Z-A GROUPS WHICH SATISFY THE mth ENGEL CONDITION

BY

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I. Introduction

Suppose that A and B are subgroups of a group G. If there exists a positive integer m such that the commutator

$$(\cdots((a, \underbrace{b), \cdots), b}_{m}) = 1$$

for all a in A and b in B, then we write $A\ |e{:}m|\ B$. A group G which satisfies $G\ |e{:}m|\ G$ is said to satisfy the m^{th} Engel condition.

The problem of determining for what groups the m^{th} Engel condition implies nilpotence has been studied by several authors. For example, K. Gruenberg in [2] has shown that finitely generated soluble groups which satisfy the m^{th} Engel condition are nilpotent. R. Baer in [1] adds groups which satisfy the maximal condition to the list. In [3] Gruenberg includes the torsion-free soluble groups

This paper grew out of an investigation of the commutator structure of Z-A groups, that is groups in which G itself is a term of its upper central series. The class of a Z-A group is the smallest ordinal n such that $Z_n = G$ where Z_n denotes the n^{th} term in the upper central series of G. The investigation resulted in a curious classification of Z-A groups. This classification is based on a class of Z-A groups which it seemed natural to call Z-A(q) groups for integer q. We will show that Z-A(1) is equal to the above class of groups and that Z-A$(1) >$ Z-A$(2) >$ Z-A(3). The class of Z-A(3) groups proved to be interesting. For instance, an example of a metabelian Z-A(3) group is found which has exponent 4 and satisfies the 3^{rd} Engel condition, but is not nilpotent. However, every Z-A(3) group with prime exponent is automatically nilpotent. It may not be significant but no example of a Z-A(3) group has been found which is not of class $\omega + 1$ and where Z_ω is not abelian. The following pages will investigate under what conditions the Engel condition implies nilpotence for Z-A(3) groups.

We will recall some definitions and notations. If x and y are elements of a group, then denote the product $x^{-1}\cdot y^{-1}\cdot x\cdot y$ of a group by the commutator (x, y). We define commutators of higher order by the recursive rule $(x_1, \cdots, x_{n-1}, x_n) = ((x_1, \cdots, x_{n-1}), x_n)$. Define the weight $w(c)$ of the commutator c constructed from the elements $x_1, \cdots, x_n$ recursively by defining the weight of the elements $x_1, \cdots, x_n$ to be 1, and if $c = (c_i, c_j)$

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then $w(c) = w(c_i) + w(c_j)$ where both c_i and c_j are commutators in $x_1, \cdots, x_n$. For the sake of convenience, designate the commutator

$$(x, \underbrace{y, \cdots, y}_{k}) \quad \text{by} \quad (x, {}_k y).$$

If A and B are two subgroups of G then the subgroup generated by the commutators (a, b) where a is in A and b is in B will be designated by (A, B).

Suppose that G is a Z-A group of class n for some ordinal n. If for some positive integer q we have $(Z_{\alpha+q}, Z_\beta) \leqq Z_\alpha$ for all ordinals α, β with $\alpha + q, \beta < n$ then G will be called a Z-A(q) group.

Suppose G is a Z-A group of class n. Since for all ordinals α and β, $(Z_{\alpha+1}, Z_\beta) \leqq Z_\alpha$ we have that G is a Z-A(1) group. Obviously Z-A(q) $\geqq$ Z-A($q + 1$).

There are examples of nilpotent groups of class 3 which have a nonabelian upper central term Z_2. For instance consider the group of 2 by 2 integral matrices with components reduced modulo 4 of the form $I + P + 2Q$ where I is the identity, P is an integral matrix with zeros in every row except the last and in the main diagonal and Q is an integral matrix. Hence Z-A(1) > Z-A(2).

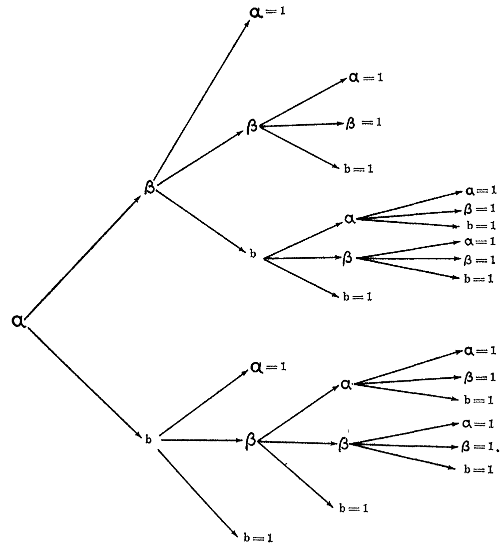
The following example presents a Z-A(2) group G which is not a Z-A(3) group. The example G will be a semidirect product of an abelian group A by a nilpotent group N. Let A be a torsion-free abelian group generated by the elements a_1, a_2, a_3 and b.

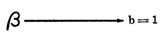
We define the following automorphisms on A.

$A \to A^\alpha$	$A \to A^\beta$
$a_1^\alpha = a_1$,	$a_1^\beta = a_1$,
$a_2^\alpha = a_2 \cdot a_1$,	$a_2^\beta = a_2 \cdot a_1$,
$a_3^\alpha = a_3$,	$a_3^\beta = a_3 \cdot a_2$,
$b^\alpha = b \cdot a_3$,	$b^\beta = b$.

Let N be the automorphism group generated by α and β. Since G is the semidirect product of A by N then A is a normal subgroup of G and N is a subgroup of G whose elements are the coset representatives of G/A. From the definitions of α and β we have $(b, \alpha) = a_3$, $(a_3, \beta) = a_2$ and $(a_2, \beta) = a_1$. Consequently G is generated by the elements α, β and b.

It will be convenient to represent the commutator (x, y) by $x \to y$ in order to diagram the commutators in the elements α, β and b. Of course we mean $x \to y \to z$ to be $(x \to y) \to z$. For $x \neq 1$ and $y \neq 1$, if $(x, y) = 1$ we write $x \to y = 1$. The accompanying diagrams will show the values of all of the commutators in the elements α, β and b.





The following tables of automorphisms will be included in order to verify these diagrams.

$$
\begin{array}{llllll}
a_1^{(\alpha,\beta)} & = a_1 , & a_1^{(\alpha,\beta,\beta)} & = a_1 , & a_1^{(\beta,\alpha,\alpha)} & = a_1 , \\
a_2^{(\alpha,\beta)} & = a_2 , & a_2^{(\alpha,\beta,\beta)} & = a_2 , & a_2^{(\beta,\alpha,\alpha)} & = a_2 , \\
a_3^{(\alpha,\beta)} & = a_3 \cdot a_1 , & a_3^{(\alpha,\beta,\beta)} & = a_3 , & a_3^{(\beta,\alpha,\alpha)} & = a_3 , \\
b^{(\alpha,\beta)} & = b \cdot a_2 a_1 , & b^{'\alpha,\beta,\beta)} & = b \cdot a_1 , & b^{(\beta,\alpha,\alpha)} & = b .
\end{array}
$$

The terms of the lower central series of G are generated from the commutators of its generators. Hence the diagrams show that G is nilpotent of class 4. If B is a group, we designate the r^{th} term of the lower central series by B_r. By using P. Hall's collection process [4, pp. 165–168] we can express every element x of G by the product $\alpha^p \cdot \beta^q \cdot b^r \cdot (\alpha, \beta)^s \cdot (\alpha, b)^t \cdot z$ where z is in G_2.

In the calculations that follow we will make repeated use of the commutator identity, which appears in [4, Theorem 10.2.12, p. 150]:

$$
(1) \qquad (x \cdot y, z) = (x, z) \cdot (x, z, y) \cdot (y, z).
$$

Therefore, if x and z commute, we have $(x \cdot y, z) = (y, z)$.

If H designates the group generated by elements x and (x, a), then for any integer n by [4, Theorem 12.49, p. 185] we have

$$
(2) \qquad (x^n, a) \equiv (x, a)^n \mod H_1 .
$$

The diagrams show that α, β and b are not in Z_3. Suppose that for some p, q and r, $\alpha^p \cdot \beta^q \cdot b^r$ is in Z_3. Then $(\alpha^p \cdot \beta^q \cdot b^r, \alpha) \equiv 1 \bmod Z_2$. But from (1) and (2) we have

$$\begin{aligned}(\alpha^p \cdot \beta^q \cdot b^r, \alpha) &= (\beta^q \cdot b^r, \alpha)\\ &\equiv (\beta^q, \alpha)\cdot(b^r, \alpha) \mod Z_2\\ &\equiv (\beta, \alpha)^q \cdot (b, \alpha)^r \mod Z_2 .\end{aligned}$$

Consequently we must have that $(\beta, \alpha)^q \cdot (b, \alpha)^r$ is in Z_2. Therefore by (1)

$$\begin{aligned}((\beta, \alpha)^q \cdot (b, \alpha)^r, b) &= ((\beta, \alpha)^q, b)\cdot(((\beta, \alpha)^q, b, (b, \alpha)^r)\cdot((b, \alpha)^r, b)\\ &\equiv ((\beta, \alpha)^q, b) \mod Z_1\\ &\equiv (\beta, \alpha, b)^q \mod Z_1 .\end{aligned}$$

But from the tables we have

$$(\beta, \alpha, b)^q = [b^{-1(\beta,\alpha)} \cdot b]^q = a_2^q \cdot a_1^q .$$

Since $(a_2^q \cdot a_1^q, \alpha) = (a_2^q, \alpha) = a_1^q$, we have that $a_2^q \cdot a_1^q$ is not in Z_1 unless $q = 0$. If $\alpha^p \cdot b^r \equiv 1 \bmod Z_3$, then by (2)

$$\begin{aligned}(\alpha^p \cdot b^r, b) = (\alpha^p, b) &= (\alpha, b)^p = a_3^{-p}\\ &\equiv 1 \mod Z_2 .\end{aligned}$$

But $(a_3^{-p}, \beta, \beta) = a_1^{-p} \neq 1$. Thus $p = 0$ if $\alpha^p \cdot b^r \equiv 1 \bmod Z_3$. Now $(b^r, \alpha) = a_3^r$ is not in Z_2. If an element x is in Z_3 it must be represented by the product $(\alpha, \beta)^s \cdot (\alpha, \beta)^t \cdot z$ where z is in G_2 since $\alpha^p \cdot \beta^q \cdot b^r$ is not in Z_3 unless $p = q = r = 0$. Suppose the product $(\alpha, \beta)^s \cdot (\alpha, \beta)^t \equiv 1 \bmod Z_2$. Then by (1) and (2) we have

$$\begin{aligned}((\alpha, \beta)^s \cdot (\alpha, b)^t, b) = ((\alpha, \beta)^s, b) &= (\alpha, \beta, b)^s\\ &= a_2^{-s} \cdot a_1^{-s}\\ &\equiv 1 \mod Z_1 .\end{aligned}$$

Thus $s = 0$. Since $(\alpha, b)^t = a_3^{-t}$ the commutator $(\alpha, b)^t$ is not in Z_2 unless $t = 0$. Consequently $(\alpha, \beta)^s \cdot (\alpha, b)^t \equiv 1 \bmod Z_2$ only if $s = t = 0$.

Since every element x of G can be expressed in the form

$$\alpha^p \cdot \beta^q \cdot b^r \cdot (\alpha, \beta)^s \cdot (\alpha, b)^t \cdot z$$

where $z \in G_2$, then $x \equiv 1 \bmod Z_2$ only if $p = q = r = s = t = 0$. Hence Z_2 is in G_2. Also x is in Z_3 only if $p = q = r = 0$ and hence Z_3 is in G_1. Since $G_4 = 1$ we have that $(G_1, G_2) \leqq G_4 = 1$. Therefore $(Z_3, Z_2) = 1$. We also have that $(Z_3, Z_3) \leqq (G_1, G_1) \leqq G_3 \leqq Z_1$. Therefore G is a

Z-A (2) group. But (α, β) and (α, b) are in Z_3 and

$$((\alpha, \beta), (\alpha, b)) = ((\alpha, \beta), a_3^{-1}) = a_1^{-1} \neq 1.$$

Therefore $(Z_3, Z_3) \neq 1$. Hence G is not a Z-A(3) group.

Since this paper will be primarily concerned with determining the nilpotent groups from among the Z-A(3) groups, we will next present an example of a metabelian Z-A(3) group which satisfies the 3rd Engel condition and has exponent 4 but is not nilpotent.

Suppose A^* is the direct sum of a countable number of copies of the cyclic group C of order two. Designate the α^{th} summand by C_α where C_α is generated by a_α. Let A be the subgroup of A^* consisting of the direct sum of the summands C_α where for no prime p does p^2 divide α. Now for each prime p define the automorphism λ_p on A by the following equations. Suppose a_α is in A. Then if the prime p divides α we define $a_\alpha^{\lambda_p} = a_\alpha + a_{\alpha/p}$, and if p does not divide α, $a_\alpha^{\lambda_p} = a_\alpha$. If the prime p divides α where a_α is in A then $a_\alpha^{\lambda_p^2} = a^{\lambda_p} + a_{\alpha/p}^{\lambda_p} = a_\alpha$. Therefore $\lambda_p^2 = 1$ for every prime p. Suppose the primes p and p' both divide α where a_α is in A. Then

$$a_\alpha^{\lambda_p \lambda_{p'}} = a^{\lambda_{p'} \lambda_p} = a_{\alpha/p} + a_{\alpha/p'} + a_{\alpha/pp'} .$$

Obviously if only one or none of the primes divides α, the corresponding automorphisms still commute. Let B designate the abelian group generated by the automorphisms λ_p. We define H to be the semidirect product of A by B. Then A is a normal subgroup of H, H/A is isomorphic to B and H is the union of A and B. The following are some properties of H.

(a) $A\ |e{:}2|\ B$.

Let the symbol $\prod$ designate a finite product. So if $b = \prod_{i=1}^{k} \lambda_{p_i}$ then for $q = \prod_{i=1}^{k} [\lambda_{p_i} - 1]^2$ and a in A we have $(a, b, b) = a^q = 1$.

(b) $(A, A) = (B, B) = 1$.

Both A and B have been shown to be abelian.

(c) $(B, A, A) = 1$.

The subgroup A is normal in H. Hence (c) follows from (b).

(d) $A\ |e{:}2|\ H$.

In [4, Theorem 11.1–6, p. 167] we find the commutator identity,

$$(x, y \cdot z) = (x, z) \cdot (x, y) \cdot (x, y, z). \tag{3}$$

Thus (d) follows from (3), (a) and (c).

(e) H is metabelian (i.e. the second term of the derived series of H is 1).

Since (H, H) is in A, (e) follows from (b).

(f) $(h_1, h_2, h_3, h_4) = (h_1, h_2, h_4, h_3)$ for all h_1, h_2, h_3 and h_4 in H.

For H is metabelian.

(g) $(b, a \cdot b', a \cdot b') = (b, a, b')$ for all b, b' in B and a in A.

$(b, a \cdot b', a \cdot b') = (b, a \cdot b', b')$ as $(b, a \cdot b')$ and a commute, both being contained in A;

$$\begin{aligned}(b, a \cdot b', b') &= ((b, a) \cdot (b, a, b'), b') = (b, a, b') \cdot (b, a, b', b') \\ &= (b, a, b').\end{aligned}$$

(h) $B \,|e{:}3|\, H$.

Suppose that b is in B and $a \cdot b'$ is in H. By (g) we have

$$(b, a \cdot b', a \cdot b') = (b, a, b').$$

Therefore by (3), (a) and (b)

$$\begin{aligned}(b, a \cdot b', a \cdot b', a \cdot b') &= (b, a, b', b') \cdot (b, a, b', a) \cdot (b, a, b', a, b') \\ &= 1.\end{aligned}$$

(i) $(a, a_1 \cdot b_1, \cdots, a_n \cdot b_n) = (a, b_1, \cdots, b_n)$ for all $a, a_1, \cdots, a_n$.

If $n = 1$, $(a, a_1 \cdot b_1) = (a, b_1)$ as a and a_1 commute. Assume (i) is true for $n = k$. By the induction hypothesis

$$(a, a_1 \cdot b_1, a_2 \cdot b_2, \cdots, b_k \cdot a_k, a_{k+1} \cdot b_{k+1}) = (a, b_1, \cdots, b_k, a_{k+1} \cdot b_{k+1}).$$

Now $(a, b_1, \cdots, b_k)$ and a_{k+1} commute as elements of A; therefore

$$(a, b_1, \cdots, b_k, a_{k+1} \cdot b_{k+1}) = (a, b_1, \cdots, b_k, b_{k+1}).$$

For any number n let $\alpha = \prod_{i=1}^{n} p_i$ where $p_i \neq p_j$ for $i \neq j$. Suppose $b_i = \lambda_{p_i}$, $i = 1, \cdots, n - 1$. Then if $t = \prod_{i=1}^{n-1} [\lambda_{p_i} - 1]$ and $s = \prod_{i=1}^{n} p_i$,

$$(a, b_1, \cdots, b_{n-1}) = a_s^t = a_{p_n} \neq 1.$$

Therefore H is not nilpotent.

Suppose that λ_p is in B and a_α is in A, where α is the product of at most n primes. If the prime p does not divide α then $(a_\alpha, \lambda_p) = 1$. If p divides α then $(a_\alpha, \lambda_p) = a_{\alpha/p}$. Therefore $(a_\alpha, \lambda_{p_1}, \cdots, \lambda_{p_m}) = 1$ for all primes $p_1, \cdots, p_m$, $m > n$.

Suppose $a_i \in A$ and $b_i \in B$ for $i = 1, \cdots, m$. Then by (i) we have

$$(a_\alpha, a_1 \cdot b_1, \cdots, a_m \cdot b_m) = (a_\alpha, b_1, \cdots, b_m).$$

Since each b_i is the product of elements λ_p, by (1), (3) and the following identity from [5, 1.1, p. 107]

$$(x \cdot y, z) = (x, z) \cdot (x, z, y) \cdot (y, z), \tag{4}$$

we can expand $(a_\alpha, b_1, \cdots, b_m)$ into factors of the form $(a_\alpha, \lambda_{p_1}, \cdots, \lambda_{p_r})$, $r \geqq m > n$. Therefore $(a_\alpha, a_1 \cdot b_1, \cdots, a_m \cdot b_m) = 1$ and $a_\alpha \in Z_m$. Hence $A \leqq Z_\omega$.

Given λ_{p_0} and primes $p_1, \cdots, p_{n+1}$ where $p_i \neq p_0$ for $i \neq 0$ if $r = \prod_{i=0}^{n+1} p_i$ and $a_r \in A$, we have

$$(\lambda_{p_0}, a_r, \lambda_{p_1}, \cdots, \lambda_{p_n}) = -a_{p_{n+1}} \neq 1.$$

Therefore $\lambda_{p_0} \notin Z_\omega$ for every prime p_0 and hence B is not in Z_ω. Thus $A = Z_\omega$ since $H = A \cdot B$. Since $H/Z_\omega = B$ we have that $H = Z_{\omega+1}$.

Consider any two elements $a \cdot b$ and $a' \cdot b'$ of H where $a, a' \in A$ and $b, b' \in B$. By (i), (e), and (a)

$$(a \cdot b, a' \cdot b', a' \cdot b', a' \cdot b') = (a \cdot b, a' \cdot b', b', b') = 1.$$

Thus $H \,|e{:}3|\, H$.

Since $Z_\omega = A$ we have that $(Z_\alpha, Z_\omega) = 1$ for $\alpha = 1, 2, \cdots$ by (b). Therefore H is a Z-A(3) group. If $a \cdot b$ is any element of H where a is in A and b is in B, then since $A^2 = B^2 = 1$ we have

$$[a \cdot b]^2 = a \cdot b \cdot a \cdot b = a \cdot b^2 \cdot a \cdot (a, b) = (a, b).$$

Since A is normal in H, $(a, b) \in A$. Therefore $H^4 = 1$ since $A^2 = 1$.

II. The derived module and ring of a Z-A(2) group

The verification that Z-A(2) groups cannot be of class equal to a limit ordinal is trivial and therefore omitted. We will assume throughout the following discussion that G is a Z-A(2) group of class $n + 1$. We define the derived module M of G to be the direct sum of the abelian groups $Z_{\alpha+1}/Z_\alpha$ for $0 \leqq \alpha < n$. The elements of $Z_{\alpha+1}/Z_\alpha$ will be called homogeneous of degree $\alpha + 1$.

If $x \in G$ then there exists only one quotient group $Z_{\alpha+1}/Z_\alpha$ in which x represents a nonunit coset. Designate the coset by $\bar{x}$. If $\bar{x}$ and $\bar{y}$ are both homogeneous of degree $\alpha + 1$ then the sum of $\bar{x}$ and $\bar{y}$ in M is their quotient group product.

Suppose that $\bar{t} \in Z_{n+1}/Z_n$ and $\bar{x} \in Z_{\alpha+1}/Z$ for $\alpha < n$. If α is not a limit ordinal, define $\bar{x}\bar{t}$ to be the coset in $Z_\alpha/Z_{\alpha-1}$, which is represented by the commutator (x, t). Otherwise $\bar{x}\bar{t} = 0$. The operation $\bar{x}\bar{t}$ is well defined. Suppose that y is in Z_α and z is in Z_n. Then $(x \cdot y, z)$ is in $Z_{\alpha-1}$ since $(Z_{\alpha+1}, Z_n)$ is in $Z_{\alpha-1}$. On expanding commutators we also find that

$$(x \cdot y, t) \equiv (x, t) \bmod Z_{\alpha-1} \quad \text{and} \quad (x \cdot y, t, z) \equiv 1 \bmod Z_{\alpha-1}.$$

Consequently $(x \cdot y, t \cdot z) \equiv (x, t) \bmod Z_{\alpha-1}$.

Suppose that $\bar{x}$ and $\bar{y}$ are homogeneous of degree $\alpha + 1$ where $1 \leqq \alpha + 1 \leqq n$ and $\bar{t}$ is homogeneous of degree $n + 1$. Since

$$(x \cdot y, t) = (x, t) \cdot (y, t) \mod Z_{\alpha-1}$$

then $\bar{t}$ represents a homomorphism of $Z_{\alpha+1}/Z_\alpha$ into $Z_\alpha/Z_{\alpha-1}$. We extend the domain of $\bar{t}$ to M by linearity so that $\bar{t}$ is an endomorphism of M. The derived ring Γ over M is the endomorphism ring generated by elements of Z_{n+1}/Z_n. Since

$$\bar{x}(\bar{t}_1 + \bar{t}_2) = (\overline{x, t_1}) + (\overline{x_1\, t_2}) = (\overline{x_1\, t_1 \cdot t_2})$$

then endomorphism addition of elements from Z_{n+1}/Z_n coincides with the quotient group multiplication. Γ is of course an associative ring, since endomorphism multiplication is associative.

The important connection between a Z-A(2) group and its derived ring is stated in the following theorem.

THEOREM 1. *If G is a Z-A(2) group of class $n + 1$ and if the derived ring Γ is nilpotent of class k then $k = n + 1$.*

We state first the following lemma.

LEMMA 1. *If G is a Z-A(2) group of class $n + 1$ and if $\bar{x}\bar{t}_1 \cdots \bar{t}_k = 0$ for x in Z_{k+1} and all $\bar{t}_1, \cdots, \bar{t}_k$ in Z_{n+1}/Z_n, then x is in Z_k.*

If x is not in Z_k then $\bar{x}$ is homogeneous of degree $k + 1$. Thus $\bar{x}\bar{t}_1 \cdots \bar{t}_k = 0$ implies that for all homogeneous elements $\bar{t}_1, \cdots, \bar{t}_k$ of degree $n + 1$, the commutator $(x, t_1, \cdots, t_k)$ is the unit of $Z_1/Z_0 = 1$. But since G is a Z-A(2) group we have $(x, Z_{\alpha_1}, \cdots, Z_{\alpha_k}) = 1$ if $Z_{\alpha_j} \leqq Z_n$ for some $j = 1, \cdots, k$. Therefore

$$(x, \underbrace{G, \cdots, G}_{k}) = 1,$$

and x is in Z_k.

If Γ is nilpotent of class k, then for x in Z_{k+1} we must have

$$\bar{x}\bar{t}_1 \cdots \bar{t}_k = 0$$

for all $\bar{t}_1, \cdots, \bar{t}_k$ in Z_{n+1}/Z_n. Thus by the lemma Z_{k+1} is Z_k and hence $G = Z_k$. Since Γ is nilpotent of class k there must be an element x in Z_k and elements $\bar{t}_1, \cdots, \bar{t}_{k-1}$ such that $\bar{x}\bar{t}_1 \cdots \bar{t}_{k-1} \neq 0$. Hence $(x, t_1, \cdots, t_{k-1}) \neq 1$ and G is nilpotent of class k.

Of course if G is nilpotent of class k then it is a trivial matter to show that Γ is nilpotent of class k.

The following arguments will show that the derived ring of a Z-A(3) group is commutative. We will demonstrate later that this is an important property of Z-A(3) groups.

THEOREM 2. *Suppose G is a Z-A(2) group of class $n + 1$. If $\bar{x}$ is in $Z_{\alpha+1}/Z_\alpha$ for $\alpha < n$ and both $\bar{t}_1$ and $\bar{t}_2$ are in Z_{n+1}/Z_n, then $\bar{x}\bar{t}_1\,\bar{t}_2 = \bar{x}\bar{t}_2\,\bar{t}_1 + \bar{q}$ where $\bar{q}$ is the coset in $Z_{\alpha-1}/Z_{\alpha-2}$ which is represented by the commutator $(x, (t_1, t_2))$.*

LEMMA 2. *If G is a Z-A group and if x is in $Z_{\alpha+1}$, then for all g_1 and g_2 in G we have*

$$(x, g_1, g_2) \equiv (x, g_2, g_1) \cdot (g_1, g_2, x)^{-1} \mod Z_{\alpha-2}.$$

From [5, p. 108], [4, Theorem 1.1, p. 107], and [4, Theorem 11.1–6, p. 167], the commutator identities follow respectively.

$$(5) \qquad (x, y, z^y) \cdot (y, z, x^z) \cdot (z, z, y^x) = 1.$$

$$(6) \qquad (x, y^{-1}) = (x, y, y^{-1})^{-1} \cdot (x, y)^{-1}.$$

Therefore by (5), (3) and (6) we have

$$(7) \qquad (x, g_1, g_2^{g_1}) \cdot (g_1, g_2, x^{g_2}) \cdot (g_2, x, g_1^x) = 1,$$

$$(8) \qquad (x, g_1, g_2^{g_1}) = (x, g_1, g_2 \cdot (g_2, g_1)) \equiv (x, g_1, g_2) \mod Z_{\alpha-2},$$

$$(9) \qquad (g_2, x, g_1^x) = (g_2, x, g_1 \cdot (g_1, x)) \equiv (g_2, x, g_1) \mod Z_{\alpha-2},$$

$$(10) \qquad (g_2, x, g_1) = ((x, g_2)^{-1}, g_1) \equiv (x, g_2, g_1)^{-1} \mod Z_{\alpha-2}.$$

Then by (9) and (10)

$$(11) \qquad (g_2, x, g_1^x) \equiv (x, g_2, g_1)^{-1} \mod Z_{\alpha-2}.$$

It follows from (3) that

$$(12) \qquad \begin{aligned} (g_1, g_2, x^{g_2}) &= (g_1, g_2, x \cdot (x, g_2)) \\ &\equiv (g_1, g_2, (x, g_2)) \cdot (g_1, g_2, x) \mod Z_{\alpha-2}. \end{aligned}$$

But

$$(g_1, g_2, (x, g_2)) \equiv 1 \mod Z_{\alpha-2}.$$

Therefore by (12)

$$(13) \qquad (g_1, g_2, x^{g_2}) \equiv (g_1, g_2, x) \mod Z_{\alpha-2}.$$

The lemma follows from (7), (8), (9) and (13).

If $\bar{x}$ is in $Z_{\alpha+1}/Z_\alpha$ and $\bar{g}_1$ and $\bar{g}_2$ are in Z_{n+1}/Z_n, Theorem 2 follows from the lemma.

Theorem 2 shows that Γ is commutative on $Z_{\alpha+1}/Z_\alpha$ if and only if $(x, (t_1, t_2)) \equiv 1 \mod Z_{\alpha-2}$ for all elements x, t_1, and t_2 such that $\bar{x}$ is in $Z_{\alpha+1}/Z_\alpha$ and both $\bar{t}_1$ and $\bar{t}_2$ are in Z_{n+1}/Z_n. If G is a Z-A(3) group of class $n + 1$, then $(Z_{\alpha+1}, Z_n)$ is in $Z_{\alpha-2}$ for every $\alpha < n$. Thus if $\bar{x}$ is in $Z_{\alpha+1}/Z_\alpha$ and both $\bar{t}_1$ and $\bar{t}_2$ are in Z_{n+1}/Z_n, it follows that $(x, (t_1, t_2)) \equiv 1 \mod Z_{\alpha-2}$, and we have the following theorem.

THEOREM 3. *The derived ring of a Z-A(3) group is commutative.*

Theorem 3 certainly is not true for Z-A(2) groups. In the example of a Z-A(2) group given above, $\bar{a}_3$ is in Z_3/Z_2 and both $\bar{\alpha}$ and $\bar{\beta}$ are in Z_4/Z_3, but $\bar{a}_3\, \bar{\alpha}\bar{\beta} = 0$ and $\bar{a}_3\, \bar{\beta}\bar{\alpha} = \bar{a}_1$.

III. Z-A(2) groups with a commutative derived ring

A Z-A(2) group G of class $n + 1$ with a commutative derived ring means of course that elements of Z_{n+1}/Z_n operate commutatively on the direct sum of the groups $Z_{\alpha+1}/Z_\alpha$ for $\alpha < n$. Denote the above class of groups by Z-A$_c$(2). Theorem 3 shows that Z-A(3) $\leqq$ Z-A$_c$(2). Whether or not this is really an equality is still unknown. It seems unlikely, but as of yet the evidence is still inconclusive.

Let $C_{m,i}$ designate the binomial coefficient of m with i. The symbol $\prod$ will denote a product and (m, j) will designate the greatest common divisor of the integers m and j. We shall also use H_α for the set of elements x of a Z-A(2) group where $\bar{x}$ is homogeneous of degree α.

The following theorem is a generalization of [3, Lemma 4.1].

THEOREM 4. *Suppose that $G \in Z\text{-}A_c(2)$ and $G\,|e{:}m|\,G$; then G/Z_{2^m-1} is periodic where the periods divide some power of*

$$k = \prod_{i=0}^{m-2} (C_{m-i,1}, \cdots, C_{m-i,m-i-1}).$$

The proof will consist of first proving that $k\Gamma^{2^m-1} = 0$ where Γ is the derived ring of G and from this the theorem will be shown to follow.

LEMMA 3. *If G is a $Z\text{-}A_c(2)$ group of class $n + 1$ and x is in $H_{\alpha+1}$, $\alpha + 1 < n + 1$, then for all t_1 and t_2 in H_{n+1} we have*

$$(x, {}_it_1, {}_jt_2) \equiv (x, {}_jt_2, {}_it_1) \mod Z_{\alpha-i-j}.$$

Since $G \in$ Z-A$_c$(2) the derived ring is commutative.

The lemma then follows from the equation

$$\bar{x}\underbrace{\bar{t}_1 \cdots \bar{t}_1}_{i}\underbrace{\bar{t}_2 \cdots \bar{t}_2}_{j} = \bar{x}\underbrace{\bar{t}_2 \cdots \bar{t}_2}_{j}\underbrace{\bar{t}_1 \cdots \bar{t}_1}_{i}.$$

LEMMA 4. *If $x_1, \cdots, x_k$ are elements of a group G which are located in the upper central term $Z_{\alpha+1}$, then for all $g_1, \cdots, g_r$ in G we have*

$$\left(\prod_{i=1}^{k} x_i, g_1, \cdots, g_r\right) \equiv \prod_{i=1}^{k} (x_i, g_1, \cdots, g_r) \mod Z_{\alpha-r}.$$

The proof will use an induction on k and r. If $k = r = 1$ the lemma is trivial. For $k = q + 1$ and $r = 1$ by using (1) we have

$$\left(\prod_{i=1}^{q+1} x_i, g\right) \equiv \left(\prod_{i=1}^{q} x_i, g\right)\cdot(x_{q+1}, g) \mod Z_{\alpha-1}.$$

Thus the lemma follows by the induction hypothesis. If $r = m + 1$ we have by the induction hypothesis

$$\begin{aligned}\left(\prod_{i=1}^{k} x_i, g_1, \cdots, g_m, g_{m+1}\right) &\equiv \left(\prod_{i=1}^{k} (x_i, g_1, \cdots, g_m), g_{m+1}\right) \mod Z_{\alpha-m-1} \\ &\equiv \prod_{i=1}^{k} (x_i, g_1, \cdots, g_m, g_{m+1}) \mod Z_{\alpha-m-1}.\end{aligned}$$

LEMMA 5. *Suppose that G is a Z-$A_c(2)$ group of class $n+1$, x is in $H_{\alpha+1}$ for $\alpha < n$, and t_1 and t_2 are both in H_{n+1}. Then*

$$(x, {}_m[t_1\, t_2]) \equiv \prod_{i=0}^{m} (x, {}_{m-i}t_1, {}_it_2)^{C_{m,n}} \mod Z_{\alpha-m}.$$

Since each factor $(x, {}_{m-i}t_1, {}_it_2)$ is in $Z_{\alpha+1-m}$ they must commute modulo $Z_{\alpha-m}$. Thus the order of the factors in the above product is immaterial.

Since $(x, t_1 \cdot t_2) = (x, t_2) \cdot (x, t_1) \cdot (x, t_1, t_2)$, the lemma is true for $m = 1$. For $m = q + 1$ if we designate $(x, {}_{q+1}[t_1 \cdot t_2])$ by A we have

$$\begin{aligned} A &= (x, {}_q[t_1 \cdot t_2], t_1 \cdot t_2) \\ &= (x, {}_q[t_1 \cdot t_2], t_2) \cdot (x, {}_q[t_1 \cdot t_2], t_1) \cdot (x, {}_q[t_1 \cdot t_2], t_1, t_2), \\ A &\equiv (x, {}_q[t_1 \cdot t_2], t_2) \cdot (x, {}_q[t_1 \cdot t_2], t_1) \mod Z_{\alpha-q-1}. \end{aligned}$$

If we apply the induction hypothesis, we get

$$A \equiv \left(\prod_{i=0}^{q} (x, {}_{q-i}t_1, {}_it_2)^{C_{q,i}}, t_2\right) \cdot \left(\prod_{i=0}^{q} (x, {}_{q-i}t_1, {}_it_2),^{C_{q,i}}, t_1\right) \mod Z_{\alpha-q-1}.$$

By Lemma 4 we have

$$A \equiv \prod_{i=0}^{q} (x, {}_{q-i}t_1, {}_{i+1}t_2)^{C_{q,i}} \cdot \prod_{i=0}^{q} (x, {}_{q-i}t_1, {}_it_2, t_1)^{C_{q,i}} \mod Z_{\alpha-q-1}.$$

If we use Lemma 3 to permute $t_1 \bmod Z_{\alpha-q-1}$ past the elements ${}_it_2$ in $(x, {}_{q-i}t_1, {}_it_2, t_1)^{C_{q,i}}$ we get

$$\begin{aligned} A &\equiv \prod_{i=0}^{q} (x, {}_{q-i}t_1, {}_{i+1}t_2)^{C_{q,i}} \cdot \prod_{i=0}^{q} (x, {}_{q+1-i}t_1, {}_it_2)^{C_{q,i}} \mod Z_{\alpha-q-1} \\ &\equiv \prod_{i=1}^{q+1} (x, {}_{q+1-i}t_1, {}_it_2)^{C_{q,i-1}} \cdot (x, {}_{q+1}t_1) \cdot \prod_{i=1}^{q} (x, {}_{q+1-i}t_1, {}_it_2)^{C_{q,i}} \\ &\qquad \mod Z_{\alpha-q-1}. \end{aligned}$$

Since the factors commute modulo $Z_{\alpha-q-1}$ we have

$$\begin{aligned} A &\equiv (x, {}_{q+1}t_2) \cdot (x, {}_{q+1}t_1) \prod_{i=1}^{q} (x, {}_{q+1-i}t_1, {}_it_2)^{C_{q,i-1}} \prod_{i=1}^{q} (x, {}_{q+1-i}t_1, {}_it_2)^{C_{q,i}} \\ &\qquad \mod Z_{\alpha-q-1} \\ &\equiv (x, {}_{q+1}t_2) \cdot (x, {}_{q+1}t_1) \prod_{i=1}^{q} (x, {}_{q+1-i}t_1, {}_it_2)^{C_{q,i-1}+C_{q,i}} \mod Z_{\alpha-q-1} \\ &\equiv \prod_{i=0}^{q+1} (x, {}_{q+1-i}t_1, {}_it_2)\, C_{q+1,i} \mod Z_{\alpha-q-1}. \end{aligned}$$

This completes the induction.

COROLLARY. *Suppose that N is a Γ-invariant submodule of the derived module M of a Z-$A_c(2)$ group of class $n+1$. Further suppose that $N\bar{t}^m = 0$ for all $\bar{t}$ in Z_{n+1}/Z_n and for some integer m which is independent of $\bar{t}$. Then $qN\bar{t}_1^{m-1}\bar{t}_2^{m-1} = 0$ for all $\bar{t}_1$ and $\bar{t}_2$ in Z_{n+1}/Z_n where $q = (C_{m,1}, \cdots, C_{m,m-1})$.*

Every element of N can be expressed in the form

$$\bar{x}_1 + \cdots + \bar{x}_i + \cdots + \bar{x}_j + \cdots + \bar{x}_k$$

where for $i \neq j$, $\bar{x}_i$ and $\bar{x}_j$ are in different summands of the derived module M.

If for instance $\bar{x}_i$ and $\bar{x}_j$ are in $Z_{\alpha+1}/Z$ then combine them. But

$$(\bar{x}_1 + \cdots + \bar{x}_k)\bar{t}^m = 0$$

implies that $\bar{x}_j \bar{t}^m = 0$ for $j = 1, \cdots, k$. Suppose that $\bar{t}_1$ and $\bar{t}_2$ are in Z_{n+1}/Z_n. The group product $t_1 \cdot t_2$ may or may not be in Z_n. If $t_1 \cdot t_2$ is in Z_n then, since $G \in$ Z-A$_c(2)$, we have that $(x_j, {}_m t_1 \cdot t_2) \equiv 1 \bmod Z_{\alpha_j - m}$, where $\bar{x}_j$ is in $Z_{\alpha_j+1}/Z_{\alpha_j}$. Should $t_1 \cdot t_2$ not be in Z_n, then $\bar{x}_j\overline{(t_1 \cdot t_2)}^m = 0$ implies that

$$(x_j, {}_m[t_1 \cdot t_2]) \equiv 1 \bmod Z_{\alpha_j - m}.$$

Thus in either case we have $(x_j, {}_m[t_1 \cdot t_2]) \equiv 1 \bmod Z_{\alpha_j - m}$ for $\bar{t}_1$ and $\bar{t}_2$ in Z_{n+1}/Z_n, x_j in $Z_{\alpha_j+1}/Z_{\alpha_j}$. But by Lemma 5

$$(x_j, {}_m[t_1 \cdot t_2]) \equiv \textstyle\prod_{i=0}^{m} (x_j, {}_{m-i}t_1, {}_i t_2)^{C_{m,i}} \quad \bmod Z_{\alpha_j - m}.$$

Then for $l = 1, 2, \cdots, m-1$ we have

$$(x_j, {}_m[t_1 \cdot t_2], {}_{l-1}t_1, {}_{m-l-1}t_2)$$
$$\equiv (\textstyle\prod_{i=0}^{m} (x_j, {}_{m-i}t_1, {}_i t_2)^{C_{m,i}}, {}_{l-1}t_1, {}_{m-l-1}t_2) \quad \bmod Z_{\alpha_j - 2m+2}.$$

If we use Lemma 4 we have

$$\equiv \textstyle\prod_{i=0}^{m} (x_j, {}_{m-i}t_1, {}_i t_2, {}_{l-1}t_1, {}_{m-l-1}t_2)^{C_{m,i}} \quad \bmod Z_{\alpha_j - 2m+2}.$$

But by Lemma 3 we can permute the elements ${}_{l-1}t_1$ past ${}_i t_2$ in

$$(x_j, {}_{m-i}t_1, {}_i t_2, {}_{l-1}t_1, {}_{m-l-1}t_2)$$

to get

$$\equiv \textstyle\prod_{i=0}^{m} (x_j, {}_{m+l-i-1}t_1, {}_{m-l+i-1}t_2)^{C_{m,i}} \quad \bmod Z_{\alpha_j - 2m+2}.$$

Thus since $(x_j, {}_m[t_1 \cdot t_2]) \equiv 1 \bmod Z_{\alpha_j - m}$ we have

$$(14) \qquad 1 \equiv \textstyle\prod_{i=0}^{m} (x_j, {}_{m+l-i-1}t_1, {}_{m-l+i-1}t_2)^{C_{m,i}} \quad \bmod Z_{\alpha_j - 2m+2}.$$

But we assumed that $N\bar{t}_1^m = 0$. This means that $(x_j, {}_m t_1) \equiv 1 \bmod Z_{\alpha_j - m}$. Therefore if $i < l$ then $m - i + l - 1 \geqq m$ and

$$(15) \qquad 1 \equiv (x_j, {}_{m+l-i-1}t_1, {}_{m-l+i-1}t_2) \quad \bmod Z_{\alpha_j - 2m+2}.$$

By Lemma 3, we have

$$(x_j, {}_{m+l-i-1}t_1, {}_{m-l+i-1}t_2) \equiv (x_j, {}_{m-l+i-1}t_2, {}_{m+l-i-1}t_1) \quad \bmod Z_{\alpha_j - 2m+2}.$$

Using the assumption $n\bar{t}_2^m = 0$ for all n in N we have that

$$(x_j, {}_m t_2) \equiv 1 \quad \bmod Z_{\alpha_j - m}.$$

Then if $i > l$ and thus $m - l + i - 1 \geqq m$, we have that

$$(16) \qquad 1 \equiv (x_j, {}_{m-l+i-1}t_2, {}_{m+l-i-1}t_1) \quad \bmod Z_{\alpha_j - 2m+2}.$$

Using (14), (15) and (16) we get

$$1 \equiv (x_j, {}_{m-1}t_1, {}_{m-1}t_2)^{C_{m,l}} \quad \bmod Z_{\alpha_j - 2m+2} \qquad \text{for } l = 1, \cdots, m-1.$$

Therefore $C_{m,l}N\bar{t}_1^{m-1}\bar{t}_2^{m-1} = 0$ and the corollary follows.

LEMMA 6. *Suppose that G is a Z-$A_c(2)$ group of class $n + 1$ and N is a Γ-invariant submodule of the derived module M where Γ is the derived ring. Further suppose that $N\bar{t}^m = 0$ for all $\bar{t}$ in Z_{n+1}/Z_n and for some integer m which is independent of $\bar{t}$. Then*

$$kN\Gamma^{2^{m-1}} = 0 \quad \textit{where} \quad k = \prod_{i=1}^{m-2} (C_{m-i,1}, \cdots, C_{m-i,m-i-1})^{2^i}.$$

If $m = 1$ the proof is obvious. Suppose that $m = r + 1$. By the corollary of Lemma 5, $(C_{r+1,1}, \cdots, C_{r+1,r})N\bar{t}_1^r\bar{t}_2^r = 0$ for all $\bar{t}_1$ and $\bar{t}_2$ in Z_{n+1}/Z_n. Define N_1 to be the submodule $(C_{r+1,1}, \cdots, C_{r+1,r})N\bar{t}_1^r$ for $\bar{t}_1$ in Z_{n+1}/Z_n. Obviously N_1 is Γ-invariant since Γ is commutative, and N is Γ-invariant. But $N_1\bar{t}^r = 0$ for all $\bar{t}$ in Z_{n+1}/Z_n. By the induction hypothesis

$$bN_1\Gamma^{2^{r-1}} = 0 \quad \text{where} \quad b = \prod_{i=1}^{r-2} (C_{r-i,1}, \cdots, C_{r-i,r-i-1})^{2^i}.$$

Since Γ is commutative

$$hN^{2^{r-1}}\bar{t}_1^r = 0$$

for every $\bar{t}_1$ in Z_{n+1}/Z_n where $h = (C_{r+1,1}, \cdots, C_{r+1,1})\cdot b$. Let $N_2 = hN\Gamma^{2^{r-1}}$. Then N_2 is Γ-invariant and $N_2\bar{t}^r = 0$ for $\bar{t}$ in Z_{n+1}/Z_n. The induction hypothesis implies that $dN_2\Gamma^{2^{r-1}} = 0$ for $d = \prod_{i=1}^{r-2} (C_{r-i,1}, \cdots, C_{r-i,r-i-1})^{2^i}$ and the lemma follows.

LEMMA 7. *Suppose that G is a Z-$A_c(2)$ group of class $n + 1$. If for some integer q,*

$$(Z_{\alpha+1}^k, \underbrace{H_{n+1}, \cdots, H_{n+1}}_{q}) \equiv 1 \mod Z_{\alpha-q}$$

then $Z_{\alpha+1}^k \leqq Z_\alpha$.

Suppose that x is in $Z_{\alpha+1}$. Since G is a Z-$A_c(2)$ group we have

$$(x^k, Z_{\alpha_1}, Z_{\alpha_2}, \cdots, Z_{\alpha_j}, \cdots, Z_{\alpha_q}) \equiv 1 \mod Z_{\alpha-q}$$

if $Z_{\alpha_j} \leqq Z_n$ for some j. But since

$$(x^k, \underbrace{H_{n+1}, \cdots, H_{n+1}}_{q}) \equiv 1 \mod Z_{\alpha-q}$$

it follows that $(x^k, \underbrace{G, \cdots, G}_{q}) \equiv 1 \mod Z_{\alpha-q}$. Therefore

$$(x^k, \underbrace{G, \cdots, G}_{q-1}) \equiv 1 \mod Z_{\alpha+1-q}.$$

Lemma 7 follows from $q - 1$ repetitions of this last step.

COROLLARY. *Suppose that G is a Z-$A_c(2)$ group of class $n + 1$. If there* *exist positive integers k and q such that for all $\alpha < n$,*

$$(H_{\alpha+1}^k, \underbrace{H_{n+1}, \cdots, H_{n+1}}_{q}) \equiv 1 \mod Z_{\alpha-q}$$

then G/Z_q is periodic and the periods are powers of k.

Suppose x is in H_{α_0+1} for $q < \alpha_0 + 1 < n + 1$. Since

$$(x^k, \underbrace{H_{n+1}, \cdots, H_{n+1}}_{q}) \equiv 1 \mod Z_{\alpha_0-q},$$

the element x^k is in H_{α_1} for $\alpha_1 < \alpha_0$ by Lemma 7. Repeating this argument on the element x^k we have that x^{k^2} is in H_{α_2} where $\alpha_2 < \alpha_1$. Continuing this process we arrive at a sequence $x^k, x^{k^2}, x^{k^3}, \cdots, x^{k^i}, x^{k^{i+1}}, \cdots$ where x^{k^i} is in H_{α_i} and $\alpha_i > \alpha_{i+1}$. But this sequence is finite since the upper central series is well ordered.

We return now to proof of Theorem 4. Since $G\,|e{:}m|\,G$ we have $M\bar{l}^m = 0$ for all $\bar{l}$ in Z_{n+1}/Z_n, where M is the derived module of G. By Lemma 6 we have $kM\Gamma^{2^{m-1}} = 0$ where $k = \prod_{i=1}^{m-2} (C_{m-i,1}, \cdots, C_{m-i,m-i-1})^{2^i}$. But this means for all $\alpha < n$,

$$(H_{\alpha+1}^k, \underbrace{H_{n+1}, \cdots, H_{n+1}}_{2^{m-1}}) \equiv 1 \mod Z_{\alpha-2m-1}.$$

Therefore the theorem follows from the corollary of Lemma 7.

The following corollary states an obvious consequence of Theorem 4.

COROLLARY. *If G is a Z-$A_c(2)$ group where $G\,|e{:}m|\,G$ and in addition if $G/Z_{2^{m-1}}$ is k-torsion-free where k is defined as above, then G is nilpotent.*

By Theorem 4, every Z-$A_c(2)$ group which satisfies the Engel condition of class m is periodic modulo $Z_{2^{m-1}}$. It is a simple matter to show that if G is a Z-$A_c(2)$ group which satisfies the Engel condition of class m then so must G/Z_α for every ordinal α. So it seems natural to study periodic Z-$A_c(2)$ groups which satisfy the Engel condition.

THEOREM 5. *Suppose that $G \in$ Z-$A_c(2)$ and $G\,|e{:}m|\,G$. If in addition G is also periodic where every element x of G has a period $q(x)$ such that all of the prime divisors of $q(x)$ are larger than those of m, then G is nilpotent.*

Since G is periodic then $G/Z_{2^{m-1}}$ must also be. Every element x of $G/Z_{2^{m-1}}$ must have a period dividing $q(x)$ where the prime divisors of $q(x)$ are larger than those of m Hence $q(x)$ and k are relatively prime where

$$k = \prod_{i=1}^{m-2} (C_{m-i,1}, \cdots, C_{m-i,m-i-1})^{2^i}.$$

Consequently $G/Z_{2^{m-1}}$ is k-torsion-free. The theorem follows from the corollary of Theorem 4.

The condition on the periods $q(x)$ in Theorem 5 are necessary when $q(x)$

is not a prime exponent for the group G. We presented an example of a Z-A(3) group H such that $H\,|e{:}3|\,H$ and $H^4 = 1$ but H is not nilpotent. However we next show that every Z-A(3) group of prime exponent is nilpotent.

THEOREM 6. *If G is a Z-$A_c(2)$ group of class $n+1$ and $G^p = 1$ for prime p then G is nilpotent.*[2]

Suppose that $x \in H_{\alpha+1}$ for $\alpha + 1 \leqq n$, and $t \in H_{n+1}$. In [4, equation 18.4.13, p. 327] M. Hall showed that $(x, {}_{p-1}t)$ can be expressed as a product of commutators of the form $(x, y_1, \cdots, y_p)$ where y_i is x or t. But

$$(x, y_1, \cdots, y_p) \equiv 1 \mod Z_{\alpha+1-p}$$

and hence $(x, {}_{p-1}t) \equiv 1 \bmod Z_{\alpha+1-p}$. But in terms of the derived module M, this means that $\bar{x}\bar{t}^{p-1} = 0$. Thus $M\bar{t}^{p-1} = 0$ for all $\bar{t}$ in Z_{n+1}/Z_n. Therefore by Lemma 6, $kM\Gamma^{2^{p-2}} = 0$ where $k = \prod_{i=1}^{p-3} (C_{m-i,1}, \cdots, C_{m-i,m-i-1})^{2^i}$ Thus

$$(H^k_{\alpha+1}, \underbrace{H_n, \cdots, H_n}_{2^{p-2}}) \equiv 1 \mod Z_{\alpha-2^{p-2}}.$$

Then by the corollary of Lemma 7 we have that $G/Z_{2^{p-2}}$ is periodic and the periods divide powers of k. But the elements of $G/Z_{2^{p-2}}$ have period p. Since k and p are relatively prime $G \leqq Z_{2^{p-2}}$.

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[2] The author is grateful to the referee for suggesting Hall's equation [4, equation 18.4.13] in order to simplify the proof of Theorem 6.