GROUPS WHOSE IRREDUCIBLE REPRESENTATIONS HAVE DEGREES DIVIDING p^e

BY

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Let G be a finitely generated group and $C[G]$ its group algebra over the complex numbers C. In this paper we consider groups with the property that the degrees of all the irreducible representations of $C[G]$ divide a fixed prime p to the power e. This is a special case of the situation studied in [4]. In fact, our result, Theorem I, is a sharper version of Theorem III of that paper. In the more special case $e = 1$, Theorem II gives necessary and sufficient conditions on the structure of the group. For $p = 2$ this yields in particular Theorem 3 of [1].

Our main results are:

THEOREM I. *Let G be a finitely generated group and p a prime. Suppose that all irreducible representations of G over the complex numbers have degrees dividing p^e. Then G has a subinvariant series*

$$G = A_e \supseteq A_{e-1} \supseteq \cdots \supseteq A_0$$

such that A_0 is abelian and A_i/A_{i-1} is elementary abelian p with not more than $2i + 1$ generators. Hence G has an abelian subgroup A_0 whose index divides $p^{e(e+2)}$.

THEOREM II. *Let G be a finitely generated group all of whose irreducible representations have degree 1 or p. Then G is one of the following types:*

1. *G is abelian.*
2. *G has a normal abelian subgroup of index p.*
3. *G has a center Z with G/Z being a group of order p^3 and period p.*

Conversely, let G be one of the above. If G is finite then all of its irreducible representations have degree 1 or p. If G is finitely generated then G at least has a complete set of representations of degree 1 or p.

In Section 4 we give examples to show that all of the above types can occur.

1. In this section we fix nomenclature and give some character-theoretic propositions which are basic to the rest of the paper. All groups in this paper are assumed to be finite unless otherwise stated.

Let χ be an irreducible character of a group G and φ an irreducible character of a subgroup H of G. φ induces a character φ^* of G and χ restricts to a character $\chi \mid H$ of H. From the Frobenius Reciprocity Theorem [3, Theorem 38.8] we can conclude that the multiplicity of χ as a constituent of φ^* is equal to the multiplicity of φ as a constituent of $\chi \mid H$.

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Supose $H \lhd G$ (H is normal in G). Then G acts on the characters of H by conjugation. That is for $g \in G$ and all $x \in H$, $\varphi^g(x) = \varphi(gxg^{-1})$. The subgroup T fixing a given character φ of H is called the inertia group of φ. Clearly $T \supseteq H$. If $t = [G:T]$ then φ has precisely t distinct conjugates $\varphi = \varphi_1, \varphi_2, \cdots, \varphi_t$. If φ is irreducible and χ is a constituent of φ^* of multiplicity a then $\chi \mid H = a(\varphi_1 + \varphi_2 + \cdots + \varphi_t)$.

If $H \lhd G$ then a character β of G/H can be regarded as a character of G with kernel containing H. Conversely every character of G with kernel containing H arises in this manner. We shall use the same symbol to denote the character whether viewed in G or G/H. The precise situation will be clear from context.

For convenience we quote here three propositions from [4].

(1.1) PROPOSITION. *Let $H \lhd G$, $[G:H] = q$, a prime. If χ is an irreducible character of G then $\chi \mid H$ is either irreducible or the sum of q distinct conjugate irreducible characters of H.*

(1.2) PROPOSITION. *Let $H \lhd G$. If χ is a character of G with $\chi \mid H$ irreducible and β is any irreducible character of G/H, then $\beta\chi$ is irreducible.*

(1.3) PROPOSITION. *Let G have a faithful irreducible representation of degree n. If the center Z of G contains the commutator G', then $[G:Z] = n^2$.*

These are Propositions 1.2, 1.1, and 4.1 respectively of [4].

(1.4) PROPOSITION. *Let $H \lhd G$ and let χ be an irreducible character of G with $\chi \mid H = a(\varphi_1 + \cdots + \varphi_t)$ where the φ_i are distinct conjugate characters of H. If T is the inertia group of $\varphi = \varphi_1$, then there is an irreducible character ψ of T with $\psi \mid H = a\varphi$ and $\psi^* = \chi$.*

Proof. Let $\chi \mid T = b_1 \zeta_1 + b_2 \zeta_2 + \cdots + b_r \zeta_r$. Since φ is a constituent of $\chi \mid H$ it follows that φ is a constituent of some of the $\zeta_i \mid H$, say $\zeta_1, \cdots, \zeta_s$. Because T is the inertia group of φ, all the conjugates of φ in T are equal and thus $\zeta_i \mid H = c_i \varphi$ for $i = 1, 2, \cdots, s$. Thus the multiplicity of φ in $\chi \mid H$ is $a = b_1 c_1 + b_2 c_2 + \cdots + b_s c_s$.

Put $\psi = b_1 \zeta_1 + \cdots + b_s \zeta_s$. Then $\psi \mid H = a\varphi$. By Frobenius Reciprocity χ is a constituent of ζ_1^* and thus of ψ^*. However

$$\deg \chi = at \deg \varphi = [G:T] \deg \psi = \deg \psi^*.$$

Thus $\chi = \psi^*$. Since χ is irreducible, ψ must be also.

2. In this section we obtain a proof of Theorem I. For the remainder of this paper let p be a fixed prime number.

(2.1) DEFINITION. A group G is said to have r.x. e (representation exponent e) if all the irreducible representations of G have degrees dividing p^e.

(2.2) LEMMA. *Let $N \lhd G$ where G has* r.x. *e. Then N has* r.x. *e. If G/N is nonabelian then N has* r.x. $(e - 1)$.

Proof. Let φ be an irreducible character of N and let χ be an irreducible constituent of φ^*. Then

$$\chi \mid N = a(\varphi_1 + \varphi_2 + \cdots + \varphi_t)$$

where the φ_i are the conjugates of φ. Since $\deg \varphi_i = \deg \varphi$ we have $\deg \chi = at \deg \varphi$. Since $\deg \chi$ divides p^e, so does $\deg \varphi$.

Assume now that G/N is nonabelian. If either a or t is > 1, then since they divide p^e we would have $p \deg \varphi$ divides p^e. If $a = 1 = t$, then $\varphi = \chi \mid N$ is irreducible. Let β be a nonlinear irreducible character of G/N. Then by Proposition 1.2, $\beta\chi$ is irreducible and hence $\deg \beta \deg \chi = \deg \beta\chi$ divides p^e. But since $\deg \beta > 1$ we have $\deg \beta \geqq p$ and $p \deg \chi = p \deg \varphi$ divides p^e. In either case then, $\deg \varphi$ divides p^{e-1}.

By a p'-Hall subgroup, we mean in the following, a subgroup of a group G of order prime to p and index a power of p.

(2.3) PROPOSITION. *A group G has a normal abelian p'-Hall subgroup H if and only if the degrees of all the irreducible representations of G are powers of p.*

Proof. If G has a normal abelian p'-Hall subgroup H then by Itô's Theorem [3, Corollary 53.18] the degrees of all the irreducible representations of G divide $[G:H]$, a power of p.

Conversely, suppose the degrees of all the irreducible representations of G are powers of p. We proceed by induction on $|G|$, the order of G. If $|G|$ is relatively prime to p take $H = G$. Then since the degrees of the irreducible representations of G divide $|G|$, all the irreducible representations are linear and G is abelian.

Suppose then that p divides $|G|$. From the equation

$$|G| = [G:G'] + \sum x_i^2$$

where the x_i are the degrees of the nonlinear irreducible characters of G, we conclude that p divides $[G:G']$. Let K be the complete inverse image in G of a subgroup of index p in the abelian group G/G'. Then $K \lhd G$ and, by Lemma 2.2, K has r.x. e for some suitably large e. By the induction hypothesis then, K has a normal abelian p'-Hall subgroup H. H is clearly a p'-Hall subgroup of G and since it is characteristic in K, H is normal in G.

We note that if G is a (not necessarily finite) group with r.x. e then every quotient group of G has r.x. e. In the finite case we have

(2.4) COROLLARY. *If G has* r.x. *e then so does every subgroup of G.*

Proof. Let K be a subgroup of G. By the above proposition G has a normal abelian p'-Hall subgroup H. Then $H \cap K$ is a normal abelian

p'-Hall subgroup of K and thus the degrees of all of the irreducible representations of K are powers of p. Let φ be an irreducible character of K and let χ be a constituent of φ^*. Then φ is a constituent of $\chi \mid H$ and thus $\deg \varphi \leqq \deg \chi$. The result follows.

(2.5) PROPOSITION. *Let $N \lhd G$ with G/N nilpotent. Suppose χ is an irreducible character of G with $\chi \mid N$ reducible. Then there exists a normal subgroup T of G of prime index such that $T \supseteq N$ and $\chi = \psi^*$ for some irreducible character ψ of T.*

Proof. Let $G = N_0 > N_1 > \cdots > N_j = N$ be a normal series with quotients of prime order. Let i be the biggest subscript with $\chi \mid N_i$ irreducible. Then $0 \leqq i < j$. By Proposition 1.1

$$\chi \mid N_{i+1} = \varphi_1 + \varphi_2 + \cdots + \varphi_q$$

where the φ_i are distinct and $q = [N_i:N_{i+1}]$. Let T be the inertia group of φ_1 in G. Then $G \supseteq T \supseteq N_{i+1} \supseteq N$, $[G:T] = q$ and $T \lhd G$. By Proposition 1.4 there is an irreducible character ψ of T with $\chi = \psi^*$.

(2.6) COROLLARY. Let $N \lhd G$ *with N abelian and G/N nilpotent. If χ is an irreducible character of G, there exists a subgroup K of G containing N and a linear character λ on K with $\chi = \lambda^*$.*

Proof. We proceed by induction on $[G:N]$. If $[G:N] > 1$ let T and ψ be as in the proposition. Then $[T:N] < [G:N]$ and we can find K and λ with $\psi = \lambda^{**}$ (** means induction to T). By transitivity of induction we conclude that $\chi = \lambda^*$.

(2.7) LEMMA. *Let G have* r.x. e *and let K and H be subgroups with $H \lhd K \subseteq G$. Suppose K/H is an abelian group of order prime to p. If H has* r.x. f *then so does K.*

Proof. Let χ be an irreducible character of K. If $\chi \mid H$ is irreducible then $\deg \chi = \deg \chi \mid H$ divides p^f. *If* $\chi \mid H$ is reducible then applying Proposition 2.5 to K we have $\chi = \psi^*$ where ψ is a character of a subgroup T of index q in K and containing H. Then $\deg \chi = q \deg \psi$ and $(p, q) = 1$. By Corollary 2.4, K has r.x. e and thus $\deg \chi$ divides p^e. This is a contradiction.

(2.8) LEMMA. *Let $N \lhd G$ with G/N a p-group. Let G have* r.x. e *and N have* r.x. $(e - 1)$. *If F is the inverse image of the Frattini subgroup of G/N in G, then F has* r.x. $(e - 1)$.

Proof. $F \lhd G$ and thus by Lemma 2.2, F has r.x. e. We must show that F has no irreducible characters of degree p^e. Suppose φ is such a character. Let χ be an irreducible constituent of φ^*. Then $p^e \geqq \deg \chi \geqq \deg \varphi = p^e$ and thus $\deg \chi = p^e$ and $\chi \mid F = \varphi$ is irreducible. Since N has r.x. $(e - 1)$, $\chi \mid N$ is reducible, and by Proposition 2.5 there is a subgroup T maximal in G and containing N with $\chi = \psi^*$ for some character ψ of T. Therefore ψ is a constituent of $\chi \mid T$ which is thus reducible. But $T \supseteq F$ and $\chi \mid F$ is irreducible, a contradiction.

(2.9) LEMMA. *Let R be a group with the following properties:*
(i) *R has a nontrivial normal abelian subgroup.*
(ii) *If $1 < N \lhd R$, then $N \supseteq R'$.*
(iii) *R has an irreducible representation of degree $m > 1$.*
Then every maximal normal abelian subgroup of R has index m.

Proof. Since R has a nontrivial normal abelian subgroup, a maximal such subgroup A is not trivial. Thus $A \supseteq R'$. Let χ be an irreducible character of R of degree m. Since R/A is abelian we can apply Corollary 2.6 and conclude that $\chi = \lambda^*$ for a linear character λ of some subgroup $K \supseteq A$. Since $K \lhd R$, if $K > A$ then K is nonabelian and $1 \neq K' \lhd R$. Hence by (ii), $K' \supseteq R'$ and thus R' is included in the kernel of the linear character λ of K. Since $R' \lhd R$, R' is therefore in the kernel of $\chi = \lambda^*$. Since χ is irreducible it must be linear, a contradiction. Hence $K = A$ and

$$m = \deg \chi = \deg \lambda^* = [R:K] = [R:A].$$

We are now ready to prove Theorem I. First we assume G is finite and then we consider the more general case.

Proof of Theorem I *for finite groups.* We prove the result by induction on e. If $e = 0$ the group is abelian and the result is trivial. Assume then that $e \geqq 1$. It will be sufficient to show that G has a normal subgroup A_{e-1} having r.x. $(e - 1)$ and such that G/A_{e-1} is an elementary abelian p-group with $\leqq 2e + 1$ generators.

We may assume G is nonabelian. Choose $N \lhd G$ maximal with G/N nonabelian. Put $R = G/N$. R has r.x. e and thus has a normal abelian p'-Hall subgroup by Proposition 2.3. Let H be the inverse image of this subgroup in G. Since R is nonabelian, N has r.x. $(e - 1)$ by Lemma 2.2. Because of the choice of N, every nontrivial normal subgroup of R contains $R' > 1$.

There are two cases:

Case 1. H/N is a nontrivial subgroup of R. No two normal subgroups of R can be disjoint because each contains R'. Therefore every normal abelian subgroup must be a q-group for some prime q, for otherwise the Sylow subgroups of such a group would be disjoint normal subgroups of R.

Since H/N is a p'-Hall subgroup it must be a maximal abelian normal subgroup of R. If χ is an irreducible character of R of degree p^f, $1 \leqq f \leqq e$ then by Lemma 2.9

$$[R:H/N] = [G:H] = p^f.$$

Since N has r.x. $(e - 1)$, we can conclude by Lemma 2.7 that H has r.x. $(e - 1)$.

Let A_{e-1} be the inverse image of the Frattini subgroup of G/H. By Lemma 2.8, A_{e-1} has r.x. $(e - 1)$. $A_{e-1} \lhd G$ and G/A_{e-1} is an elementary abelian p-group of order $\leqq p^f$ and thus has $\leqq f \leqq e$ generators.

Case 2. H/N is trivial and R is a p-group. Let χ be an irreducible character of R of degree p^f, $1 \leqq f \leqq e$. The kernel of χ must be trivial, for otherwise it would contain R'. Thus χ is a faithful irreducible character. Therefore the center Z of R is cyclic. Moreover $Z \supseteq R'$ and thus by Proposition 1.3, $[R:Z] = p^{2f}$.

Let A_{e-1} be the inverse image of the Frattini subgroup F of R. Then

$$[G:A_{e-1}] = [R:F] \leqq [R:(F \cap Z)] = [R:Z][Z:(F \cap Z)].$$

Since Z is cyclic $[Z:(F \cap Z)] = 1$ or p and thus $[G:A_{e-1}] \leqq p^{2f+1}$.

The result now follows as in Case 1.

Proof of Theorem I. Let G be a finitely generated group all of whose irreducible representations have degrees dividing p^e. By a theorem of M. Hall [5, page 56] there are only finitely many subgroups of G of index $\leqq p^{e(e+2)}$. Suppose that $L_1, L_2, \cdots, L_s$ are all of those which are nonabelian. Choose $x_i, y_i \in L_i$ with the commutator $z_i = [x_i, y_i] \neq 1$.

By Theorem V of [6], G is a subdirect product of finite groups and thus we can find a normal subgroup N of finite index in G such that $z_i \notin N$ for $i = 1, 2, \cdots, s$. Then G/N is a finite group having r.x. e and thus there is a subinvariant series $G = A_e \supseteq A_{e-1} \supseteq \cdots \supseteq A_0 \supseteq N$ of G with A_i/A_{i-1} an elementary abelian p-group with $\leqq 2i + 1$ generators such that A_0/N is abelian. $[G:A_0] \leqq p^{e(e+2)}$ and thus if A_0 is not abelian it is one of the L_i. However by the choice of N, each L_i/N is nonabelian and therefore A_0 is abelian. This proves the theorem.

3. Here we study in more detail groups having r.x. 1 and work toward a proof of Theorem II.

(3.1) PROPOSITION. *Let G be a group with an abelian p-Sylow subgroup. If G has* r.x. *e then G has a subinvariant series*

$$G = A_e \supseteq A_{e-1} \supseteq \cdots \supseteq A_0$$

such that A_0 is abelian and A_i/A_{i-1} is an elementary abelian p-group with $\leqq i$ generators. Hence G has an abelian subgroup A_0 whose index divides $p^{e(e+1)/2}$.

Proof. The result follows from the fact that Case 2 of the proof of Theorem I for finite groups cannot occur because a homomorphic image of G which is a p-group must be abelian. Case 1 yields $[A_i:A_{i-1}] \leqq p^i$.

(3.2) DEFINITION. A subgroup A of a group G having r.x. 1 is said to be *special* if

(i) A is abelian and normal in G;

(ii) G/A is an elementary abelian p-group;
(iii) if $B > A$ then B is nonabelian.

Note that Theorem I guarantees the existence of a special subgroup of index dividing p^3.

We now prove a lemma which seems to be crucial in determining the structure of G. The notation $C(a)$ means the centralizer of a in G.

(3.3) LEMMA. *Let A be a special subgroup of G. Then every element of A is either central in G or commutes with nothing outside of A. That is if $a \in A$ then either $C(a) = G$ or $C(a) = A$.*

Proof. The result is trivial if $[G:A] = p$, so we assume $[G:A] \geqq p^2$. Suppose $a \in A$ with $A < C(a) < G$. Choose $x \in C(a) - A$ and $y \in G - C(a)$. Set $K = \langle A, x, y\rangle$. By (ii) of Definition 3.2 it is clear that $[K:A] = p^2$. Since $x \notin A$, $\langle A, x\rangle$ is nonabelian; thus there exists $b \in A$ with $x \notin C(b)$. Therefore $u = x^{-1}bxb^{-1}$ and $v = y^{-1}aya^{-1}$ are nonidentity elements of A. Since in the group algebra of A, $1 + uv \neq u + v$ we have $(1 - u)(1 - v) \neq 0$. Thus there is some irreducible (linear) representation λ of the algebra with $\lambda(1 - u)\cdot\lambda(1 - v) \neq 0$. Then λ is a linear character of A different from 1 at both u and v.

Let T be the inertia group of λ in G. Since G has r.x. 1, we can conclude from Proposition 1.4 that $[G:T] = 1$ or p. Therefore $T \cap K > A$. Let $z \in (T \cap K) - A$. Since K/A is elementary abelian we have $z = x^r y^s c$ for some $c \in A$ and integers $r, s < p$. If $s \neq 0$ then by taking a power of z we can assume $s = 1$. Then $x^r y \in T$ and

$$\begin{aligned} \lambda(a) &= \lambda^{(x^r y)^{-1}}(a) = \lambda(y^{-1}x^{-r}ax^r y) \\ &= \lambda(y^{-1}ay) = \lambda(y^{-1}aya^{-1})\lambda(a) \neq \lambda(a) \end{aligned}$$

since $\lambda(y^{-1}aya^{-1}) \neq 1$. If $s = 0$ then $x^r \in T \cap K$ and thus $x \in T$ and

$$\begin{aligned} \lambda(b) &= \lambda^{x^{-1}}(b) = \lambda(x^{-1}bx) \\ &= \lambda(x^{-1}bxb^{-1})\lambda(b) \neq \lambda(b) \end{aligned}$$

since $\lambda(x^{-1}bxb^{-1}) \neq 1$. In either case we have the desired contradiction.

(3.4) LEMMA. *Let G have* r.x. 1. *If the p'-Hall subgroup H of G is not central then G has a normal abelian subgroup of index p.*

Proof. Let A be a special subgroup with index dividing p^3. We assume $[G:A] > p$ and show that H is central. Since $H \subseteq A$ we can write $A = QH$ where Q is the p-Sylow subgroup of A and is thus normal in G. The group G/Q has r.x. 1 and has an abelian p-Sylow subgroup and thus has an abelian subgroup of index 1 or p by Proposition 3.1. Let B be the inverse image of this subgroup in G. Then $[G:B] = 1$ or p and $[G:A] > p$ and thus $A \not\supseteq B$. Choose $x \in B - A$. Since B/Q is abelian and $H \subseteq B$ we have for all $h \in H$,

$hxh^{-1}x^{-1} \in Q$. On the other hand $H \lhd G$ and thus $hxh^{-1}x^{-1} \in H$. Since $H \cap Q = 1$ we have $x \in C(h)$ and $x \notin A$. By Lemma 3.3 then, $C(h) = G$, that is H is central.

(3.5) PROPOSITION. *Let A be a special subgroup of p-group G of index p^t.*

(a) *If $t > 1$, then the center Z of G has index p in A.*

(b) *If $t = 1$, then $p \cdot |Z| \cdot |G'| = |G|$.*

Proof. Since A is special $Z < A$. By Lemma 3.3, $C(a) = G$ or A for each $a \in A$. The first possibility occurs for the $z = |Z|$ elements of Z. Each of the remaining elements of A thus has p^t conjugates in G. Suppose there are r such classes. Then

$$|A| = w = z + p^t r.$$

The $r + z$ conjugacy classes of G contained in A are the orbits of the action of G on A.

G acts also on the linear characters of A, fixing some of them and permuting the others in orbits of size p. Say there are k characters fixed and m classes of size p. Then

$$w = k + pm.$$

We claim that the number of orbits in these two actions of G are equal. Let $\mathbf{X}$ be the character matrix of A. G permutes either the rows or the columns to give the same result. We then have for each $y \in G$, $\mathfrak{P}(y)\mathbf{X} = \mathbf{X}\mathfrak{Q}(y)$ where $\mathfrak{P}$ and $\mathfrak{Q}$ are the two permutation representations of G. Since $\mathbf{X}$ is nonsingular, $\mathfrak{P}$ and $\mathfrak{Q}$ are similar and thus have the same character. It follows from Theorem 32.3 of [3] that the number of transitivity classes is the same. Thus

$$k + m = r + z.$$

We can eliminate m and r from the three equations involving k, m, r, z and w and obtain

$$z(p^t - 1) = w(p^{t-1} - 1) + k(p^t - p^{t-1}).$$

If $t > 1$ we have $z(p^t - 1) \geqq w(p^{t-1} - 1)$ and

$$w/z \leqq (p^t - 1)/(p^{t-1} - 1) \leqq p + 1.$$

Since w/z is a power of p we have $w/z = p$. This proves (a).

If $t = 1$ we have $z = k$. If λ is a linear character of A which is fixed by G then λ^* has only linear constituents. This follows since if χ is a nonlinear constituent of λ^* then by Proposition 1.1, $\chi \mid A$ is a sum of p distinct conjugate characters. Since by Frobenius Reciprocity, each constituent of λ^* has multiplicity 1, there are p linear characters of G which restrict to λ on A.

Conversely, every linear character of G restricts to a character of A fixed by the action of G. Thus the number of linear characters of G equals pk. Since this is also equal to $[G:G']$ the result follows.

(3.6) PROPOSITION. *Let G be a p-group having* r.x. 1. *Then G satisfies one of the following*:

(1) *G is abelian.*

(2) *G has a maximal abelian subgroup of index p.*

(3) *G has a center Z of index* p^3.

Proof. By Theorem I, G has a special subgroup A of index dividing p^3. If the index is 1 then (1) holds. If $[G:A] = p$ then (2) holds. Finally if $[G:A] = p^2$ then (3) holds by Proposition 3.5. We will show that these exhaust the possibilities. Suppose then that G has no special subgroups of index $< p^3$ and that A is special with $[G:A] = p^3$. Then the center Z of G has index p^4.

We claim that G/Z is elementary abelian. From the list of groups of order p^4 on page 145 of [2] we see that every such group other than the elementary abelian one satisfies one of the following:

(i) It has a normal cyclic subgroup of order $\geqq p^2$ with an elementary abelian quotient.

(ii) It has a Frattini subgroup of order $\geqq p^2$.

(iii) It has a nonabelian quotient of order p^3.

In case (i) we can extend Z by the generator of the cyclic subgroup and get a normal abelian subgroup of G of index $\leqq p^2$ which is either special or can be extended to a special subgroup.

In case (ii) we can apply Lemma 2.8 to conclude that Z extended by the Frattini subgroup of the quotient is abelian and normal of index $\leqq p^2$. Again this is either special or can be extended to a special subgroup.

In case (iii), G has a normal abelian subgroup N with nonabelian quotient of order p^3. Then by Lemma 2.8 the inverse image of the Frattini subgroup of G/N has r.x. 0 and thus is abelian. Again its index is $\leqq p^2$ and it is either special or can be extended to a special subgroup.

In each of these cases we have a contradiction of the assumption that there exists no special subgroup of index $\leqq p^2$. The only remaining possibility is that G/Z is elementary abelian.

Now let $x \notin Z$. Then $\langle Z, x\rangle$ is a normal abelian subgroup of index p^3. Since its quotient is elementary abelian it can be extended to a special subgroup. However no special subgroup has index $< p^3$ and thus $\langle Z, x\rangle$ is itself special. Since $x \notin Z$ we have by Lemma 3.3, $C(x) = \langle Z, x\rangle$ which has index p^3 in G. Therefore every conjugacy class of G has either 1 or p^3 elements.

G has $z = |Z|$ classes of size 1 and $(g - z)/p^3$ classes of size p^3 where $g = |G|$. The total number of classes of G then is $c = z + (g - z)/p^3$. Now G has $g' = [G:G']$ linear characters and $c - g'$ irreducible characters of degree p. Hence $g = g' + p^2(c - g')$. Since $g = p^4z$ solving for g' yields

$$g' = z(-p^5 + p^4 + p^3 - 1)/(p^3 - p) < 0.$$

This is the desired contradiction.

Proof of Theorem II *for finite groups.* Suppose G has r.x. 1. If the p'-Hall subgroup H of G is not central then G is type (2) by Lemma 3.4. Otherwise $G = H \times P$ where P is the p-Sylow subgroup of G. P has r.x. 1 and thus, by Proposition 3.6, P must be one of three types. If P is abelian then G is type (1); if P has a maximal abelian subgroup of index p then G is type (2) and if the center of P has index p^3 then the center Z of G has index p^3. If G/Z has an element of order p^2 then again G is of type (2). The only remaining possibility is type (3).

Conversely, let G be one of the three types. If G is type (1) then all irreducible representations are linear. If G is type (2) then by Itô's Theorem G has r.x. 1. Finally, let G be type (3). By Itô's Theorem G has r.x. 3. Let χ be a character of G. Then $\chi \mid Z = a\lambda$ where λ is a linear character of Z and $a = \deg \chi$. Hence χ has multiplicity a in λ^* and

$$a \deg \chi = (\deg \chi)^2 \leqq \deg \lambda^* = p^3.$$

We must then have $\deg \chi = 1$ or p.

Proof of Theorem II. Let G be a finitely generated group all of whose irreducible representations are of finite degree 1 or p. By M. Hall's Theorem, G has only finitely many subgroups of any given finite index. Let $A_1, A_2, \cdots, A_r$ be the normal subgroups of index p, if any, and $Z_1, Z_2, \cdots, Z_s$ be those of index p^3 if any. Suppose that G is nonabelian, has no abelian normal subgroup of index p and no central subgroup of index p^3. Choose $g, h \in G$, $a_i, b_i \in A_i$, $c_j \in G$ and $z_j \in Z_j$ with $[g, h]$, $[a_i, b_i]$ and $[c_j, z_j]$ all different from 1.

By Theorem V of [6], G is a subdirect product of finite groups and thus we can find a normal subgroup N of finite index in G which does not contain any of the above commutators. G/N has r.x. 1 and thus our theorem applies. However, by the choice of N we see that G/N cannot be any of the three types. We conclude from this that either G is abelian or it has a normal abelian subgroup of index p or it has a central subgroup of index p^3. The result then follows.

Conversely, let G be finitely generated and one of the three types. Then G has an abelian subgroup of finite index which by Schreier's Theorem [5, page 36] is also finitely generated and thus is a subdirect product of finite groups. Hence G is also of this form.

Every quotient group of G is one of the three types and thus every irreducible character of G whose kernel has finite index is of degree 1 or p. Since G is a subdirect product of finite groups, these characters form a complete set.

4. In this section we give examples to show that all of the different types of groups in Theorem II can occur.

(4.1) *Example.* Let G be the group of order p^7 generated by elements u, v, w, x as follows: The elements $u, v,$ and w all commute and span a normal abelian subgroup $A = \langle u, v, w\rangle$ of index p. We have

$$u^{p^2} = v^{p^2} = w^{p^2} = 1, \qquad x^p = 1,$$

$$xux^{-1} = u^{1+p}, \qquad xvx^{-1} = v^{1+p}, \qquad xwx^{-1} = w^{1+p}.$$

Clearly $G' = \langle u^p, v^p, w^p\rangle$ so $|G'| = p^3$. By Proposition 3.5(b)

$$[G:Z] = |G|/|Z| = p\,|G'| = p^4.$$

Thus G is type (2) but not type (1) or (3).

(4.2) *Example.* Let G be the group of order p^6 generated by elements u, v, w, x, y, z as follows: The elements u, v, and w are central and span a subgroup $N = \langle u, v, w\rangle$ having order p^3. We have

$$u^p = v^p = w^p = 1, \qquad x^p = y^p = z^p = 1,$$

$$xyx^{-1} = uy, \qquad yzy^{-1} = vz, \qquad zxz^{-1} = wx.$$

Clearly $G' = \langle u, v, w\rangle$ so $|G'| = p^3$ and $Z \supseteq N$. G is of course nonabelian. If G had a normal abelian subgroup of index p then by Proposition 3.5(b)

$$p^6 = |G| = p\,|Z|\,|G'| \geqq p\,|N|\,|G'| = p^7,$$

a contradiction. Thus G is not type (1) or (2). If $Z > N$ then $[G:Z] \leqq p^2$ and we see immediately that G would have a normal abelian subgroup of index p. Since this is not the case, $Z = N$. Thus G is type (3) and G/Z is elementary abelian p.

(4.3) *Example.* Let G be the group of order p^5, for $p > 2$, generated by elements u, v, x, y, z as follows: The elements u and v are central and span $N = \langle u, v\rangle$ a subgroup of order p^2. We have

$$u^p = v^p = 1, \qquad x^p = y^p = z^p = 1,$$

$$yxy^{-1} = ux, \qquad zxz^{-1} = vx, \qquad zyz^{-1} = xy.$$

Clearly $G' = \langle u, v, x\rangle$ so $|G'| = p^3$ and $Z \supseteq N$. If G had a normal abelian subgroup of index p then by Proposition 3.5(b)

$$p^5 = |G| = p\,|Z|\,|G'| \geqq p\,|N|\,|G'| = p^6,$$

a contradiction. Just as in the previous example this implies also that $Z = N$. Thus G is type (3) but not type (1) or (2) and G/Z is the non-abelian group of order p^3 and period p.

We remark that all the groups given above are just multiple semidirect products. Thus it is not difficult to show that they exist.

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