# INTERVAL FUNCTIONS AND ABSOLUTE CONTINUITY

BY

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## 1. Introduction

Suppose $[a, b]$ is a number interval.

The author [1] has shown the following theorem:

THEOREM A. *If each of h and m is a real-valued nondecreasing function on* $[a, b]$, *and H is a real-valued bounded function of subintervals of* $[a, b]$ *such that the integral (Section 2)*

$$\int_{[a,b]} H(I)\, dm$$

*exists, then the integral*

$$\int_{[a,b]} H(I) \int_I (dh)^p\, (dm)^{1-p}$$

*exists for each number p such that* $0 < p < 1$.

We note that in the above theorem the function $w$ on $[a, b]$ such that

$$w(a) = 0 \quad \text{and} \quad w(x) = \int_{[a,x]} (dh)^p (dm)^{1-p} \quad \text{for} \quad a < x \leqq b,$$

is absolutely continuous with respect to $m$. This suggests an extension of Theorem A, and in this paper we prove (Theorem 3) that if each of $h$ and $m$ is a real-valued nondecreasing function on $[a, b]$, then the following four statements are equivalent:

(1) If $H$ is a real-valued bounded function of subintervals of $[a, b]$ such that $\int_{[a,b]} H(I)\, dm$ exists, then $\int_{[a,\, b]} H(I)\, dh$ exists.

(2) $\int_{[a,b]} (dh)^p (dm)^{1-p} \to h|_a^b$ as $p \to 1$ for $0 < p < 1$.

(3) $\int_{[a,b]} \left| dh - \int_I (dh)^p (dm)^{1-p} \right| \to 0$ as $p \to 1$ for $0 < p < 1$.

(4) $h$ is absolutely continuous with respect to $m$.

## 2. Preliminary lemmas and definitions

Suppose $[a, b]$ is a number interval.

Throughout this paper all integrals discussed are Hellinger [2] type limits of the appropriate sums, i.e., if $K$ is a real-valued function of subintervals of $[a, b]$, and $[r, s]$ is a subinterval of $[a, b]$, then $\int_{[r,s]} K(I)$ denotes the limit, for successive refinements of subdivisions, of sums $\sum_E K(I)$, where $E$ is a subdivision of $[r, s]$ and the sum is taken over all intervals $I$ of $E$. We see that $\int_{[a,b]} K(I)$ exists if and only if for each subinterval $[u, v]$ of $[a, b]$, $\int_{[u,v]} K(I)$ exists, so that if $a \leqq u < v < w \leqq b$, then

$$\int_{[u,w]} K(I) = \int_{[u,v]} K(I) + \int_{[v,w]} K(I).$$

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The definitions and theorems of this paper can be extended to "many-valued" interval functions.

We state a lemma whose proof follows by conventional methods.

LEMMA 1. *If $H$ is a real-valued bounded function of subintervals of $[a, b]$, and $h$ is a real-valued function on $[a, b]$, then the following two statements are equivalent:*

(1) $\int_{[a,b]} H(I)\, dh$ *exists.*

(2) *For each positive number $c$, there is a real-valued function $g$ on $[a, b]$ such that $\int_{[a,b]} |dh - dg| < c$ and $\int_{[a,b]} H(I)\, dg$ exists.*

In order to maintain the interval-function context of this paper, we now use interval-function methods to prove a known [3, p. 50] lemma about a nondecreasing function absolutely continuous with respect to a nondecreasing function.

LEMMA 2. *If each of $h$ and $m$ is a real-valued nondecreasing function on $[a, b]$, and $h$ is absolutely continuous with respect to $m$, and $c$ is a positive number, then there are a number $W > 0$ and a real-valued function $g$ on $[a, b]$ such that if $I$ is a subinterval of $[a, b]$, then*

$$0 \leqq \Delta g \leqq \min\{\Delta h, W\Delta m\}, \quad \textit{and} \quad h|_a^b - g|_a^b < c.$$

*Proof.* There is a number $k > 0$ such that if $E$ is a subset of a subdivision of $[a, b]$ and $\sum_E \Delta m < k$, then $\sum_E \Delta h < c/2$.

For each subinterval $I$ of $[a, b]$ let $H(I)$ denote $\min\{\Delta h, W\Delta m\}$, where $W = [(h|_a^b)/k] + 1$.

If $[u, v]$ is a subinterval of $[a, b]$, and $L[u, v]$ is the least upper bound of all sums $\sum_D H(I)$, where $D$ is a subdivision of $[u, v]$, then

$$L[u, v] \leqq \min\{h|_u^v, Wm|_u^v\}.$$

We see that if $S$ is a refinement of the subdivision $T$ of the subinterval $[r, s]$ of $[a, b]$, then $0 \leqq \sum_S L(I) \leqq \sum_T L(I)$, so that

$$\int_{[r,s]} L(I) \leqq \min\{h|_r^s, Wm|_r^s\}.$$

Let $g$ denote the function on $[a, b]$ such that

$$g(a) = 0 \quad \text{and} \quad g(x) = \int_{[a,x]} L(I) \quad \text{for } a < x \leqq b.$$

There is a subdivision $D$ of $[a, b]$ such that if $E$ is a refinement of $D$, then $0 \leqq \sum_E [L(I) - \Delta g] < c/8$. For each $I$ in $D$, there is a subdivision $S_I$ of $I$ such that $0 \leqq L(I) - \sum_{S_I} H(J) < c/(8N)$, where $N$ is the number of intervals in $D$, so that

$$0 \leqq \sum_D \sum_{S_I} [L(J) - H(J)] \leqq \sum_D [L(I) - \sum_{S_I} H(J)] < c/8.$$

Now

$$\begin{aligned} 0 \leqq h|_a^b - g|_a^b &= \textstyle\sum_D \sum_{S_I} [\Delta h - \Delta g] \\ &\leqq \left| \textstyle\sum_D \sum_{S_I} [\Delta h - H(J)] \right| + \left| \textstyle\sum_D \sum_{S_I} [L(J) - H(J)] \right| \\ &\qquad + \left| \textstyle\sum_D \sum_{S_I} [L(J) - \Delta g] \right| \\ &< \left| \textstyle\sum_Q [\Delta h - H(J)] \right| + c/8 + c/8, \end{aligned}$$

where $Q$ is the set (if any) of all $J$ such that for some $I$ in $D$, $J$ is in $S_I$ and $\Delta h \neq H(J)$, so that $H(J) = W\Delta m$. Therefore

$$W \textstyle\sum_Q \Delta m = \sum_Q H(J) \leqq \sum_Q \Delta h \leqq h|_a^b,$$

so that $\sum_Q \Delta m \leqq (h|_a^b)/W < k$, and therefore $\sum_Q \Delta h < c/2$. Therefore

$$0 \leqq \textstyle\sum_Q [\Delta h - H(J)] \leqq \sum_Q \Delta h < c/2,$$

so that $h|_a^b - g|_a^b < c/2 + c/8 + c/8 = 3c/4 < c$.

## 3. A convergence theorem

We now prove a theorem about the convergence of the integral $\int_{[a,b]} (dg)^p (dm)^{1-p}$ as $p \to 1$ for $0 < p < 1$.

THEOREM 2. *If each of $g$ and $m$ is a real-valued nondecreasing function on the number interval $[a, b]$, and $g$ is such that for some positive number $W$, $\Delta g \leqq W\Delta m$ for each subinterval $I$ of $[a, b]$, then*

$$\int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| \to 0 \quad as \quad p \to 1$$

*for* $0 < p < 1$.

*Proof.*[1] We first demonstrate the theorem for the case that $\Delta g \leqq \Delta m$ for each subinterval $I$ of $[a, b]$.

Suppose $0 < p < 1$.

If $I$ is a subinterval of $[a, b]$, then

$$0 \leqq (\Delta g)^p (\Delta m)^{1-p} - \Delta g \leqq p\Delta g + (1 - p)\Delta m - \Delta g = (1 - p)(\Delta m - \Delta g).$$

Therefore if $E$ is a subdivision of the subinterval $[u, v]$ of $[a, b]$, then $0 \leqq \sum_E [(\Delta g)^p (\Delta m)^{1-p} - \Delta g] \leqq (1 - p) \sum_E [\Delta m - \Delta g] = (1 - p)[m|_u^v - g|_u^v]$, so that

$$0 \leqq \int_{[u,v]} [(dg)^p (dm)^{1-p} - dg] \leqq (1 - p)[m|_u^v - g|_u^v].$$

If $D$ is a subdivision of $[a, b]$, then

[1] The author wishes to thank the referee for valuable suggestions incorporated in the paper in general and this proof in particular.

$$\sum\nolimits_D \left| \Delta g - \int_I (dg)^p (dm)^{1-p} \right| = \sum\nolimits_D \int_I [(dg)^p (dm)^{1-p} - dg] \\ \leqq \sum\nolimits_D (1 - p)[\Delta m - \Delta g],$$

so that

$$\int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| \leqq (1 - p)[m|_a^b - g|_a^b] \to 0 \quad \text{as} \quad p \to 1.$$

We now prove the theorem for the general case.

If $0 < p < 1$, then

$$\int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right|$$

$$= W \int_{[a,b]} \Big| d(g/W) - \int_I [d(g/W)]^p (dm)^{1-p}$$

$$+ [1 - W^{p-1}] \int_I [d(g/W)]^p (dm)^{1-p} \Big|$$

$$\leqq W \int_{[a,b]} \left| d(g/W) - \int_I [d(g/W)]^p (dm)^{1-p} \right|$$

$$+ W \mid 1 - W^{p-1} \mid \int_{[a,b]} [d(g/W)]^p (dm)^{1-p}$$

$$\to W0 + W \mid 1 - 1 \mid (g|_a^b)/W \quad \text{as} \quad p \to 1.$$

Therefore

$$\int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| \to 0 \quad \text{as} \quad p \to 1.$$

## 4. The characterization theorem

In this section we prove the second theorem mentioned in the introduction.

THEOREM 3. *If each of $h$ and $m$ is a real-valued nondecreasing function on the number interval $[a, b]$, then the following four statements are equivalent:*

(1) *If $H$ is a real-valued bounded function of subintervals of $[a, b]$ such that $\int_{[a,b]} H(I)\, dm$ exists, then $\int_{[a,b]} H(I)\, dh$ exists.*

(2) $\int_{[a,b]} (dh)^p (dm)^{1-p} \to h|_a^b$ *as $p \to 1$ for $0 < p < 1$.*

(3) $\int_{[a,b]} \mid dh - \int_I (dh)^p (dm)^{1-p} \mid \to 0$ *as $p \to 1$ for $0 < p < 1$.*

(4) *$h$ is absolutely continuous with respect to $m$.*

*Proof.* We first show that (4) implies (3). Suppose $c$ is a positive number. By Lemma 2, there are a real-valued function $g$ on $[a, b]$ and a number $W > 0$ such that if $I$ is a subinterval of $[a, b]$, then $0 \leqq \Delta g \leqq \min \{\Delta h, W\Delta m\}$ and $h|_a^b - g|_a^b < c/8$.

By Theorem 2, there is a positive number $k < 1$ such that if $k < p < 1$, then

$$\int_{[a,b]} \left| dg - \int_I (dg)^p (dm)^{1-p} \right| < c/8$$

and such that furthermore $(c/8)^p < c/4$ and $(m|_a^b)^{1-p} < 2$, so that if $D$ is a subdivision of $[a, b]$, then

$$\sum_D \left| \Delta h - \int_I (dh)^p (dm)^{1-p} \right|$$

$$\leqq \sum_D | \Delta h - \Delta g | + \sum_D \left| \Delta g - \int_I (dg)^p (dm)^{1-p} \right|$$

$$+ \sum_D \left| \int_I (dh)^p (dm)^{1-p} - \int_I (dg)^p (dm)^{1-p} \right|$$

$$\leqq c/8 + c/8 + \sum_D \int_I (dh - dg)^p (dm)^{1-p}.$$

By Hölder's inequality

$$c/8 + c/8 + \sum_D \int_I (dh - dg)^p (dm)^{1-p} \leqq c/4 + \sum_D (\Delta g - \Delta h)^p (\Delta m)^{1-p}$$

$$\leqq c/4 + (h|_a^b - g|_a^b)^p (m|_a^b)^{1-p} < c/4 + (c/8)^p(2) < c/4 + (c/4)(2),$$

so that

$$\int_{[a,b]} \left| dh - \int_I (dh)^p (dm)^{1-p} \right| \leqq (3c)/4 < c.$$

Therefore (4) implies (3).

It is obvious that (3) implies (2).

We now show that (2) implies (4). Suppose that (2) is true, but that $h$ is not absolutely continuous with respect to $m$. We see that $m|_a^b \neq 0$.

There are a number $W > 0$ and a sequence $\{D_k\}_{k=1}^{\infty}$ of proper subsets of subdivisions of $[a, b]$ such that $\sum_{D_n} \Delta m \to 0$ as $n \to \infty$, but for each positive integer $n$, $\sum_{D_n} \Delta h \geqq W$. We see that for each positive integer $n$, there is a subset $C_n$ of a subdivision of $[a, b]$ such that $D_n$ and $C_n$ are mutually exclusive and $D_n + C_n$ is a subdivision of $[a, b]$.

If $n$ is a positive integer, then $\sum_{C_n} \Delta h = h|_a^b - \sum_{D_n} \Delta h \leqq h|_a^b - W$, so that if $0 < p < 1$, then

$$\int_{[a,b]} (dh)^p (dm)^{1-p} \leqq \sum_{D_n} (\Delta h)^p (\Delta m)^{1-p} + \sum_{C_n} (\Delta h)^p (\Delta m)^{1-p}$$

$$\leqq (\textstyle\sum_{D_n} \Delta h)^p (\sum_{D_n} \Delta m)^{1-p} + (\sum_{C_n} \Delta h)^p (\sum_{C_n} \Delta m)^{1-p}$$

$$\leqq (h|_a^b)^p (\textstyle\sum_{D_n} \Delta m)^{1-p} + (h|_a^b - W)^p (m|_a^b - \sum_{D_n} \Delta m)^{1-p}$$

$$\to (h|_a^b)^p (0) + (h|_a^b - W)^p (m|_a^b - 0)^{1-p} \quad \text{as} \quad n \to \infty;$$

so that

$$\int_{[a,b]} (dh)^p (dm)^{1-p} \leqq (h|_a^b - W)^p (m|_a^b)^{1-p} \to h|_a^b - W \quad \text{as} \quad p \to 1.$$

Therefore, since $\int_{[a,b]} (dh)^p (dm)^{1-p} \to h|_a^b$ as $p \to 1$ for $0 < p < 1$, it follows that $h|_a^b \leqq h|_a^b - W$, a contradiction. Therefore (2) implies (4).

We now show that (3) implies (1). Suppose $H$ is a real-valued bounded function of subintervals of $[a, b]$ such that $\int_{[a,b]} H(I)\,dm$ exists.

If $c$ is a positive number, then there is a positive number $p < 1$ such that

$$\int_{[a,b]} \left| dh - \int_I (dh)^p\,(dm)^{1-p} \right| < c.$$

By Theorem A, $\int_{[a,b]} H(I) \int_I (dh)^p (dm)^{1-p}$ exists.

Therefore, by Lemma 1, $\int_{[a,b]} H(I)\,dh$ exists. Therefore (3) implies (1).

Finally, we show that (1) implies (4). Suppose (1) is true but that $h$ is not absolutely continuous with respect to $m$.

We first show that if $a \leqq y < b$, and $m$ is continuous from the right at $y$, then so is $h$. Suppose this is not true. Then there is a sequence of numbers $\{y_k\}_{k=1}^{\infty}$ of $(y, b]$ such that $y_n - y + m(y_n) - m(y) \to 0$ as $n \to \infty$, but for some number $V > 0$, and each positive integer $n$, $h(y_n) - h(y) \geqq V$. There is a real-valued function $H$ of subintervals of $[a, b]$ such that

$$\begin{aligned} H(I) &= 1 \quad \text{if } I \text{ is } [y, y_n] \text{ for some } n,\\ &= 0 \quad \text{otherwise.} \end{aligned}$$

We see that $\int_{[a,b]} H(I)\,dm = 0$. However, if $D$ is a subdivision of $[a, b]$, then there are refinements $E$ and $E'$ of $D$ such that for some $N$, $[y, y_N]$ is in $E$ and for no $n$ is $[y, y_n]$ in $E'$, so that

$$\left| \sum_E H(I)\Delta h - \sum_{E'} H(I)\Delta h \right| = h(y_N) - h(y) \geqq V,$$

so that $\int_{[a,b]} H(I)\,dh$ does not exist, a contradiction.

In a similar manner it follows that if $a < y \leqq b$, and $m$ is continuous from the left at $y$, then so is $h$.

Now from the supposition that $h$ is not absolutely continuous with respect to $m$ it follows that there are a number $W > 0$ and a sequence $\{D_k\}_{k=1}^{\infty}$ of subdivisions of $[a, b]$ such that for each positive integer $n$, the following conditions are satisfied:

(a) Each interval of $D_{n+1}$ is a proper subset of some interval of $D_n$.

(b) There is a subset $E_n$ of $D_n$ such that $\sum_{E_n} \Delta h \geqq W$ and $\sum_{E_n} \Delta m < 2^{-n}$.

(c) $\max\{v - u \text{ for } [u, v] \text{ in } D_n\} < 1/n$.

There is a real-valued function $H$ of subintervals of $[a, b]$ such that

$$\begin{aligned} H(I) &= 1 \quad \text{if } I \text{ is in } E_n \text{ for some } n,\\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Suppose $c$ is a positive number. There is a positive integer $N$ such that $2^{1-N} < c$. If $E$ is a refinement of $D_N$, and $I$ is in $E$ and $E_n$ for some $n$, then $n \geqq N$. If we let $E'$ denote the set (if any) of all $I$ in $E$ and $E_n$ for some $n$, it follows that

$$0 \leqq \sum_E H(I)\Delta m = \sum_{E'} \Delta m \leqq \sum_{k=N}^{\infty} 2^{-k} = 2^{1-N} < c.$$

Therefore $\int_{[a,b]} H(I)\,dm = 0$.

Now suppose $D$ is a subdivision of $[a, b]$.

Let $M$ denote the set of all $x$ such that for some $[u, v]$ in $D$, $x$ is $u$ or $v$.

For each positive integer $n$, let $E_n^*$ denote the set (if any) of all $[u, v]$ in $E_n$ such that for some $x$ in $M$, $u < x < v$.

Let $M^*$ denote the set (if any) of all $x$ in $M$ such that for each positive integer $n$, there is a positive integer $w > n$ such that for some $[u, v]$ in $E_w^*$, $u < x < v$.

For each positive integer $n$, let $E_n^{**}$ denote the set (if any) of all $[u, v]$ in $E_n$ such that for some $x$ in $M^*$, $u < x < v$.

Now, since for each positive integer $n$, $\sum_{E_n^{**}} \Delta m \leqq \sum_{E_n} \Delta m < 2^{-n} \to 0$ as $n \to \infty$, it follows that $m$ is continuous at each number of $M^*$, so that $h$ is continuous at each number of $M^*$, and therefore $\sum_{E_n^{**}} \Delta h \to 0$ as $n \to \infty$.

There is a positive integer $N$ such that if $x$ is in $M$ and not in $M^*$, and $n$ is a positive integer $\geqq N$, then there is no $[u, v]$ in $E_n$ such that $u < x < v$; so that if $I$ is in $E_n^*$, then $I$ is in $E_n^{**}$, and therefore $E_n^*$ is $E_n^{**}$.

There is a positive integer $n > N$ such that $\sum_{E_n^*} \Delta h = \sum_{E_n^{**}} \Delta h < W/2$, so that $E_n^*$ is a proper subset of $E_n$, and $E_n - E_n^*$ is therefore a subset of some refinement $S$ of $D$, so that $\sum_S H(I)\Delta h \geqq \sum_{E_n - E_n^*} \Delta h > W/2$.

Now the set of all $x$ such that for some $n$ and some $[u, v]$ in $E_n$, $x$ is $u$ or $v$, is countable. Therefore, since each interval $I$ of $D$ is uncountable, there is a refinement $T$ of $D$ such that for no $n$ is $I$ in $T$ and $E_n$. This implies that $\sum_T H(I)\Delta h = 0$, so that $\left| \sum_S H(I)\Delta h - \sum_T H(I)\Delta h \right| > W/2$.

Therefore $\int_{[a,b]} H(I)\, dh$ does not exist, a contradiction. Therefore (1) implies (4).

Therefore (1), (2), (3), and (4) are equivalent.

## References

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