ON AUTOMORPHIC FORMS OF NEGATIVE DIMENSION[1]

BY

JOSEPH LEHNER

1.[2] An automorphic form of positive dimension on an H-group is completely determined by its principal parts at the parabolic cusps; a form of zero dimension is determined up to an additive constant. The classical circle method of Hardy-Ramanujan-Rademacher-Zuckerman yields explicit expressions for the Fourier coefficients of forms of nonnegative dimension on the modular group and on certain of its subgroups. Recently we showed how this method could be modified to cover all H-groups, although for forms of zero dimension the Fourier coefficients are given only up to a bounded error term [1, Ch. IX], [2], [3], [4].

The situation is quite different when we consider automorphic forms of *negative* dimension. There may exist nonconstant forms of negative dimension that are regular everywhere including the cusps; in particular, there may exist *cusp* forms, that is, forms which vanish at the parabolic cusps. Hence in general the Fourier coefficients of a form of negative dimension can only be determined by its principal parts up to the order of magnitude of the Fourier coefficients of an everywhere regular form.

It is the purpose of this paper to show that the circle method suffices to determine the Fourier coefficients of forms of negative dimension also, insofar as these are determined by their principal parts. The circle method is thus revealed as a uniform method, valid for all dimensions, for extracting all possible information from the principal parts of an automorphic form. As we remark at the end of this section, the same statement holds for automorphic *integrals*.

Let

$$g(m, r) = \begin{cases} m^{r/2} & \text{if } 0 < r < 2, \\ m \log m & \text{if } r = 2, \\ m^{r-1} & \text{if } r > 2. \end{cases} \tag{1}$$

If $c_m^{(k)}$ is the m^{th} Fourier coefficient of an everywhere regular form $G(\tau) \in \{\Gamma, -r, v\}$, i.e., of dimension $-r$ and multiplier v, it is known that ([5], cf. also [6])

$$c_m^{(k)} = O(g(m, r)). \tag{2}$$

For $r > 2$ this estimate is best possible, as the Eisenstein series show. At any rate (2) is the best result presently obtainable for all H-groups by the

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[2] For definitions and notation, cf. [1, Chapters VIII, IX]; also Section 2, below.

circle method, and so we cannot expect, by this method, to determine the Fourier coefficients of a general form F of dimension $-r$ more precisely than (2). An exception to this statement arises, however, in case F, after subtraction of its principal parts, vanishes at the cusps. Then we can reduce the error term to the order of magnitude of the coefficients of a cusp form, which has long been known to be $O(m^{r/2})$. (Cf. [1, Ch. VIII, 3J].) Moreover, this situation always occurs when the multiplier system of the group is such that certain parameters κ_j, defined in Section 2, are all positive. In that case an everywhere regular automorphic form is already a cusp form.

We shall now state our results. Let $F \in \{\Gamma, -r, v\}$, that is, $F(\tau)$ is a meromorphic function in the upper half-plane $\mathfrak{H}$, it satisfies the transformation equation[3]

$$F(M\tau) = v(M)(c\tau + d)^r F(\tau), \qquad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{3}$$

and it tends to a limit (which may be infinite) on approach to a parabolic cusp of Γ. Here r is real, v is independent of τ, and $|v| = 1$. Suppose, moreover, F is regular in $\mathfrak{H}$. At each finite parabolic cusp p_k there is an expansion[4]

$$\begin{aligned} (\tau - p_k)^r e(-\kappa_k A_k \tau/\lambda_k)F(\tau) &= f_k(t_k), \qquad t_k = e(A_k \tau/\lambda_k), \\ f_k(t) &= \textstyle\sum_{m=-\mu_k}^{\infty} a_m^{(k)} t^m, \qquad \mu_k \geqq 0, \end{aligned} \tag{4}$$

valid for $\tau \in \mathfrak{H}$ (or $|t| < 1$). The $\{a_m^{(k)}\}$ are called the Fourier coefficients of F at p_k. A similar expansion holds at $p_0 = i\infty$; the factor $(\tau - p_k)^r$ is to be replaced by 1. This is of course the usual Fourier series.

THEOREM 1. *If $F(\tau) \in \{\Gamma, -r, v\}$ is regular in $\mathfrak{H}$ and $r > 0$, then*

$$\begin{aligned} a_m^{(k)} = 2\pi i^{-r} \lambda_k^{-1} \sum_{j=1}^{s} \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \sum_{\substack{c \in C_{jk} \\ 0<c<m^{1/2}}} c^{-1} A(c, m_k, \nu_j) \\ \times M(c, m_k, \nu_j, r) + O(g(m, r)), \qquad m > 0, \quad k = 1, \cdots, s. \end{aligned} \tag{5}$$

When $r > 2$, the sum in the right member can be extended to $c < \infty$ without affecting the error term.

The new symbols appearing in this theorem are defined in Section 2 and (40)–(42).

If F is regular in $\mathfrak{H}$, all principal parts vanish and we get (2).

THEOREM 2. *If*

$$\kappa_j > 0, \qquad j = 1, 2, \cdots, s,$$

[3] The branch of $(c\tau + d)^r$ is fixed by restricting the argument to the range $-\pi \leqq \arg < \pi$.

[4] Cf. [1, p. 273, formulas (14), (14a); also Note 30].

then

$$a_m^{(k)} = {}^*a_m^{(k)} + O(m^{r/2}), \qquad r > 0 \tag{6}$$

where ${}^*a_m^{(k)}$ *is the finite sum in* (5).

For $r > 2$, these results are available from the Petersson theory.

We turn now to *integrals*. An integral is an analytic function $f(\tau)$ that is meromorphic in $\mathfrak{H}$ and satisfies the functional equation

$$f(M\tau) = v(M)f(\tau) + C_M, \qquad M \in \Gamma, \quad \tau \in \mathfrak{H} \tag{7}$$

where C_M, the *period*, is independent of τ. We consider only integrals that are regular in $\mathfrak{H}$. It is clear that the derivative $f'(\tau)$ belongs to $\{\Gamma, -2, v\}$. The Fourier coefficients of f' are therefore given by Theorems 1 and 2, and from them we obtain by integration the Fourier coefficients of f. Hence we have

THEOREM 3. *Let*

$$f(\tau) = b^{(k)}A_k\tau + \sum_{m=-\mu_k}^{\infty} b_m^{(k)} e((m + \kappa_k)A_k\tau/\lambda_k), \quad k = 1, \cdots, s \tag{8}$$

be the Fourier expansions of the integral $f(\tau)$. *Then*

$$b_m^{(k)} = {}^*a_m^{(k)}/2\pi i m_k + O(\log m), \qquad m > 0, \quad m_k = (m + \kappa_k)/\lambda_k. \tag{9}$$

If $\kappa_j > 0, j = 1, \cdots, s$, *then*

$$b_m^{(k)} = {}^*a_m^{(k)}/2\pi i m_k + O(1), \qquad m > 0. \tag{10}$$

The integral f is said to be of the *first kind* if it is regular everywhere, including the cusps . Necessary and sufficient for this to be the case is that

$$b^{(k)} = 0, \qquad b_m^{(k)} = 0, \quad m = -1, \cdots, -\mu_k, \qquad k = 1, \cdots, s.$$

Then f' is a cusp form, and its Fourier coefficients $a_m^{(k)}$ have the estimate

$$a_m^{(k)} = O(m).$$

Hence

$$b_m^{(k)} = O(1).$$

THEOREM 4. *The Fourier coefficients* $b_m^{(k)}$ *of an integral of the first kind are bounded.*

2. We shall make use of the notation and results of [1], which we summarize here. Let Γ be an H-group, and let $p_0 = i\infty, p_1, p_2, \cdots$ be the parabolic cusps of Γ. Define

$$A_j = \begin{pmatrix} 0 & -1 \\ 1 & -p_j \end{pmatrix}, \quad j > 0; \qquad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11}$$

If P_j generates the cyclic subgroup of Γ each element of which fixes p_j, so do $-P_j$, $\pm P_j^{-1}$. Then $A_j P_j^{-1} A_j^{-1}$ is a translation, and so also for the other generators. Denote by P_j that generator for which

$$A_j P_j A_j^{-1} = \begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix}, \qquad j > 0$$

has $\lambda_j > 0$. This also defines λ_j.

Let v be a multiplier system belonging to Γ and the dimension $-r$. Define κ_j by

$$e(\kappa_j) = v(P_j), \qquad 0 \leqq \kappa_j < 1.$$

Here and throughout

$$e(u) = \exp 2\pi i u.$$

Let $\tilde{R}$ be a fundamental region of Γ touching the real axis only in finite points and such that each parabolic cycle consists of a single vertex [1, p. 270]. We denote the cycles in $\tilde{R}$ by $p_1, \cdots, p_s$; all p_i are finite. Since Γ is an H-group, s is finite.

Let $F \in \{\Gamma, -r, v\}$ be regular in the upper half-plane $\mathcal{H}$. In the expansion (4) of F the finite sum

$$\sum_{n=-\mu_k}^{-1} a_n^{(k)} t_k^n,$$

which may be empty, is called the principal part of F at p_k.

Fix k in the range $1 \leqq k \leqq s$. We shall find an asymptotic formula for $\{a_m^{(k)}, m \geqq 1\}$ in terms of the principal parts of F at the cusps p_j, $j = 1, 2, \cdots, s$.

3. For this purpose we put the transformation equation (3) into a different form (cf. [1, Ch. IX, 1B]). Let

$$(12) \qquad M^* = A_j M A_k^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad M \in \Gamma, \quad j = 1, \cdots, s.$$

Note. In an effort to simplify the typography we are using the above notation instead of[5] $\tilde{M} = (a' \; b' \mid c' \; d')$ as we did in [1].

Then with the f_j of (4) we have for $j = 1, \cdots, s$

$$(13) \quad f_k(e(w/\lambda_k)) = v^{-1}(M)(cw + d)^{-r} e(\kappa_j w'/\lambda_j - \kappa_k w/\lambda_k) f_j(e(w'/\lambda_j)),$$

with

$$w = A_k \tau, \qquad w' = M^* w$$

provided $b < 0$, $d > 0$. Moreover, (13) is valid for $j = 0$ (i.e., $p_0 = i\infty$) and *all* $M \in \Gamma$ if we admit a factor of absolute value 1 in the right member.

Cauchy's theorem applied to (4) gives

$$(14) \qquad \lambda_k a_m^{(k)} = \int_L f_k(e(w/\lambda_k)) e(-mw/\lambda_k)\, dw,$$

where L is any horizontal line segment of length λ_k lying in $\mathcal{H}$.

[5] We sometimes write $(a \; b \mid c \; d)$ for the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The particular L that we use, and its partition, are described in [1, Ch. IX, 2D–2F]. For $h > 0$ we construct the image $K(M^*)$ of the horizontal line $\operatorname{Im} \tau = h$ by M^{*-1}. This is a circle of diameter $1/c^2h$ tangent to the real axis at $-d/c$:

$$(15) \qquad K(M^*) : |\tau - (d/c + i/2c^2h)| = 1/2c^2h.$$

Let $N > 0$ be arbitrary, and suppose $K(M^*)$ cuts the horizontal line $l_N : \operatorname{Im} \tau = N^{-2}$; this implies $c^2h < N^2$. The intersection of $K(M^*)$ and l_N is an interval

$$(16) \qquad I(M^*) : (-d/c + iN^{-2} - \vartheta, -d/c + iN^{-2} + \vartheta),$$

where

$$(17a) \qquad \vartheta(M^*) = \vartheta = N^{-1}c^{-1}h^{-1/2}(1 - c^2h/N^2)^{1/2}.$$

Hence

$$(17b) \qquad \vartheta < 1/cNh^{1/2}.$$

If h exceeds a certain positive constant depending only on Γ, the circles $K(M^*)$ do not intersect. The system of circles is periodic modulo λ_k, and we can select an interval on l_N of length λ_k that meets a complete set of circles belonging to one period. Call this interval $\mathbf{k}$.

The circles meeting $\mathbf{k}$ can be characterized. Define

$$(18) \qquad \begin{aligned} \mathbf{T} &= \{M^* \mid 0 < d \leqq c\lambda_k, -c\lambda_j \leqq a < 0; j = 1, \cdots, s\}, \\ \mathbf{T}_N &= \{M^* \in \mathbf{T} \mid 0 < c < Nh^{-1/2}\}. \end{aligned}$$

A circle $K(M^*)$ cuts $\mathbf{k}$ if and only if $M^* \in \mathbf{T}_N$. Let

$$\mathbf{j} = \bigcup \{I(M^*) \mid M^* \in \mathbf{T}_N\};$$

$\mathbf{j}$ is a proper subset of $\mathbf{k}$. The complement of $\mathbf{j}$ is partitioned by

$$\mathbf{k} - \mathbf{j} = \{I(M') \mid M' \in \mathbf{T}_N^0\}.$$

(In [1] we wrote $\bar{M}$ instead of M'.) Here $I(M')$ is a finite union of intervals, $M' = MA_k^{-1}$, and $\mathbf{T}_N^0$ is a finite set for each N.

Since the sets of the partition are disjoint, we have

$$\sum_{M^* \in \mathbf{T}_N} |I(M^*)| + \sum_{M' \in \mathbf{T}_N^0} |I(M')| = \lambda_k,$$

$|I|$ denoting the measure of I.

For future reference we note the following. Partition $\mathbf{T}_N$ into the sets

$$(19) \qquad \begin{aligned} \mathbf{T}_N^{(1)} &= \{M^* \mid 0 < c < 2^{-1}Nh^{-1/2}\}, \\ \mathbf{T}_N^{(2)} &= \{M^* \mid 2^{-1}Nh^{-1/2} \leqq c < Nh^{-1/2}\}. \end{aligned}$$

Then from (17a) we get

$$(20) \qquad \vartheta > 1/2cNh^{1/2}, \qquad M^* \in \mathbf{T}_N^{(1)}.$$

4. In the estimations of the later sections we shall need the following result.

LEMMA. *Let r be real and $\beta > \alpha > 0$. Let $\sum_{\alpha,\beta}$ denote a sum over those pairs (c, d) for which $M^* = (a\ b \mid c\ d) \in \mathbf{T}$ and*

$$\alpha \leqq c \leqq \beta.$$

Then with $B_i = B_i(\Gamma, r)$ we have

$$\begin{aligned} \textstyle\sum_{\alpha,\beta} c^{-r} &< B_1(B_2\beta^{2-r} - \alpha^{2-r}) \quad \text{if} \quad r < 2, \\ &< B_3 + 2\log \beta/\alpha \qquad \text{if} \quad r = 2, \\ &< B_4(\alpha^{2-r} - B_5\beta^{2-r}) \quad \text{if} \quad r > 2. \end{aligned} \tag{21}$$

Proof. The elements of $\mathbf{T}$ fall into s classes. It is therefore sufficient to prove the result for a given class, say $M^* \in A_1\Gamma A_k^{-1} = \Gamma_1$. The system of matrices Γ_1, though not a group, has many properties of a discontinuous group, of which the most important for us is the following: *There is a disk K_1 such that no two images of K_1 by distinct elements of Γ_1 intersect.* The proof is immediate. Indeed, the group Γ admits a disk with this property, say K; we have only to select K lying entirely in the interior of a fundamental region of Γ. Set $K_1 = A_k K$. Then $A_1 M_1 A_k^{-1} K_1 = A_1 M_1 K$ cannot meet $A_1 M_2 A_k^{-1} K_1 = A_1 M_2 K$; otherwise $M_1 K$ would meet $M_2 K$.

Consider $M^* = (a\ b \mid c\ d)$ with $c \neq 0$; let $\tau_0 = x_0 + iy_0 \in K_1$. We have by (18),

$$|\operatorname{Re} M^*\tau_0| = \left|\frac{a}{c} - \left(\frac{d}{c} + x_0\right)\frac{1}{(cx_0 + d)^2 + c^2y_0^2}\right| \leqq \lambda_j + \frac{\lambda_k + x_0}{\tilde{c}^2 y_0^2} = B_6,$$

where [1, Ch. VIII, 2D]

$$0 < \tilde{c} = \min\left\{c \,\middle|\, \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix} \in \mathbf{T},\ c > 0\right\}. \tag{22}$$

Hence the real parts of all $M^*\tau_0$ are bounded. Moreover, the diameters of K_1 and its images are also bounded, for the same noneuclidean area means smaller euclidean area near the real axis. Thus there is a B_7 such that the strip $|x| \leqq B_7$ contains the images of K_1 by all $M^* \in \Gamma_1$ with $c \neq 0$.

Write $M^*\tau = x' + iy'$. It is easily checked that for $\tau \in K_1$,

$$B_8\beta^{-2} \leqq B_8 c^{-2} \leqq y' \leqq B_9 c^{-2} \leqq B_9\alpha^{-2}.$$

Hence the region

$$D : |x| \leqq B_7, \quad B_8\beta^{-2} \leqq y \leqq B_9\alpha^{-2}$$

contains all images of K_1 by transformations M^* appearing in the sum under consideration.

Let $r < 2$. Then

$$\sum_{\alpha,\beta} \int_{M^*K_1} y^{r/2-2}\,dx\,dy < \iint_D y^{r/2-2}\,dx\,dy = B_{10}(B_{11}\beta^{2-r} - \alpha^{2-r}). \tag{23}$$

On the other hand we find, remembering the invariance of $y^{-2}\,dx\,dy$ under M^*,

$$\int_{M^*K_1} = \int_{K_1} |c\omega + d|^{-r} v^{r/2-2}\,du\,dv,$$

with $\tau = M^*\omega$, $\omega = u + iv$. Since

$$B_{12} \leqq |c\omega + d|\cdot c^{-1} \leqq B_{13}, \qquad \omega \in K_1$$

uniformly in M^*, this gives

$$\sum_{\alpha,\beta} \int_{M^*K_1} y^{r/2-2}\,dx\,dy > B_{14} \sum_{\alpha,\beta} c^{-r} \iint_K v^{r/2-2}\,du\,dv$$
$$= B_{15} \sum_{\alpha,\beta} c^{-r}.$$

This concludes the proof when $r < 2$, and the argument in the other cases differs only in the evaluation of the integral in the right member of (23).

COROLLARY. *With $B = B(\Gamma, r)$ we have*

$$\begin{aligned} &\textstyle\sum_{0,\beta} c^{-r} < B\beta^{2-r}, && r < 2,\\ (24)\qquad &\textstyle\sum_{0,\beta} c^{-2} < B(1 + \log\beta), &&\\ &\textstyle\sum_{0,\beta} c^{-r} < B, && r > 2. \end{aligned}$$

In particular the series $\sum_{0,\infty} c^{-r}$, $r > 2$, converges. (*Cf.*[6].)

Put $\alpha = \tilde{c} > 0$ in the lemma, and note from (22) that $\sum_{\tilde{c},\beta} = \sum_{0,\beta}$.

5. We return to (14) and have

$$\begin{aligned} \lambda_k a_m^{(k)} &= \sum_{M^*\in\mathbf{T}} \int_{I(M^*)} f_k(e(w/\lambda_k))e(-mw/\lambda_k)\,dw \\ (25)\qquad &\quad + \sum_{M'\in\mathbf{T}^0} \int_{I(M')} f_k(e(w/\lambda_k))e(-mw/\lambda_k)\,dw = S_1 + S_2, \end{aligned}$$

where we have suppressed the parameter N in $\mathbf{T}_N$ and $\mathbf{T}_N^0$. We recall from (16) that

$$\operatorname{Im} w = N^{-2}, \qquad w \in I(M^*).$$

At this point we introduce the assumption

$$r > 0,$$

that is, we are considering forms of *negative* dimension only. We also assume $m \geqq 1$.

The estimation of S_2 is the same as in [1, Ch. IX, 2G] except at one point:

$$|c''w + d''|^2 = \operatorname{Im} w/\operatorname{Im} w'' \geqq N^{-2}h_1^{-1}, \qquad w'' = MA_k^{-1}w, \quad h_1 > 0.$$

Hence as in [1],

$$(25a) \qquad |S_2| < CN^r \exp(CmN^{-2}),$$

C denoting throughout a general positive constant independent of m and N.

We split up S_1 :

$$S_1 = S_1' + S_1'', \qquad S_1' = \textstyle\sum_{M^* \epsilon \mathbf{T}^{(1)}}, \qquad S_1'' = \textstyle\sum_{M^* \epsilon \mathbf{T}^{(2)}}$$

(cf. (19)). In each integral of S_1'' apply the transformation equation (13) and get

$$S_1'' = \sum_{M^* \epsilon \mathbf{T}^{(2)}} v^{-1}(M) \int_{I(M^*)} (cw + d)^{-r} e(\kappa_j w'/\lambda_j - (m + \kappa_k)w/\lambda_k) \times f_j(e(w'/\lambda_j))\, dw$$

Now with $w = x + iy$, $w' = M^* w = x' + iy'$, we have, since $M^* \epsilon \mathbf{T}^{(2)}$,

$$y = N^{-2}, \qquad y' = y/((cx + d)^2 + c^2y^2) \leqq 1/c^2y \leqq 4h,$$
$$|cw + d|^2 \geqq c^2y^2 \geqq 4^{-1}N^2h^{-1}.$$

Hence[6]

$$(26) \quad |S_1''| \leqq CN^r \exp(CmN^{-2}) \textstyle\sum_{M^* \epsilon \mathbf{T}^{(2)}} |I(M^*)| < CN^r \exp(CmN^{-2}).$$

In S_1', on the other hand, it is necessary to introduce the Fourier series of f_j, for its principal part will make an essential contribution. So we write

$$S_1' = \sum_{M^* \epsilon \mathbf{T}^{(1)}} v^{-1}(M) \sum_{\nu=1}^{\mu_j} a_{-\nu}^{(j)} \int_{I(M^*)} (cw + d)^{-r} e(-\nu_j w' - m_k w)\, dw$$

$$(27) \quad + \sum_{M^* \epsilon \mathbf{T}^{(1)}} v^{-1}(M) \int_{I(M^*)} (cw + d)^{-r} \sum_{n=0}^{\infty} a_n^{(j)} e((n + \kappa_j)w'/\lambda_j - m_k w)\, dw$$

$$= S_{11}' + S_{12}',$$

where in the first sum we replaced n by $-\nu$, and where we have set

$$\nu_j = (\nu - \kappa_j)/\lambda_j, \qquad m_k = (m + \kappa_k)/\lambda_k.$$

We estimate S_{12}'.

On the path of integration we have $y = N^{-2}$, as before. Moreover $y' \geqq h$, for $I(M^*)$ lies within $K(M^*)$, and reference to the lines preceding (15) shows that the interior of $K(M^*)$ is the image by M^{*-1} of the half-plane Im $\tau > h$. Therefore

$$|S_{12}'| \leqq C \exp(CmN^{-2}) \sum_{n=0}^{\infty} |a_n^{(j)}| \exp(-Chn)$$

$$(28) \qquad \cdot \sum_{M^* \epsilon \mathbf{T}^{(1)}} \int_{I(M^*)} |cw + d|^{-r}\, dw$$

$$\leqq C \exp(CmN^{-2}) \sum_{M^* \epsilon \mathbf{T}^{(1)}} \int_{I(M^*)} |cw + d|^{-r}\, dw,$$

[6] We also have $y' \geqq h$—see the paragraph immediately preceding (28). From $h \leqq y' \leqq 4h$ we can conclude that $f_j(e(w'\lambda_j))$ is bounded.

since the infinite series converges to a sum independent of N. Setting $w = -d/c + iN^{-2} + x$, we can write

$$(29) \qquad \int_{I(M^*)} = 2c^{-r} \int_0^{\vartheta} (x^2 + N^{-4})^{-r/2}\, dx,$$

with the ϑ of (17a). The sum over M^* in (28) can be expressed as a sum over c, d, where

$$0 < c < 2^{-1}Nh^{-1/2}, \qquad 0 < d \leqq c\lambda_k .$$

We shall bound the sum of the integrals.

From (29) we get

$$\int_{I(M^*)} < 2c^{-r}\left\{ \int_0^{N^{-2}} N^{2r}\, dx + \int_{N^{-2}}^{\vartheta} x^{-r}\, dx \right\}.$$

There are now two cases to consider; in each case we use the inequality (17b) for ϑ and estimate the sum over c, d by means of (24). The function $g(N^2, r)$ is defined in (1). We call attention to the fact that the case $r = 1$ was incorrectly handled in [5].

(i) $0 < r < 2$.

Since $x^2 + N^{-4} \geqq 2xN^{-2}$ and $-r/2 + 1 > 0$, we have

$$\int_{I(M^*)} < CN^r c^{-r} \int_0^{\vartheta} x^{-r/2}\, dx < CN^r c^{-r} \vartheta^{-r/2+1}$$
$$< CN^{3r/2-1} c^{-r/2-1};$$

since $r/2 + 1 < 2$, this gives

$$\sum_{c,d} \int_{I(M^*)} < CN^{3r/2-1} N^{1-r/2} = CN^r = Cg(N^2, r).$$

(ii) $r \geqq 2$.

$$\sum_{c,d} \int_{I(M^*)} < 2 \sum c^{-r}(N^{2r-2} + C(N^{2r-2} - \vartheta^{-r+1}))$$
$$< C \sum c^{-r} N^{2r-2} < CN^{2r-2} \begin{cases} \log N, & r = 2 \\ 1, & r > 2 \end{cases}$$
$$= Cg(N^2, r).$$

In every case, then, we obtain

$$(30) \qquad \sum_{M^*} \int_{I(M^*)} < Cg(N^2, r), \qquad r > 0.$$

Insertion of this result in (28) yields

$$(31) \qquad | S'_{12} | \leqq Cg(N^2, r) \exp(CmN^{-2}).$$

6. The next step is to treat the integrals of S'_{11} (cf. (27)). Let $w + d/c = iz$, and note that

$$w' = M^*w = a/c - 1/c(cw + d) = a/c + i/c^2z.$$

We then have

$$(32) \qquad \begin{aligned} \int_{I(M^*)} &= i^{1-r} e((m_k d - \nu_j a)/c) \cdot I_c , \\ I_c &= c^{-r} \int_{N^{-2}-i\vartheta}^{N^{-2}+i\vartheta} z^{-r} \exp \{2\pi(m_k z + \nu_j/c^2z)\}\, dz. \end{aligned}$$

In order to handle the many-valued function z^{-r} we cut the plane along the negative real axis and require that $|\arg z| < \pi$. Then

$$(33) \qquad I_c = L_c - \{J_1 + \cdots + J_6\},$$

where L_c is a loop integral that starts from $-\infty$ on the lower side of the real axis, circles about 0 in the positive sense, and ends at $-\infty$ on the upper side. Here

$$J_6 = \int_{-\infty}^{-N^{-2}}, \qquad J_5 = \int_{-N^{-2}}^{-N^{-2}-i\vartheta}, \qquad J_4 = \int_{-N^{-2}-i\vartheta}^{N^{2}-i\vartheta},$$

$$J_3 = \int_{N^{-2}+i\vartheta}^{-N^{-2}+i\vartheta}, \qquad J_2 = \int_{-N^{-2}+i\vartheta}^{-N^{-2}}, \qquad J_1 = \int_{-N^{-2}}^{-\infty},$$

all paths are straight, and the integrands are all the same as in I_c. We must estimate the J's.

We recall that $m_k > 0$. We shall assume $\nu \geqq 1$ so that $\nu_j > 0$. In J_2 and J_5 we have

$$z = -N^{-2} + iy, \qquad \text{Re}\,(1/z) < 0,$$

and this gives

$$|J_2|, |J_5| < c^{-r} \int_0^{\vartheta} (y^2 + N^{-4})^{-r/2}\, dy,$$

which is the same as (29). Hence by (30),

$$(34) \qquad \textstyle\sum_{c,d} |J_2|, \sum_{c,d} |J_5| < Cg(N^2, r).$$

On the path of J_3, J_4 we have $z = x \pm i\vartheta$, respectively, with

$$\text{Re}\, z = x < N^{-2}, \qquad \text{Re}(1/c^2z) = x/c^2(x^2 + \vartheta^2) < N^{-2}/c^2 \cdot 4^{-1}c^{-2}N^{-2}h^{-1} = 4h.$$

Hence

$$\begin{aligned} |J_3|, |J_4| &< 2c^{-r} \exp(CmN^{-2}) \int_0^{N^{-2}} (x^2 + \vartheta^2)^{-r/2}\, dx \\ &< C \exp(CmN^{-2}) \cdot c^{-r}N^{-2}\vartheta^{-r}, \end{aligned}$$

which, because of (20) and (24), yields

$$(35) \qquad \textstyle\sum_{c,d} |J_3|, \ \sum_{c,d} |J_4| < CN^r \exp(CmN^{-2}).$$

Next,[7]

$$J_1 + J_6 = c^{-r}\cdot 2i \sin \pi r \int_{N^{-2}}^{\infty} z^{-r} \exp\{-2\pi(m_k z + \nu_j/c^2 z)\}\, dz,$$

$$| J_1 + J_6 | \leqq 2c^{r-2} \int_0^{N^2/c^2} y^{r-2} \exp\{-2\pi(m_k/c^2 y + \nu_j y)\}\, dy.$$

Later we shall make the choice

$$N = 2(mh)^{1/2}; \tag{36}$$

hence we have, since $c < 2^{-1}Nh^{-1/2}$,

$$| J_1 + J_6 | < Cc^{r-2} \int_0^{\infty} y^{r-2} \exp\{-2\pi(C/y + Cy)\}\, dy = Cc^{r-2},$$

and so by (21),

$$\textstyle\sum_{c,d} | J_1 + J_6 | < CN^{2-(2-r)} = CN^r. \tag{37}$$

Finally the loop integral equals [7, p. 181]

$$L_c = 2\pi i c^{-1}(m_k/\nu_j)^{(r-1)/2}\, I_{r-1}(4\pi \sqrt{\nu_j\, m_k}/c), \tag{38}$$

with I_{r-1} the Bessel function of the first kind of pure imaginary argument. Combining all results from (25) on, we get

$$\begin{aligned} \lambda_k\, a_m^{(k)} = 2\pi i^{-r} \textstyle\sum_{M^* \epsilon \mathrm{T}(1)} v^{-1}(M^*) e((m_k\, d - \nu_j\, a)/c) & \\ \times\, c^{-1}(m_k/\nu_j)^{(r-1)/2}\, I_{r-1}(4\pi \sqrt{\nu_j\, m_k}/c) & \\ + \, Cg(N^2, r) \exp(CmN^{-2}), \qquad m > 0. & \end{aligned} \tag{39}$$

7. The right member of (39) can be simplified somewhat [1, Ch. IX, 2K]. Define

$$\begin{aligned} C_{jk} &= \left\{ c \;\middle|\; \begin{pmatrix} \cdot & \cdot \\ c & \cdot \end{pmatrix} \epsilon\, A_j \Gamma A_k^{-1} \right\}, \\ D_c = D_c(j, k) &= \left\{ d \;\middle|\; \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} \epsilon\, A_j \Gamma A_k^{-1},\, 0 < d \leqq c\lambda_k \right\}. \end{aligned} \tag{40}$$

The summation over M^* can now be written

$$\sum_{M^*} = \sum_{j=1}^{s} \sum_{\substack{c \epsilon C_{jk} \\ 0 < c < 2^{-1}h^{-1/2}N}} \sum_{d \epsilon D_c} .$$

The summation over d can be carried out by defining

$$A(c, m_k, \nu_j) = \textstyle\sum_{d \epsilon D_c} v^{-1}(M) e((m_k\, d - \nu_j\, a)/c), \quad M = A_j^{-1} M^* A_k. \tag{41}$$

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[7] If r is an integer, $J_1 + J_6 = 0$, since the integrand is single-valued. It is unnecessary to cut the plane in this case.

Also let

$$M(c, \sigma, \rho, r) = (\sigma/\rho)^{(r-1)/2} I_{r-1}(4\pi \sqrt{\sigma\rho}/c), \qquad \sigma > 0. \tag{42}$$

Inserting these new notations in (39) and fixing N as in (36), we obtain Theorem 1.

We observe that the sum on c in the right member of (5) can be extended over all $c \in C_{jk}$, $c > 0$, provided $r > 2$. Indeed, the elementary estimate

$$I_{r-1}(u) < C|u|^{r-1}, \qquad |u| < u_0, \quad r > 1$$

combined with (24), shows the sum over $c \geqq \sqrt{m}$ to be $O(m^{r-1})$.

8. If all κ_j, $j = 1, \cdots, s$ are > 0 rather than only $\geqq 0$, we can improve the error term to $O(m^{r/2})$, as in Theorem 2. Obviously we need consider only $r \geqq 2$. It is necessary to replace all $O(N^{2r-2})$ or $O(N^{2r-2} \log N)$ terms by $O(N^r)$.

The first such term occurs in the estimation of S'_{12} (cf. (27)). Since $\kappa_j > 0$, we can replace the integral in (28) by

$$\int_{I(M^*)} |cw + d|^{-r} \exp(-2\pi\kappa_j y'/\lambda_j)\, dw$$
$$= 2c^{-r} \int_0^{\vartheta} (x^2 + N^{-4})^{-r/2} \exp\left\{ -2\pi \frac{\kappa_j}{N^2c^2\lambda_j} \frac{1}{x^2 + N^{-4}} \right\} dx.$$

Replace the integrand by its maximum:

$$\int_{I(M^*)} < 2c^{-r}\vartheta(CN^{-2}c^{-2})^{-r/2} < CN^{r-1}c^{-1}.$$

Hence

$$\sum_{c,d} \int_{I(M^*)} < CN^{r-1} \cdot N = CN^r,$$

as promised.

The second and last error term requiring improvement arises in connection with J_2 and J_5—cf. (34)—and is handled in the same way. This completes the proof of Theorem 2.

Note added in proof. The division of S_1 into S'_1 and S''_1 is unnecessary (cf. the lines following (25a)). The estimates used in the treatment of S'_1 apply also to S''_1.

References

1. J. Lehner, *Discontinuous groups and automorphic functions*, Amer. Math. Soc. Mathematical Surveys, no. 8, 1964.
2. ———, *The Fourier coefficients of automorphic forms belonging to a class of horocyclic groups*, Michigan Math. J., vol. 4 (1957), pp. 265–279.
3. ———, *The Fourier coefficients of automorphic forms on horocyclic groups, II*, Michigan Math. J., vol. 6 (1959), pp. 173–193.

4. ———, *The Fourier coefficients of automorphic forms on horocyclic groups, III*, Michigan Math. J., vol. 7 (1960), pp. 65–74.
5. ———, *Magnitude of the Fourier coefficients of automorphic forms of negative dimension*, Bull. Amer. Math. Soc., vol. 67 (1961), pp. 603–606.
6. H. Petersson, *Über Betragmittelwerte und die Fourier-Koeffizienten der ganzen automorphen Formen*, Arch. Math., vol. 9 (1958), pp. 176–182.
7. G. N. Watson, *A treatise on the theory of Bessel functions*, 2nd ed., Cambridge, The University Press, 1944.

Michigan State University
East Lansing, Michigan
University of Maryland
College Park, Maryland