$. The spaces X which have the preceding extension property with $\varepsilon = 0$ form a much smaller class (all finite-dimensional subspaces of X must have a polyhedron as unit cell). The situation is the same for related "into" extension properties, even for "into" extension properties for operators with a 2-dimensional range. For some results in this direction we refer to [12], [13], [14]. Nothing like this occurs with the "from" extension properties of Theorems 1 and 2. For example it is easily seen from the proof of (iv) $\Rightarrow$ (i) in Theorem 1 that if for a Banach space X there is, for every two Banach spaces Z and Y with $Z \supset X$, $\dim Z/X = 1$, $\dim Y = 3$, every operator T from X into Y and every $\varepsilon > 0$, an extension $\tilde{T}$ of T from Z into X with $\| \tilde{T} \| \leqq (1 + \varepsilon) \| T \|$ then $X^* = L_1$.

[2] By "sufficiently close" we mean that the norm $\| \; \|$ in the space satisfies $\| x \| \leqq ||| x ||| \leqq (1 + \varepsilon)\| x \|$ for every x where $||| \; |||$ is an l_1 norm and ε is a suitable positive number.

From this it follows that properties (ii) and (iii) of Theorem 1 are also equivalent to their "almost norm-preserving" versions.

2. The spaces whose conjugates are L_1 spaces do not have an "into" extension property for weakly compact operators similar to property (iii) of Theorem 1. For a discussion of this question we refer to [10].

3. We have shown here that the "from" extension property for operators with a 3-dimensional range implies the "from" extension property for compact operators but that the "from" extension property for operators with a 2-dimensional range is a strictly weaker property. In the corresponding "into" situation it seems likely that the 2-dimensional extension property already implies the extension property for compact operators. We have been able to show that this is the case if we restrict ourselves to spaces whose unit cells have at least one extreme point (cf. [14] for a precise formulation of this result).

We conclude this section with a problem concerning extension of operators from X into itself. It is proved in [12] that if X has the "metric approximation property" and if for every $Z \supset X$ with $\dim Z/X = 1$ and every operator T from X into itself with a finite-dimensional range there is a norm-preserving extension of T from Z into X then X^* is an L_1 space. We do not know whether this result still holds if we consider only operators T with a 3-dimensional range.

5. Lifting of operators

In this section we prove, using the results of the previous sections and of [14], some results concerning lifting of operators. These results complement those of Köthe and Grothendieck (cf. [3] and also Nachbin [16]). We first define the relevant notions. Let X, Y and Z be Banach spaces with Y being a quotient space of Z (with the quotient norm). We denote by φ the quotient map from Z onto Y (quotient maps will be denoted by φ throughout this section). Let T be an operator from X into Y. We say that T can be lifted in a norm-preserving manner to Z if there is an operator $\tilde{T}$ from X into Z satisfying $\| \tilde{T} \| = \| T \|$ and $\varphi\tilde{T} = T$. We say that T can be lifted in an almost norm-preserving manner if for every $\varepsilon > 0$ there is an operator $\tilde{T}$ from X into Z satisfying $\| \tilde{T} \| \leqq (1 + \varepsilon)\| T \|$ and $\varphi\tilde{T} = T$.

Our main theorem concerning lifting is the following. ((i) $\Leftrightarrow$ (iii) is due to Grothendieck [3].)

THEOREM 4. *Let X be a Banach space. The following statements are equivalent.*

(i) *X is an $L_1(\mu)$ space for some measure μ.*

(ii) *For every two Banach spaces Z and Y with Y being a quotient space of Z and* $\dim Y = 2$, $\dim Z = 3$, *every operator from X into Y can be lifted in a norm-preserving manner to Z.*

(iii) *For every two finite-dimensional Banach spaces Z and Y with Y being* *a quotient space of Z, every operator from X into Y can be lifted in a norm-preserving manner to Z.*

(iv) *For every Banach space Z having X as a quotient space and such that* $\dim \varphi^{-1}(0) = 1$ *(where φ is the quotient map), and for every 3-dimensional Y, every operator from Y into X can be lifted in an almost norm-preserving manner to Z.*

(v) *For every Banach space Z having X as a quotient space and every Banach space Y, every compact operator from Y into X can be lifted in an almost norm-preserving manner to Z.*

Proof. (iii) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (iv) are trivial. (i) $\Rightarrow$ (iii) was proved by Grothendieck [3] and thus we have only to show that (ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (v), and (iv) $\Rightarrow$ (i).

Proof of (ii) $\Rightarrow$ (i). Let X satisfy (ii). By passing to the dual it follows that X^* has the following extension property. For every two Banach spaces $Z \supset Y$ with $\dim Z = 3$, $\dim Y = 2$, every operator from Y into X^* has a norm-preserving extension from Z into X^*. From Theorem 1 of [14] we deduce that X^{**} is an L_1 space. Hence, X is an L_1 space (this is an easy and well-known consequence of the results in [3]).

Proof of (i) $\Rightarrow$ (v). Let X be $L_1(\mu)$. If μ is purely atomic it is known that X satisfies (v) (even for noncompact T, cf. Nachbin [16, p. 346]). In the general case let a compact T from Y into X and an $\varepsilon > 0$ be given. It follows easily from the compactness of T that there are a subspace B of X isometric to l_1^n for some finite n, and a projection P_1 of norm 1 from X onto B such that $\| P_1 T - T \| \leqq \varepsilon/4$. Since (v) is known to hold if we replace X by B it follows that there is an operator T' from Y into Z such that $\| P_1 T' \| \leqq \| T \| + \varepsilon/4$, $\varphi P_1 T' = P_1 T$. Put $T_1 = P_1 T$ and $\tilde{T}_1 = P_1 T'$. Next we find a projection P_2 of norm 1 from X onto a finite-dimensional subspace isometric to some l_1^n such that $\| T - T_1 - T_2 \| \leqq \varepsilon \cdot 4^{-2}$, where $T_2 = P_2(T - T_1)$. Repeating the same argument we get a sequence of operators $\{T_n\}_{n=1}^{\infty}$ from Y into X and a sequence of operators $\{\tilde{T}_n\}_{n=1}^{\infty}$ from Y into Z such that $\sum_n T_n = T$, $\varphi \tilde{T}_n = T_n$ and $\| \tilde{T}_n \| \leqq \| T_n \| + \varepsilon \cdot 4^{-n}$ for every n, and $\| T_n \| \leqq \varepsilon \cdot 4^{-n+1}$ for $n \geqq 2$. The operator $\tilde{T} = \sum_n \tilde{T}_n$ satisfies $\varphi \tilde{T} = T$ and $\| \tilde{T} \| \leqq \| T \| + \varepsilon$. It is clear that in our construction we get a compact $\tilde{T}$.

Proof of (iv) $\Rightarrow$ (i). Let X satisfy (iv). Passing to the dual we obtain that X^* has the following extension property. Let U be any Banach space having X as a quotient space such that $\dim U^*/X^* = 1$. Then for every 3-dimensional Y, for every $\varepsilon > 0$, and for every operator T from X^* into Y which is a transpose of an operator from Y^* into X there is an extension $\tilde{T}$ of T from U^* into Y with $\| \tilde{T} \| \leqq (1 + \varepsilon)\| T \|$. Thus X^* has a property which is similar to but (formally) weaker than property (iv) of Theorem 1. We shall now point out the modification needed in the proof of (iv) $\Rightarrow$ (i) of Theorem 1 in order to show that also here (iv) $\Rightarrow$ (i).

a. We have to show that in Lemma 1 if X is a conjugate space and if the set A is finite (we actually need only the case in which A consists of 4 elements) then the space Z can be chosen to be a conjugate space such that on X the w^* topology induced from Z coincides with the given w^* topology. The simplest way to do this is to use Grünbaum's proof of the Lemma [5, proof of Theorem 1]. In this proof the unit cell S_Z of Z is chosen to be the convex hull of S_X and a finite set of points (Z as a vector space is, of course, taken to be the direct sum of X and a 1-dimensional space, and S_Z is then chosen so that, among other requirements, $S_Z \cap X = S_X$). It follows that S_Z is compact if we take in Z the product of the given w^* topology on X and the usual topology of the line. Hence this Z has the required properties.

b. The fact that we assume the existence of an almost norm-preserving extension rather than a norm-preserving one does not cause any difficulty—see paragraph 1 of Section 4.

c. The fact that, for conjugate X, it is sufficient in (iv) of Theorem 1 to consider only transposed operators is a consequence of the following version of Lemma 2 (applied to X with its w^* topology).

LEMMA 6. *Let X be a locally convex Hausdorff linear topological space of dimension $\geqq n$. Let $\{S_i\}_{i=1}^{n+1}$ be a collection of closed convex sets in X at least one of which is compact, such that $\bigcap_{i=1}^{n+1} S_i = \emptyset$. Then there is a closed subspace V of X with* $\dim X/V = n$ *such that neither V nor any translate of it intersects all the sets S_i.*

Proof. We assume that S_1 is compact. For every i we can represent S_i as $\bigcap_{\alpha \in A_i} Q_{i,\alpha}$ where the $Q_{i,\alpha}$ are closed half spaces (that is sets of the form $\{x; f(x) \leqq \lambda\}$ for some $f \in X^*$ and real λ), containing S_i in their interior. Hence

$$\bigcap_{i=2}^{n+1} \bigcap_{\alpha \in A_i} Q_{i,\alpha} \cap S_1 = \emptyset.$$

By the compactness of S_1 it follows that for every $i \geqq 2$ there is finite set (which we may clearly assume to be nonempty) $B_i \subset A_i$ such that

$$\bigcap_{i=2}^{n+1} \bigcap_{\alpha \in B_i} Q_{i,\alpha} \cap S_1 = \emptyset.$$

It follows that there is a closed half space Q_1 containing S_1 in its interior such that

$$\bigcap_{i=2}^{n+1} \bigcap_{\alpha \in B_i} Q_{i,\alpha} \cap Q_1 = \emptyset.$$

Put $C_1 = \mathrm{Int}(Q_1)$ and $C_i = \bigcap_{\alpha \in B_i} \mathrm{Int}(Q_{\alpha,i})$ for $i \geqq 2$. The C_i are open sets and we have $C_i \supset S_i$ for every i and $\bigcap_i C_i = \emptyset$. To these C_i we can apply without any change the arguments in (3.1) and (3.2) of [8]. This concludes the proof of the lemma and hence also of Theorem 4.

Remark 1. The lifting property obtained from (iv) by replacing the requirement $\dim Y = 3$ by $\dim Y = 2$ is strictly weaker than (iv). For example it follows from Theorem 3 that every l_∞^n satisfies this weaker version of (iv).

Remark 2. In properties (iv) and (v) we cannot in general assert that T has a norm-preserving lifting. This is obvious from the following trivial example. Let X be 1-dimensional and Y and Z any Banach spaces. For every $f \in Z^*$ with $\| f \| = 1$ we may consider the mapping φ from Z into X defined by $\varphi(z) = f(z)$ as a quotient map. Let $g \in Y^*$ be a nonzero functional which attains its supremum on S_Y. The operator $Ty = g(y)$ from Y into X can be lifted in a norm-preserving manner to Z if and only if f attains its supremum on S_Z, that is if $\varphi S_Z = S_X$. Thus if $\varphi S_Z \neq S_X$ there does not exist a norm-preserving lifting even in the most simple situation. The question whether there exists a norm-preserving lifting if we assume that $\varphi S_Z = S_X$ is treated in

THEOREM 5. *Let X be a Banach space. The following statements are equivalent.*

(i) *X is an $L_1(\mu)$ space with μ purely atomic.*

(ii) *For every Banach space Z having X as a quotient space such that $\varphi S_Z = S_X$, and every 3-dimensional Y, every operator from Y into X has a norm-preserving lifting to Z.*

(iii) *The same as* (ii) *but without any restriction on* dim Y.

Proof. (i) $\Rightarrow$ (iii). Let $X = l_1(I)$ for a set I and for each $i \in I$ let e_i be the characteristic function of this point. Let $\{z_i\}_{i \in I}$ satisfy $\varphi z_i = e_i$ and $\| z_i \| = 1$ for every i. Let $y \in Y$ and let $Ty = \sum \lambda_i e_i$ (in this sum at most a countable number of terms are different from 0). Define $\tilde{T}y = \sum \lambda_i z_i$. In this way we get a norm-preserving lifting of T (this proof is a slight modification of a proof due to Köthe of a related result, cf. [3]). (iii) $\Rightarrow$ (ii) is clear.

Proof of (ii) $\Rightarrow$ (i). Let X satisfy (ii). By Theorem 4, X is an $L_1(\mu)$ space. If μ is not purely atomic X contains a subspace isometric to $L_1(0, 1)$ and hence by Corollary 2 of Theorem 3, X has a subspace B isometric to the 2-dimensional inner product space. Let Y be a 3-dimensional subspace of X containing B, and let T be the identity map from Y into X. Let Z be $l_1(S_X)$ (S_X plays here the role of an index set). The map φ from Z onto X defined by $\varphi(\sum \lambda_i e_{x_i}) = \sum \lambda_i x_i$, $\| x_i \| \leqq 1$, $\sum | \lambda_i | < \infty$ is a quotient map and satisfies $\varphi S_Z = S_X$. T has no norm-preserving lifting to Z since Z does not contain a subspace isometric to B. That l_1 does not contain any smooth subspace (and in particular no subspace isometric to B) follows from the fact that every nonconstant function of the form $f(\lambda) = \sum_{i=1}^{\infty} | a_i + \lambda b_i |$ is nondifferentiable for some λ (this was pointed out to me by Professor S. Kakutani).

In this connection it should be remarked, perhaps, that l_1 has strictly convex subspaces. This follows from the fact that if the set $\{a_i/b_i\}_{i=1}^{\infty}$ is dense in the line then $f(\lambda)$ is not linear on any interval.

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