

OBSTRUCTIONS TO IMPOSING DIFFERENTIABLE STRUCTURES

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A classical problem of differential topology is the following, which has been called the *regularity problem*: Given a topological n -manifold M , does M possess a differentiable structure? Recently, M. Kervaire has found an example in dimension 10 which shows the answer is "not always" [6]. On the other hand, the affirmative answer is known to hold for $n \leq 3$, at least if M is nonbounded (i.e., if $\text{Bd } M$ is empty); one combines the triangulation theorem of E. E. Moise [8] with work of S. S. Cairns [2].

To obtain further results, one usually adds more hypotheses to the problem, requiring M to possess a triangulation which is nice, in some sense. Specifically, one requires that the triangulation make M into a *combinatorial manifold*, in which the closed star of every vertex has a subdivision which is isomorphic with a rectilinear subdivision of the standard n -simplex. (It follows readily that some subdivision of this triangulation is a *Brouwer triangulation* [1]. This means that the closed star of every vertex is imbeddable in R^n by a homeomorphism which is linear on each simplex, and if the vertex lies on $\text{Bd } M$, the image of the open star is an open subset of the half-space $x_1 \geq 0$.) This extra hypothesis is reasonable, since if M has a differentiable structure, it also has a compatible triangulation [12], [14], which is automatically combinatorial (see 8.4 of [12]).

In this paper, we apply our previously developed techniques [11] to this problem. Roughly, our approach is to assume a Brouwer triangulation of M , and take the imbeddings $l_v : \text{Cl}(\text{St } v) \rightarrow R^n$ as a first try at coordinate systems covering M . These do not overlap differentiably, but we attempt to "smooth them out" so that they will. Obstructions to this smoothing are encountered, which appear in $\mathfrak{H}_{m-1}(M, \text{Bd } M; \Gamma^{n-m})$. Here \mathfrak{H} denotes infinite homology, with twisted coefficients in the nonorientable case. Γ^{n-m} is the group of orientation-preserving diffeomorphisms of S^{n-m-1} , modulo those extendable to the ball B^{n-m} . In special cases (e.g., if M is contractible) all these homology groups vanish, and our techniques suffice to construct a differentiable structure on M . A list of such cases appears in 2.12.

Once one has strengthened the hypotheses, one may also wish to strengthen the conclusion of the problem, and require that the differentiable structure obtained should be compatible with the given triangulation, or some subdivision of it. The differentiable structures we construct do not have this property. For example, they may possess *conical points*: If we imbed a

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2-simplex in R^3 as the join of a differentiable curve, in the x - y plane, with a point p not in this plane, p will be a conical point of the imbedding. We conjecture that the differentiable structures we obtain may be modified so as to satisfy the compatibility conditions, but it remains to be seen whether this is correct.

R. Thom has also outlined an obstruction theory to attack this problem [13]; he states that the differentiable structures his methods obtain do satisfy the compatibility condition. Since this paper was written, Milnor has invented his theory of microbundles [7], which also gives rise to an obstruction theory for the imposing of compatible differentiable structures. In addition, M. Hirsch has outlined an obstruction theory both for the imposing of compatible differentiable structures and for their equivalence [5].

1. The obstruction theory

The following assumptions hold throughout the paper: M is a connected, oriented n -manifold, possibly with boundary, not necessarily compact, provided with a triangulation making it into a Brouwer manifold. (In 2.9, we indicate what happens in the nonorientable case.) For each vertex v of M , there is given an orientation-preserving homeomorphism $l_v : \text{Cl}(\text{St } v) \rightarrow R^n$ which is linear on each simplex; we denote the image complex by K_v . (If $v \in \text{Bd } M$, $l_v(\text{St } v)$ is to be open in H^n , which denotes the half-space with non-negative first coordinate.)

We shall attempt to smooth these coordinate systems step by step, beginning with neighborhoods of the $(n - 1)$ -simplices and working down. The general induction assumption for this procedure is given in the following definition.

1.1. DEFINITION. For each vertex v of M , let $f_v : \text{Cl}(\text{St } v) \rightarrow K_v$ be a homeomorphism such that

(1) f_v agrees with l_v on $\text{Lk } v = \text{Cl}(\text{St } v) - \text{St } v$, and on all simplices of dimension $\leq m$;

(2) $f_w f_v^{-1}$ is a diffeomorphism on $f_v((\text{St } v \cap \text{St } w) - M^{(m)})$ for every pair of vertices v, w , where $M^{(m)}$ denotes the m -skeleton of M ;

(3) for each vertex v of M , each simplex σ of $\text{Cl}(\text{St } v)$ of dimension $\leq m$, and each vertex w of σ , there is a neighborhood V of σ in $\text{St } v$ such that $f_w f_v^{-1}$ is smooth on $f_v(V - M^{(m)} - \text{Bd } M)$ near $f_v(\sigma)$ (see 2.2 of [11]).

The homeomorphisms f_v are then said to define a *differentiable structure mod $M^{(m)}$* . In the case $m = n - 1$, the maps $f_v = l_v$ satisfy these hypotheses. If $m = 0$, the maps f_v define a differentiable structure on M .

We seek to redefine the maps f_v to obtain a differentiable structure mod $M^{(m-1)}$. There is an obstruction to this smoothing, which we now describe.

1.2. DEFINITION. Let $\mathbf{f} = \{f_v\}$ be a differentiable structure mod $M^{(m)}$. Choose a partial ordering of the vertices of M which induces a linear ordering

on the vertices of each simplex. Given the m -simplex σ of $\text{Int } M$, let its vertices be v_0, \dots, v_m in the given ordering, and denote f_{v_i} by f_i for the moment. The map $f_{i+1}f_i^{-1}$ is defined in a neighborhood V of $f_i(\sigma)$ and is smooth on $V - \sigma$ near σ . Hence the element $\gamma(f_{i+1}f_i^{-1}) = \gamma_{i+1}$ of Γ^{n-m} is defined, as in 3.4 of [11]. (In order to define γ_{i+1} , we must choose coordinates so that $f_i(\sigma)$ and $f_{i+1}(\sigma)$ are contained in $0 \times R^m \subset R^{n-m} \times R^m$; the orientations induced by the ordering v_0, \dots, v_m are to be compatible with the natural orientation of R^m . By 3.5 of [11], γ_{i+1} does not depend on these choices.)

The *obstruction chain* $\lambda_m \mathbf{f}$ (where $\mathbf{f} = \{f_v\}$) is the function assigning to each such σ the m -tuple $(\gamma_1, \dots, \gamma_m)$ of elements of Γ^{n-m} .

1.3. PROPOSITION. *If $\lambda_m \mathbf{f} = 0$, then the maps f_v may be approximated by maps g_v which define a differentiable structure mod $M^{(m-1)}$.*

Proof. First case. Assume the notation of 1.2. Consider the homeomorphism $f_i f_0^{-1}$ ($1 \leq i \leq m$) of a neighborhood U_0 of $f_0(\sigma)$ in K_0 into a neighborhood U_i of $f_i(\sigma)$ in K_i . Now $\gamma(f_i f_0^{-1}) = \gamma_1 + \gamma_2 + \dots + \gamma_i$, by 3.5 of [11], and all $\gamma_j = 0$ by assumption. Hence by 4.1 (Case I) of [11], there is a homeomorphism $H_i : \bar{U}_0 \rightarrow \bar{U}_i$ which agrees with $f_i f_0^{-1}$ except in a small neighborhood of σ and is smooth on U_0 near each proper face of σ . We define $g_i = H_i f_0$ on $f_i^{-1}(U_i)$ and $g_i = f_i$ otherwise; we define $g_0 = f_0$. Since the coordinate maps g_0 and g_i overlap differentiably near σ , so do g_i and g_j for any pair i, j .

Second case. If $\sigma \subset \text{Bd } M$, we use the same argument, except that no coefficients γ_i are involved. Case II of 4.1 of [11] is applied to construct the map H_i ; everything else goes through as above.

1.4. DEFINITION. Let K denote the semisimplicial complex whose elements are the ordered simplices $v_0 \dots v_m$ of M ; face and degeneracy operators are defined as usual ([9, §1]). $\text{Bd } K$ denotes the subcomplex based on simplices of $\text{Bd } M$. K_N denotes the set obtained by deleting all degenerate simplices; $K/\text{Bd } K$, the set $K - \text{Bd } K$.

Given an abelian group Γ , let $\bar{W}(\Gamma)$ denote the following FD-complex [3]:

$$\bar{W}(\Gamma)_0 = 0,$$

$$\bar{W}(\Gamma)_m = \Gamma \times \Gamma \times \dots \times \Gamma, \quad m \text{ times,}$$

$$\partial_0(\gamma_1, \dots, \gamma_m) = (\gamma_2, \dots, \gamma_m),$$

$$\partial_i(\gamma_1, \dots, \gamma_m) = (\gamma_1, \dots, \gamma_{i-1}, \gamma_i + \gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_m), \quad 0 < i < m,$$

$$\partial_m(\gamma_1, \dots, \gamma_m) = (\gamma_1, \dots, \gamma_{m-1}),$$

$$s_i(\gamma_1, \dots, \gamma_m) = (\dots, \gamma_i, 0, \gamma_{i+1}, \dots), \quad 0 \leq i \leq m.$$

(This complex is related to the Eilenberg-MacLane space $K(\Gamma, 1)$; its homology is Γ in dimension 1 and zero otherwise. This fact is easily proved, but is not needed in the sequel.)

Let $(\bar{W}(\Gamma) \times K)_m$ denote the group of all (possibly infinite) m -chains of K with coefficients in $\bar{W}(\Gamma)_m$. An elementary chain is denoted by $\gamma \times \sigma$, where $\gamma = (\gamma_1, \dots, \gamma_m)$ and $\sigma = v_0 \dots v_m$. Because K is locally-finite, face and degeneracy operators on $\bar{W}(\Gamma) \times K$ are well defined by the equations

$$\partial_i(\gamma \times \sigma) = \partial_i \gamma \times \partial_i \sigma, \quad s_i(\gamma \times \sigma) = s_i \gamma \times s_i \sigma;$$

so that $\bar{W}(\Gamma) \times K$ becomes an FD-complex.

If either Γ or K is finite, $\bar{W}(\Gamma) \times K$ is the usual cartesian product [3] of $\bar{W}(\Gamma)$ and the group of (possibly infinite) chains of K . $\bar{W}(\Gamma) \times K_N$ is defined similarly, by allowing only nondegenerate simplices of K ; it is a chain complex, but not an FD-complex. $\bar{W}(\Gamma) \times (K/\text{Bd } K)$ and $\bar{W}(\Gamma) \times (K_N/\text{Bd } K)$ are defined similarly. If \mathbf{f} is a differentiable structure mod $M^{(m)}$, $\lambda_m \mathbf{f}$ might be considered to belong to any of these groups. In order that $\lambda_m \mathbf{f}$ be a cycle, it must be one of the last two groups; in order to have a satisfactory obstruction theory, the latter of these must be chosen (see 1.6).

1.5. THEOREM. $\partial(\lambda_m \mathbf{f}) = 0$.

Proof. Let $\tau = w_1 \dots w_m$ be a simplex of $\text{Int } M$; let us compute the coefficient of $\partial(\lambda \mathbf{f})$ on τ . Let $f_i = f_{w_i}$.

Let $\gamma \times \sigma = (\gamma_1, \dots, \gamma_m) \times \sigma$ be a term of the chain $\lambda \mathbf{f}$; then the coefficient of $\partial(\gamma \times \sigma)$ on τ is $\pm(\gamma'_1, \dots, \gamma'_{m-1})$, where $\gamma'_i = \gamma(f_{i+1} f_i^{-1})(\sigma)$. (To prove this, set $\sigma = w_1 \dots w_i w w_{i+1} \dots w_m$ and note that

$$\gamma_i + \gamma_{i+1} = \gamma(f_w f_i^{-1})(\sigma) + \gamma(f_{i+1} f_w^{-1})(\sigma),$$

by definition of $\lambda \mathbf{f}$, and this equals $\gamma(f_{i+1} f_i^{-1})(\sigma)$ by 3.5 of [11]. The cases $\sigma = w w_1 \dots w_m$ and $\sigma = w_1 \dots w_m w$ must be checked similarly.) The sign is $+$ or $-$ according as the orientations of σ and τ , induced by the ordering, agree or not.

Let (a_1, \dots, a_{m-1}) be the value of $\partial(\lambda \mathbf{f})$ on τ . Then a_j is merely the sum of the coefficients $\gamma(f_{j+1} f_j^{-1})$ on σ , where j is fixed and the sum extends over all m -simplices σ in $\text{St } \tau$, and signs are chosen according to the relative orientations of σ and τ . It follows from 3.9, 5.4, and 5.5 of [11] that this sum is zero (just as in the proof in 5.6 of [11] that the obstruction chain there considered was a cycle).

1.6. THEOREM. Let $\mathbf{f} = \{f_v\}$ be a differentiable structure mod $M^{(m)}$; let $\lambda \mathbf{f} = 0$. By 1.3, we may choose maps g_v which form a differentiable structure mod $M^{(m-1)}$. If ξ is any m -chain of $\bar{W}(\Gamma^{n-m+1}) \times K_N/\text{Bd } K$, there are maps h_v which also form a differentiable structure mod $M^{(m-1)}$, such that

$$\lambda_{m-1} \mathbf{g} - \lambda_{m-1} \mathbf{h} = \partial \xi.$$

Proof. We proceed as in 4.7 and 4.8 of [11] to investigate how altering the smoothings g_v affects the obstruction chain. Let $\sigma = v_0 \dots v_m$ be an m -simplex of $\text{Int } M$, and consider the map $F = f_i f_0^{-1}$ in a neighborhood of $f_0(\sigma)$ for some

fixed i . ($f_i = f_{v_i}$, as usual; $i > 0$.) The construction of g_i involves the choice of an extension μ of the diffeomorphism of S^{n-m-1} , corresponding to σ under the map F , to a diffeomorphism of B^{n-m} . Another choice ν gives a different smoothing h_i . The approximations to $f_i f_0^{-1}$ are then $g_i f_0^{-1}$ and $h_i f_0^{-1}$ respectively, (since $f_0 = g_0 = h_0$). The composite $(h_i f_0^{-1})(g_i f_0^{-1})^{-1} = h_i g_i^{-1}$ is defined in a neighborhood of $f_i(\sigma)$, equals the identity on σ and outside a small neighborhood of σ , and is a diffeomorphism mod $f_i(\text{Bd } \sigma)$ (as in 4.8 of [11]). There is an element γ of Γ^{n-m+1} such that the value of $\gamma(h_i g_i^{-1})$ on the face $\partial_j \sigma$ equals $(-1)^j \gamma$ (i.e., the obstruction chain $\lambda(h_i g_i^{-1})$ is the boundary of the elementary chain $\gamma \sigma$, by 4.8 of [11]). By appropriate choice of ν , any element γ of Γ^{n-m+1} may so be obtained (as in 5.3 of [11]).

Let $h_v = g_v$ for all vertices of M other than v_i , and consider $\lambda \mathbf{g} - \lambda \mathbf{h}$, an $(m - 1)$ -chain of $\bar{W}(\Gamma^{n-m+1}) \times K_N/\text{Bd } K$. Its coefficient is necessarily zero except on faces of σ ; its coefficient is zero also on $\partial_i \sigma$, since the coordinate maps g_i and h_i are not involved in defining the coefficients of $\lambda \mathbf{g}$ and $\lambda \mathbf{h}$ on $\partial_i \sigma$ (by 1.2). We wish now to compute its coefficient on $\partial_j \sigma$, providing this face is not in $\text{Bd } K$.

Consider the case $i < j$. Let u be the vertex preceding v_i , and w the vertex succeeding v_i , in $\partial_j \sigma$. The entry in the i^{th} place of the value of $\lambda \mathbf{g}$ on $\partial_j \sigma$ is $\gamma(g_i g_u^{-1})(\partial_j \sigma)$; the corresponding element in $\lambda \mathbf{h}$ is $\gamma(h_i h_u^{-1})(\partial_j \sigma)$. Similarly, the elements in the $(i + 1)^{\text{th}}$ place are $\gamma(g_w g_i^{-1})(\partial_j \sigma)$ and $\gamma(h_w h_i^{-1})(\partial_j \sigma)$, respectively. Since $g_u = h_u$ and $g_w = h_w$, the coefficient of $\lambda \mathbf{g} - \lambda \mathbf{h}$ on $\partial_j \sigma$ is

$$(0, \dots, \gamma(g_i h_i^{-1}), \gamma(h_i g_i^{-1}), \dots, 0) = (-1)^j (0, \dots, -\gamma, \gamma, \dots, 0),$$

where the nonzero coefficients appear in places i and $i + 1$ of this $(m - 1)$ -tuple.

The same argument applies in the case $j < i < m$, except $-\gamma$ and γ appear in places $i - 1$ and i , respectively. In the case $i = m$, the coefficient of $\lambda \mathbf{g} - \lambda \mathbf{h}$ on $\partial_j \sigma$ is $(-1)^j (0, \dots, 0, -\gamma)$ for all $j < i$.

We conclude the following: Given i , changing the map g_i to h_i in a neighborhood of $\sigma = v_0 \dots v_m$ gives the equation

$$\lambda \mathbf{g} - \lambda \mathbf{h} = \partial((0, \dots, -\gamma, \gamma, \dots, 0) \times \sigma) \quad \text{if } 0 < i < m,$$

the nonzero coefficients appearing in places i and $i + 1$ of this m -tuple, or

$$\lambda \mathbf{g} - \lambda \mathbf{h} = \partial((0, \dots, 0, -\gamma) \times \sigma) \quad \text{if } i = m.$$

We noted before that γ may be chosen arbitrarily. Hence it follows that, since an arbitrary elementary chain $(a_1, \dots, a_m) \times \sigma$ may be written as a sum of chains of these two forms, it is possible to alter the maps g_1, \dots, g_m in a neighborhood of σ so as to change $\lambda_{m-1} \mathbf{g}$ by $\partial((a_1, \dots, a_m) \times \sigma)$. By proceeding similarly for each term in ξ , we may define \mathbf{h} so that $\lambda \mathbf{g} - \lambda \mathbf{h} = \partial \xi$.

1.7. DEFINITION. If \mathbf{f} is a differentiable structure mod $M^{(m)}$, the homology class of $\lambda \mathbf{f}$ in $\mathfrak{H}_m(\bar{W}(\Gamma^{n-m}) \times K_N/\text{Bd } K)$ is called the *obstruction class* to obtaining a differentiable structure mod $M^{(m-1)}$.

The preceding proposition shows that if these classes all vanish, M possesses a differentiable structure. (For m large, $\Gamma^{n-m} = 0$, so there is never any trouble in getting the induction started.) The obstruction class depends, among other things, on the vertex-ordering chosen for M ; this dependence is not essential, as the following proposition shows:

1.8. PROPOSITION. *A reordering of the vertices of M , giving rise to a second semisimplicial complex \bar{K} , induces an isomorphism ρ_* of $\mathfrak{H}_m(\bar{W}(\Gamma) \times K_N/\text{Bd } K)$ with $\mathfrak{H}_m(\bar{W}(\Gamma) \times \bar{K}_N/\text{Bd } \bar{K})$ which carries one obstruction class into the other.*

Proof. We define the chain map ρ . Its value on the elementary chain $(\gamma_1, \dots, \gamma_m) \times (v_0, \dots, v_m)$ of $\bar{W}(\Gamma) \times (K_N/\text{Bd } K)$ is obtained as follows: Let v'_0, \dots, v'_m be the vertices of $\sigma = v_0 \dots v_m$ in the ordering induced by \bar{K} ; let ε be the sign of the permutation involved. Let γ_0 denote $-(\gamma_1 + \dots + \gamma_m)$. Supposing that $v'_k = v_i$ and $v'_{k+1} = v_j$, define

$$\gamma'_{k+1} = \varepsilon(\gamma_{i+1} + \gamma_{i+2} + \dots + \gamma_j),$$

agreeing that in the case $j < i$, the summation extends from $i + 1$ through m and then from 0 through j . We define

$$\rho((\gamma_1, \dots, \gamma_m) \times (v_0, \dots, v_n)) = (\gamma'_1, \dots, \gamma'_m) \times (v'_0, \dots, v'_m).$$

The reader may verify that ρ carries one obstruction chain into the other.

We need to show ρ commutes with ∂ , but first we prove that the product of two reorderings induces a homomorphism which is the product of the induced homomorphisms. From this it follows at once that ρ is an isomorphism, since the identity reordering induces the identity homomorphism.

Let A denote the permutation matrix which acts on the column vector (v_0, \dots, v_m) to give the column vector (v'_0, \dots, v'_m) ; for convenience, we number the rows and columns of A from 0 to m . Define a matrix A^* as follows: Consider rows k and $k + 1$ of A ($0 \leq k < m$; if $k = m$, $k + 1$ is replaced by 0 in what follows). Let 1's appear in columns i and j , respectively, of these rows. Then row $k + 1$ of A^* is to have 1's in columns $i + 1, \dots, j$, and 0's elsewhere; so that εA^* acts on the column vector $\vec{\gamma} = (\gamma_0, \dots, \gamma_m)$ to give the column vector $(\gamma'_0, \dots, \gamma'_m)$. (The fact that $\gamma'_0 = -(\gamma'_1 + \dots + \gamma'_m)$ is proved by noting $(1 \ 1 \ \dots \ 1)A^* = (a \ a \ \dots \ a)$, where a is some integer; multiplying both sides on the right by $\vec{\gamma}$ gives $\gamma'_0 + \dots + \gamma'_m = a(\gamma_0 + \dots + \gamma_m) = 0$.)

Let A have 1's at positions (k, i) and $(k + 1, j)$ and let B have 1's at (i, p) and (j, q) ; then AB has 1's at (k, p) and $(k + 1, q)$. Since A^* has 1's in row $k + 1$ at columns $i + 1, \dots, j$, the row $k + 1$ of A^*B^* is obtained by adding together the rows $i + 1, \dots, j$ of B^* . It follows from the definition of B^* that this sum is an integer a in columns $q + 1, \dots, p$; and it is the integer $a + 1$ in columns $p + 1, \dots, q$. Hence the product of row $k + 1$ of A^*B^* with $\vec{\gamma}$ is $a(\gamma_0 + \dots + \gamma_m) + \gamma_{p+1} + \dots + \gamma_q = \gamma_{p+1} + \dots + \gamma_q$.

Hence $A^*B^*\gamma = (AB)^*\gamma$, as we desired to prove; the signs automatically come out all right.

To show ρ commutes with ∂ , it suffices to consider an elementary chain $\gamma \times \sigma = (\gamma_1, \dots, \gamma_m) \times v_0 \dots v_m$ and its faces. In view of the result just proved, it also suffices to consider the simple permutation which exchanges v_{i-1} and v_i . When $1 < i < m$, one has the formula

$$\rho(\gamma \times \sigma) = -[(\gamma_1, \dots, \gamma_m) + (0, \dots, \gamma_i, -2\gamma_i, \gamma_i, \dots, 0)] \times v_0 \dots v_i v_{i-1} \dots v_m,$$

where the nonzero elements in the second term appear in places $i - 1, i$, and $i + 1$. When $i = 1$ or $i = m$, similar formulas hold. In every case, the reader may verify the following relations:

$$\begin{aligned} \partial_j \rho(\gamma \times \sigma) &= \rho \partial_j(\gamma \times \sigma) \quad \text{for } j \neq i, i + 1, \\ \partial_i \rho(\gamma \times \sigma) &= -\partial_{i+1}(\gamma \times \sigma) = -\rho \partial_{i+1}(\gamma \times \sigma), \\ \partial_{i+1} \rho(\gamma \times \sigma) &= -\partial_i(\gamma \times \sigma) = -\rho \partial_i(\gamma \times \sigma). \end{aligned}$$

It follows at once that $\partial \rho(\gamma \times \sigma) = \rho \partial(\gamma \times \sigma)$.

2. Computation of the homology group

Our definition of the obstruction to imposing a differentiable structure has two virtues—it does work, and its use involves no further messy technicalities of a “smoothing” nature than those already carried out in [11]. It has the fault that it lies in an unfamiliar group $\mathfrak{C}_m(\bar{W}(\Gamma^{n-m}) \times K_N/\text{Bd } K)$; if we are to apply the theory, we must compute this group. In this section, we prove (2.8) that under suitable hypotheses this group is isomorphic with $\mathfrak{C}_{m-1}(M, \text{Bd } M; \Gamma^{n-m})$. In the succeeding section, we examine the image of the obstruction class under this isomorphism, and provide a description of it which clarifies the geometric nature of the obstruction.

The homology group $\mathfrak{C}_m(\bar{W}(\Gamma) \times K/\text{Bd } K)$ is easy to compute. If K is finite, it is the usual cartesian product of FD-complexes, so the Eilenberg-Zilber theorem [4] shows it isomorphic with $\mathfrak{C}_m(\bar{W}(\Gamma) \otimes K/\text{Bd } K)$. Then, since the subgroup of cycles of $\bar{W}(\Gamma)_m$ form a direct summand (easily proved), the Künneth formulas apply to show it isomorphic with

$$\sum_{p+q=m} \mathfrak{C}_p(K, \text{Bd } K; \mathfrak{C}_q(\bar{W}(\Gamma))) = \mathfrak{C}_{m-1}(K, \text{Bd } K; \Gamma).$$

Even if K is not finite, this computation holds; the isomorphism is induced by the chain map α defined below, as one may prove without much difficulty.

Unfortunately, the homology groups we wish to compute are those of the chain complex $\bar{W}(\Gamma) \times K_N/\text{Bd } K$, and this is another object altogether. The Eilenberg-Zilber and Künneth theorems provide us only with motivation for the work which follows.

2.1. DEFINITION. Let a homomorphism

$$\alpha : (\bar{W}(\Gamma) \times K_N/\text{Bd } K)_m \rightarrow \mathcal{C}_{m-1}(M, \text{Bd } M; \Gamma)$$

be defined by the equation

$$\alpha((\gamma_1, \dots, \gamma_m) \times v_0 \cdots v_m) = \gamma_1(v_1 \cdots v_m).$$

($\mathcal{C}(K, \text{Bd } K; \Gamma)$ will be used to refer to the chains of the ordered complex K ; $\mathcal{C}(M, \text{Bd } M; \Gamma)$, to the chains based on oriented simplices of M .) One readily verifies that $\alpha\partial = -\partial\alpha$.

2.2. PROPOSITION. α_* is natural with respect to the reordering isomorphism ρ_* (1.8), so that $\alpha_*[\lambda f]$ is independent of the particular ordering used to define λf .

Proof. Assume the hypotheses of 1.8. Let

$$\bar{\alpha} : (\bar{W}(\Gamma) \times \bar{K}_N/\text{Bd } \bar{K}) \rightarrow \mathcal{C}_{m-1}(M, \text{Bd } M; \Gamma)$$

be defined by the equation of 2.1. We wish to show $\alpha_* = \bar{\alpha}_*\rho_*$.

Define $D : (\bar{W}(\Gamma) \times K_N/\text{Bd } K) \rightarrow \mathcal{C}_m(M, \text{Bd } M; \Gamma)$ by defining

$$D(\gamma \times \sigma) = D(\gamma_1, \dots, \gamma_m) \times v_0 \cdots v_m = (\gamma_1 + \cdots + \gamma_i)v_0 \cdots v_m$$

if the reordering of $v_0 \cdots v_m$ makes v_i into its leading vertex. We use the same convention about the sum $\gamma_i + \cdots + \gamma_j$, if $j < i$, as in 1.8; hence if $i = 0$, $\gamma_1 + \cdots + \gamma_i = 0$. We leave it to the reader to show that $\partial D - D\partial = \alpha - \bar{\alpha}\rho$.

2.3. Remark. We have assumed throughout that K is based on a Brouwer triangulation of the n -manifold M . In the remainder of this section, we also assume that this triangulation is the first barycentric subdivision of another triangulation of M , which we denote by \tilde{M} , and that the ordering chosen in K is the natural one. That is, a simplex of K is a sequence $\tau = \sigma_1 \sigma_2 \cdots \sigma_m$ of simplices of \tilde{M} , where σ_i is a face of σ_{i-1} , for all i ; σ_i is called a vertex of τ . If σ_0 has σ_1 as a face, then the symbol $\sigma_0 \tau$ denotes the simplex $\sigma_0 \sigma_1 \sigma_2 \cdots \sigma_m$ of K .

2.4. LEMMA. If ξ is an m -cycle ($m < n$) of $\bar{W}(\Gamma) \times K_N/\text{Bd } K$ and $\alpha\xi$ is homologous to zero, then ξ is homologous to a chain η such that $\alpha\eta = 0$.

Proof. $\alpha\xi$ lies in the subdivision of the $(n - 1)$ -skeleton of \tilde{M} ; since its dimension is $m - 1$, it bounds (mod $\text{Bd } M$) an oriented chain which lies on the subdivision of this skeleton. Let $\sum a_i \tau_i$ be the corresponding ordered chain; τ_i is in K_N . For each m -simplex τ_i appearing in the chain, choose an n -simplex σ_i of \tilde{M} having the vertices of τ_i as faces. Let c be the chain $\sum (a_i, 0, \dots) \times \sigma_i \tau_i$, which lies in $(\bar{W}(\Gamma) \times K_N)_{m+1}$; let $\eta = \xi + \partial c$. Then $\alpha\eta = \alpha\xi - \partial(\alpha c) = 0$.

2.5. LEMMA. *If ξ is an m -cycle ($m < n$) of $\bar{W}(\Gamma) \times K_N/\text{Bd } K$ and $\alpha\xi = 0$, ξ is homologous to a chain whose leading coefficients are all zero.*

Proof. Consider ξ as a chain in $\bar{W}(\Gamma) \times K_N$; let $m > 0$, since otherwise $\xi = 0$.

Case I. Assume $m > 1$ for the moment. Let σ be a simplex of dimension $n - 1$; consider all simplices of K_N whose second vertex is σ . Let σ_1 be an n -simplex of \tilde{M} incident on σ ; let $\eta = \sum (a_i, \dots) \times \sigma_1 \sigma \tau_i^{m-2}$ be that part of ξ carried by simplices beginning $\sigma_1 \sigma$. We prove that $\sum a_i \tau_i^{m-2}$ is the boundary of some chain $\sum b_j \tau_j^{m-1}$, where τ_j^{m-1} is a simplex of K_N lying in $\text{Bd } \sigma$. For this, let us note that the part of $\partial\xi$ carried by simplices beginning $\sigma_1 \sigma$ must appear in $\partial\eta$, because σ_1 and σ have dimensions n and $n - 1$, respectively. This part of $\partial\eta$ is of the form $\sum (a_i, \dots) \times \sigma_1 \sigma \tau_i^{m-2}$; since ξ is a cycle mod $\text{Bd } K$, this must vanish, for no simplex beginning σ_1 lies in $\text{Bd } K$. Hence $\sum a_i \tau_i^{m-2}$ is an $(m - 2)$ -cycle on $\text{Bd } \sigma$, which is a sphere of dimension $n - 2$. (If $m = 2$, it is a 0-chain of index 0.) Thus $\sum a_i \tau_i^{m-2}$ bounds a chain lying on $\text{Bd } \sigma$, as desired.

If σ lies in $\text{Bd } \tilde{M}$, set

$$\zeta = \sum (b_j, -b_j, 0, \dots, 0) \times \sigma_1 \sigma \tau_j^{m-1}.$$

If σ does not lie in $\text{Bd } \tilde{M}$, let σ_2 be the other n -simplex incident on σ , and define

$$\zeta = \sum (b_j, -b_j, 0, \dots, 0) \times (\sigma_1 \sigma \tau_j^{m-1} - \sigma_2 \sigma \tau_j^{m-1}).$$

Then $\xi - \partial\zeta$ has leading coefficient 0 on all simplices having σ as a second vertex, and has the same coefficients as ξ on all other simplices not in $\text{Bd } K$. This is proved by direct computation, once we note that the part of ξ carried by simplices beginning $\sigma_2 \sigma$ must be of the form $\sum (-a_i, \dots) \times \sigma_2 \sigma \tau_i^{m-2}$, since $\alpha\xi \equiv 0 \pmod{\text{Bd } \tilde{M}}$, by hypothesis.

The case $m = 1$ is handled by noting that in this case, η simply equals $a \times \sigma_1 \sigma$. If σ is in $\text{Bd } \tilde{M}$, set $\zeta = (a, -a) \times \sigma_1 \sigma v$, where v is a vertex of σ ; if σ is not in $\text{Bd } \tilde{M}$, set $\zeta = (a, -a) \times (\sigma_1 \sigma v - \sigma_2 \sigma v)$.

Case II. Let τ be a fixed $(m - 1)$ -simplex of K whose leading vertex has dimension $k < n - 1$; consider all simplices of K_N of the form $\sigma_i \tau$. Let $\eta = \sum (a_i, \dots) \times \sigma_i \tau$ be that part of ξ carried by such simplices. If τ is in $\text{Bd } K$, then $\text{Lk } \tau$ is a cell of dimension $n - k - 1 > 0$ in M . Hence the 0-chain $\sum a_i \sigma_i$ bounds, mod $\text{Bd } K$, a chain $\sum b_j \tau_j$ carried on $\text{Lk } \tau$. Set $\zeta = \sum (0, b_j, 0, \dots) \times \tau_j \tau$; then

$$\partial\zeta = \sum (b_j, 0, \dots) \times (\partial\tau_j)\tau + \sum (0, b_j, 0, \dots) \times \tau_j \partial\tau.$$

Hence $\xi - \partial\zeta$ has leading coefficient zero on all simplices of the form $\sigma_i \tau$ not in $\text{Bd } K$, and has the same leading coefficients as ξ on all other simplices.

If τ is not in $\text{Bd } K$, then $\text{Lk } \tau$ is a sphere of positive dimension. $\sum a_i \sigma_i$ has

index 0, because the coefficient of $\alpha\xi$ on τ must be zero. The argument just given applies.

A combination of the chains ζ defined in Cases I and II will give us a chain ζ such that $\xi - \partial\zeta$ has leading coefficient zero on all simplices not in $\text{Bd } K$.

2.6. LEMMA. *If ξ is an m -cycle ($m < n$) of $\bar{W}(\Gamma) \times K_N/\text{Bd } K$ whose leading coefficients are all zero, then ξ bounds a chain ζ (whose leading coefficients are all zero).*

Proof. Consider ξ as a chain in $\bar{W}(\Gamma) \times K_N$. We proceed by induction; assume the first $k - 1$ coefficients of ξ are zero on all simplices not in $\text{Bd } K$ ($1 < k \leq m$). Case I disposes of that part of ξ carried by simplices of K_N whose $(k + 1)^{\text{th}}$ vertex has dimension $n - k$; Case II deals with the rest of ξ .

Case I. Let σ be a simplex of dimension $n - k$, and consider all m -simplices of K whose $(k + 1)^{\text{th}}$ vertex is σ . Assume $k < m$ for the moment. Let

$$\eta = \sum_{i,j}(0, \dots, 0, a_{ij}, \dots) \times \tau_i^{k-1} \sigma \tau_j^{m-k-1}$$

be that part of ξ carried by such simplices; a_{ij} appears in the k^{th} place of this m -tuple. τ_i^{k-1} is a principal simplex of K ; it has $k + 1$ vertices, the last of which has dimension $n - k$. We prove that η is of the form

$$\sum_{i,j}(0, \dots, 0, b_j, \dots) \varepsilon_i \times \tau_i^{k-1} \sigma \tau_j^{m-k-1}, \quad \text{where } \varepsilon_i = \pm 1.$$

To prove this, we consider that part of $\partial\xi$ carried by simplices of the form $\tau^{k-2} \sigma \tau^{m-k-1}$ and having nonzero coefficients in place $k - 1$. This part must occur also in $\partial\eta$. (To have a nonzero $(k - 1)^{\text{th}}$ coefficient, a term of $\partial\xi$ must arise from applying one of the face operators $\partial_0, \dots, \partial_{k-1}$. But $\tau^{k-2} \sigma \tau^{m-k-1} = \partial_j \tau^m$ for $0 \leq j \leq k - 1$ only if τ^m is of the form $\tau^{k-1} \sigma \tau^{m-k-1}$.) Since $\partial\xi \equiv 0 \pmod{\text{Bd } K}$, we inspect the $(k - 1)^{\text{th}}$ coefficients in $\partial\eta$ and obtain the equation $\sum_{i,j} a_{ij} (\partial \tau_i^{k-1}) \sigma \tau_j^{m-k-1} \equiv 0 \pmod{\text{Bd } K}$. Thus $\sum_i a_{ij} \tau_i^{k-1}$ is a cycle mod $\text{Bd } K$ on $\text{Lk}(\sigma \tau_j^{m-k-1})$, which is a sphere or cell of dimension $k - 1$. Hence this cycle is a multiple of the fundamental cycle: $\sum_i a_{ij} \tau_i^{k-1} = b_j \sum_i \varepsilon_i \tau_i^{k-1}$, where $\varepsilon_i = \pm 1$. Thus η is of the desired form.

Now we prove that $\sum_j b_j \tau_j^{m-k-1}$ bounds a chain carried on $\text{Bd } \sigma$. To prove this, we note that part of $\partial\xi$ carried by simplices of the form $\tau^{k-1} \sigma \tau^{m-k-2}$ must come from $\partial\eta$, since $\tau^{k-1} \sigma$ is principal. We inspect the k^{th} coefficients in that part of $\partial\eta$ to obtain the equation

$$\sum_{i,j} b_j \varepsilon_i \tau_i^{k-1} \sigma (\partial \tau_j^{m-k-1}) = 0.$$

Thus $\sum_j b_j \partial \tau_j^{m-k-1} = 0$, so that $\sum_j b_j \tau_j^{m-k-1}$ is a cycle on $\text{Bd } \sigma$, which is a sphere of dimension $n - k - 1 > 0$. (If $k = m - 1$, it is a 0-cycle of index zero.) Hence it bounds a chain $\sum c_j \tau_j^{m-k}$ carried on $\text{Bd } \sigma$, as desired. We define

$$\zeta = \sum_{i,j}(\dots, 0, c_j, -c_j, 0, \dots) \times \varepsilon_i \tau_i^{k-1} \sigma \tau_j^{m-k},$$

where c_j and $-c_j$ appear in places k and $k + 1$ of this $(m + 1)$ -tuple. Then

$$\begin{aligned} \partial\zeta &= \sum_{i,j}(\dots, c_j, -c_j, \dots) \times \varepsilon_i(\partial\tau_i^{k-1})\sigma\tau_j^{m-k} \\ &\quad + (-1)^{k+1}\sum_{i,j}(\dots, c_j, -c_j, \dots) \times \varepsilon_i\tau_i^{k-1}\sigma(\partial\tau_j^{m-k}). \end{aligned}$$

In the first term, c_j and $-c_j$ appear in places $k - 1$ and k ; in the second term, in places k and $k + 1$. The first term is zero mod $\text{Bd } K$, since $\sum_i \varepsilon_i \tau_i^{k-1}$ is a cycle mod $\text{Bd } K$. Then $\xi + (-1)^k \partial\zeta$ has zero coefficients in places $1, \dots, k$ on all simplices of the form $\tau^{k-1}\sigma\tau^{m-k-1}$, and agrees with ξ on all other simplices not in $\text{Bd } K$.

(In the case $k = m$, $\eta = \sum (0, \dots, 0, a_i) \times \tau_i^{m-1}\sigma$. As above,

$$\eta = \sum (0, \dots, 0, b) \times \varepsilon_i \tau_i^{m-1}\sigma.$$

Set $\zeta = \sum (0, \dots, b, -b) \times \varepsilon_i \tau_i^{m-1}\sigma v$, where v is an arbitrary vertex of σ .)

Case II. Let τ be an $(m - k)$ -simplex of K whose leading vertex has dimension less than $n - k$, and consider all m -simplices of K of the form $\tau_i^{k-1}\tau$. Let $\eta = \sum (0, \dots, 0, a_i, \dots) \times \tau_i^{k-1}\tau$ be that part of ξ carried by such simplices, where a_i appears in the k^{th} place. As before, that part of $\partial\xi$ carried by simplices of the form $\tau_j^{k-2}\tau$ and having nonzero coefficient in place $k - 1$ must come from $\partial\eta$. By inspecting the $(k - 1)^{\text{th}}$ coefficients of $\partial\eta$, we obtain the equation

$$\sum a_i(\partial\tau_i^{k-1})\tau \equiv 0 \pmod{\text{Bd } K}.$$

Thus $\sum a_i \tau_i^{k-1}$ is a cycle mod $\text{Bd } K$ on $\text{Lk } \tau$, which is a sphere or cell of dimension $n - (m - k) - 1 > k - 1$. Hence $\sum a_i \tau_i^{k-1} \equiv \partial \sum b_j \tau_j^k \pmod{\text{Bd } K}$, where τ_j^k lies in $\text{Lk } \tau$. Set

$$\zeta = \sum (0, \dots, b_j, \dots, 0) \times \tau_j^k \tau,$$

where b_j appears in place $k + 1$ of this $(m + 1)$ -tuple. Then $\xi - \partial\zeta$ has zero coefficients in places $1, \dots, k$ on simplices not in $\text{Bd } K$ ending with τ , and its coefficients agree with those of ξ in places $1, \dots, k$ on all other simplices of K .

2.7. LEMMA. *If c is a cycle of $\mathcal{C}_{m-1}(M, \text{Bd } M; \Gamma)$ ($m < n$) carried by the subdivision of the $(m - 1)$ -skeleton of \tilde{M} , then there is a cycle ξ of $\tilde{W}(\Gamma) \times K_N/\text{Bd } K$ such that $\alpha\xi = c$. (ξ has leading coefficient zero on every simplex σ such that $\partial_0 \sigma$ is not in the carrier of c .)*

Proof. Let $m > 1$ for the moment. Let $\bar{c} = \sum a_{ij} \sigma_i \tau_j^{m-2}$ be the ordered chain corresponding to c , where σ_i has dimension $m - 1$, and $a_{ij} = 0$ if the leading vertex of τ_j^{m-2} is not a face of σ_i , or if σ_i is in $\text{Bd } \tilde{M}$. For each i , choose an n -simplex σ_i^n of \tilde{M} having σ_i as a face. Let

$$\theta = \sum_{i,j}(a_{ij}, 0, \dots, 0) \times \sigma_i^n \sigma_i \tau_j^{m-2};$$

then $\alpha\theta = c$. Now

$$\partial\theta = \sum_{i,j}(a_{ij}, 0, \dots, 0) \times (-\sigma_i^n \tau_j^{m-2} + \sigma_i^n \sigma_i \partial\tau_j^{m-2}),$$

so that θ is not a cycle. However, the second term in $\partial\theta$ does vanish. For that part of ∂c carried by simplices beginning σ_i must come from $\partial \sum_j a_{ij} \sigma_i \tau_j^{m-2}$, since c is carried by the subdivision of the $(m - 1)$ -skeleton of \bar{M} . Hence $\sum_j a_{ij} \sigma_i \partial\tau_j^{m-2} \equiv 0 \pmod{\text{Bd } K}$, for each i , since \bar{c} is a cycle mod $\text{Bd } K$. Hence the second term in $\partial\theta$ vanishes.

We now choose a chain η so that $\theta + \eta$ will have leading coefficients zero. Let j be fixed. That part of $\partial\bar{c}$ which is carried by simplices ending τ_j^{m-2} is $\sum_i a_{ij} \tau_j^{m-2}$. If τ_j^{m-2} is not in $\text{Bd } K$, $\sum_i a_{ij} \sigma_i$ is a 0-cycle of index zero on the $(n - m + 1)$ -sphere $\text{Lk } \tau_j^{m-2}$. Thus $\sum_i a_{ij} \sigma_i$ bounds mod K on $\text{Lk } \tau_j^{m-2}$ whether τ_j^{m-2} lies in $\text{Int } K$ or $\text{Bd } K$. Let $\sum_i a_{ij} \sigma_i \equiv \partial \sum_i b_{ij} \tau_{ij} \pmod{\text{Bd } K}$. Let

$$\eta_j = \sum_i (0, b_{ij}, 0, \dots) \times \tau_{ij} \tau_j^{m-2};$$

let $\eta = \sum_j \eta_j$. Then $\partial\eta = 0$, and direct computation shows $\partial(\theta + \eta)$ is a chain whose leading coefficients are zero on simplices not in $\text{Bd } K$.

Let φ be that part of $\partial(\theta + \eta)$ carried on $K - \text{Bd } K$. By 2.6, φ bounds, mod $\text{Bd } K$, a chain ζ of $\bar{W}(\Gamma) \times K_N$ whose leading coefficients are all zero. Then $\xi = \theta + \eta - \zeta$ satisfies the demands of the lemma.

(In the case $m = 1$, $c = \sum a_i \sigma_i$ and we set $\xi = \sum a_i \times \sigma_i^n \sigma_i$. Then $\alpha\xi = c$ and ξ is automatically a cycle.)

2.8. THEOREM. Under the hypotheses of 2.3,

$$\alpha_* : \mathfrak{C}_m(\bar{W}(\Gamma) \times K_N/\text{Bd } K) \rightarrow \mathfrak{C}_{m-1}(M, \text{Bd } M; \Gamma)$$

is an isomorphism for $m < n$.

Proof. The fact that α_* is 1-1 follows from 2.4, 2.5, and 2.6; the fact that it is onto, from 2.7.

2.9. *Remark.* The result of this theorem is that we may consider the obstruction to imposing a differentiable structure on M as a homology class of $\mathfrak{C}_{m-1}(M, \text{Bd } M; \Gamma^{n-m})$. (The dimension $m = n$ does not concern us, since there are no obstructions there.) We have throughout treated only the case M orientable. There are no difficulties involved in extending the theory to the nonorientable case; the only change is that all the preceding theorems and proofs become slightly messier, because the coefficients in $\bar{W}(\Gamma) \times K$ must be twisted, as must the coefficients in $\mathfrak{C}_m(M, \text{Bd } M; \Gamma)$. The use of script \mathfrak{C} will imply, as in [11], that infinite chains are allowed and that coefficients are twisted if M is nonorientable.

The results may be translated into ordinary homology and cohomology (denoted by H) by using duality and universal coefficient theorems, as in 6.2 and 6.5 of [11].

2.10. THEOREM. *Let M be a combinatorial n -manifold. If any one of the following conditions holds, M has a differentiable structure.*

- (1) $n \leq 4$.
- (2) $n = 5$; M is open or $\text{Bd } M$ is nonempty.
- (3) $H_q(M) = 0$ for $q \geq 4$.
- (4) M is compact orientable, and $H_q(M, \text{Bd } M) = 0$ for $q \leq n - 5$.

Proof. These conditions overlap, of course. The sufficiency of condition (1) was first proved by Cairns (see [2]). The possible obstructions lie in the groups

$$\mathfrak{C}_{n-2}(M, \text{Bd } M; \Gamma^1), \dots, \mathfrak{C}_0(M, \text{Bd } M; \Gamma^{n-1}).$$

Since $\Gamma^q = 0$ for $q \leq 3$ [10], these groups vanish when $n \leq 4$. If $n = 5$, the only possible obstruction lies in $\mathfrak{C}_0(M, \text{Bd } M; \Gamma^4)$; this group vanishes under the conditions of (2). The sufficiency of conditions (3) and (4) follows from the isomorphisms mentioned in 2.9.

2.11. Remark. One disadvantage of this obstruction theory is if \mathbf{f} is a differentiable structure mod $M^{(m)}$, and M_1 is some subdivision of M , there is no natural way of obtaining from \mathbf{f} a differentiable structure mod $M_1^{(m)}$. This is a peculiarity of our definition of a “diffeomorphism mod L ”. Our obstruction theory thus depends heavily on the combinatorial structure of M , not only on its piecewise-linear structure. The obstruction theories constructed by Milnor and Hirsch are formulated so as to avoid this difficulty.

3. A direct definition of the obstruction chain

By Theorem 2.9, it suffices in applying our obstruction theory to consider the class $\alpha_*[\lambda\mathbf{f}]$ in $\mathfrak{C}_{m-1}(M, \text{Bd } M; \Gamma^{n-m})$. Independently of the author’s work, S. Smale and M. Hirsch had defined an element of this group which seemed, for geometric reasons, to be a reasonable candidate for the obstruction to imposing a differentiable structure, although technical difficulties hampered the building of an obstruction theory around it. (It is a dualized form of the definition given by Thom [13]; his obstruction class lies in $H^{q+1}(M; \Gamma^q)$.) This element turns out to be precisely $\alpha_*[\lambda\mathbf{f}]$; their definition, to which we now turn, provides the desired geometric interpretation of the obstruction to imposing a differentiable structure.

3.1. DEFINITION. Let N be an oriented differentiable manifold combinatorially equivalent to S^n , i.e., to the boundary of an $(n + 1)$ -simplex. An equivalent requirement is that N be diffeomorphic to S^n mod a finite number of points, or mod a single point, by 6.8 and 6.9 of [11]. Let $g : N \rightarrow S^n$ be a diffeomorphism mod a finite number of points, and let $\lambda(N)$ be the index of the obstruction chain $\lambda_0 g$; $\lambda(N) \in \Gamma^n$. By 5.4 and 6.11 of [11], λ sets up a 1-1 correspondence between equivalence classes of such manifolds N , the equivalence relation being that of orientation-preserving diffeomorphism, and

elements of the group Γ^n . If $-N$ denotes the opposite orientation of N , then $\lambda(-N) = -\lambda(N)$.

3.2. DEFINITION. Let $f_v : \text{Cl}(\text{St } v) \rightarrow K_v$ be a differentiable structure mod $M^{(m)}$, as defined in 1.1. Let a partial ordering be chosen for the vertices of M , so that λf may be defined. Given an m -simplex σ of M , with leading vertex v , let f_σ denote the restriction of f_v to $\text{Cl}(\text{St } \sigma)$, and let K_σ denote the image of f_σ ; similarly for any $(m - 1)$ -simplex τ . The coordinate systems $(\text{St } \sigma, f_\sigma)$ cover $M - M^{(m-1)}$ and overlap differentiably (so that it would really have been more appropriate to call $\mathbf{f} = \{f_v\}$ a differentiable structure mod the $(m - 1)$ -skeleton rather than the m -skeleton).

Let τ be an $(m - 1)$ -simplex of $\text{Int } M$, oriented by the given ordering of its vertices. The coordinate systems $(\text{St } \sigma, f_\sigma)$ impose a differentiable structure on $\text{St } \tau - \tau$ and hence induce one on $\text{St } \tau' - \tau'$, where $\tau' = f_\tau(\tau)$. They also induce an oriented differentiable structure M^{n-m} on a small sphere in the plane orthogonal to τ' (as we verify below). This manifold belongs to the class described in 3.1; the new *obstruction chain* is defined as assigning to the oriented $(m - 1)$ -simplex τ the element $\lambda(M^{n-m})$ of Γ^{n-m} . It is considered as an element of the chain group $\mathcal{C}_{m-1}(M, \text{Bd } M; \Gamma^{n-m})$.

This chain is a reasonable candidate for an obstruction chain: if $\lambda(M^{n-m}) = 0$, the differentiable structure on M^{n-m} is the ordinary one and may be extended to a differentiable structure on the ball B^{n-m+1} . It then seems likely that the differentiable structure already defined on $\text{St } \tau - \tau$ should be able to be extended to a differentiable structure that covers τ as well.

3.3. PROPOSITION. *The new obstruction chain is well defined and equals $\alpha(\lambda f)$.*

Proof. Let τ be a fixed $(m - 1)$ -simplex of $\text{Int } M$; let σ be an m -simplex having τ as a face. Let $\tau' = f_\tau(\tau)$ and $\tau_\sigma = f_\sigma(\tau)$. Let \mathcal{P}_τ be an $(n - m + 1)$ -plane orthogonal to τ' at the point of p of τ' ; let S_τ be a small sphere in this plane, with center at p , taken in the usual differentiable structure. We orient \mathcal{P}_τ by a linear map, carrying R^n onto itself, τ' onto $0 \times R^{m-1}$, and \mathcal{P}_τ onto $R^{n-m+1} \times 0$, all in orientation-preserving fashions. This imposes an orientation on S_τ as well. Similarly, let S_σ be a small oriented sphere in the plane \mathcal{P}_σ orthogonal to τ_σ at $f_\sigma f_\tau^{-1}(p)$; let C_σ be the cell $S_\sigma \cap \text{Int } K_\sigma$. We map C_σ into S_τ by first applying $f_\tau f_\sigma^{-1}$, then projecting parallel to τ' into the plane \mathcal{P}_τ , and then projecting radially from p into S_τ . The crucial fact here is that $f_\tau f_\sigma^{-1} : K_\sigma \rightarrow R^n$ is smooth on $\text{Int } K_\sigma$ near τ_σ , so that by 3.3 of [11], this map of C_σ into S_τ is an orientation-preserving homeomorphism, which is a diffeomorphism mod the single point q_σ where C_σ intersects the simplex $f_\sigma(\sigma)$.

It is in this way that the coordinate systems $(\text{St } \sigma, f_\sigma)$ induce an oriented differentiable structure M^{n-m} on the sphere S_τ ; the identity map g of M^{n-m}

onto S_τ is a diffeomorphism mod finitely many points, so that $\lambda(M^{n-m})$ is well-defined.

Let us compute $\lambda(M^{n-m})$. It is the index of the obstruction chain $\lambda_0 g$, whose coefficient we now compute at the point which is the image of q_σ . When referred to coordinates in M^{n-m} and S_τ , this is merely the obstruction coefficient assigned to q_σ for the map of C_σ into S_τ . By 3.9 of [11], in turn, this is exactly the coefficient $\gamma(f_\tau^{-1}f_\sigma)$ evaluated on $f_\sigma(\sigma)$.

If τ and σ have the same leading vertex, $f_\tau^{-1}f_\sigma$ is the identity and the coefficient is zero. Otherwise, it is the coefficient γ_1 , where $(\gamma_1, \dots, \gamma_m)$ is the coefficient of σ in λf . (For then $\sigma = v w \dots$, where w is the leading vertex of τ ; $\gamma_1 = \gamma(f_w^{-1}f_\sigma)(f_\sigma(\sigma))$ by definition; and f_τ and f_σ are restrictions of f_w and f_σ , respectively.)

Now the coefficient of $\alpha(\lambda f)$ on τ is precisely the sum of the leading coefficients in λf of m -simplices σ having τ as their terminal face. This is, as we have just shown, $\lambda(M^{n-m})$.

3.4. The last paragraph of 3.2 suggests the likelihood of the following: Even if $\alpha(\lambda f)$ is not homologous to zero, one ought to be able to extend the differentiable structure given by the coordinate systems $(St \sigma, f_\sigma)$ across those $(m - 1)$ -simplices τ not lying in the carrier of $\alpha(\lambda f)$. This is in fact the case, under suitable hypotheses.

PROPOSITION. *Assume the hypotheses of 2.3. Let f be a differentiable structure mod $M^{(m)}$; let $\lambda f = 0$. There is a differentiable structure g_v mod $M^{(m-1)}$, by 1.3. Let c be any representative cycle in $\alpha_*[\lambda g]$ whose carrier c^* is contained in the $(m - 1)$ -skeleton of \tilde{M} . There is a differentiable structure h_v mod $M^{(m-1)}$ with $\alpha(\lambda h) = c$, such that the coordinate systems $(St \tau, h_\tau)$, for all $(m - 1)$ -simplices τ not in c^* , define a differentiable structure on $M - M^{(m-2)} - c^*$.*

Proof. By 2.7, there is a cycle ξ with $\alpha\xi = c$, such that the leading coefficient of ξ is zero on each m -simplex σ such that $\partial_0 \sigma$ is not in c^* . Since α_* is an isomorphism, ξ is homologous to λg , so that there is a differentiable structure g'_v mod $M^{(m-1)}$ such that $\lambda g' = \xi$, by 1.6. For each m -simplex $\sigma = v_0 \dots v_m$ with $\partial_0 \sigma = v_1 \dots v_m$ not in c^* , we may redefine the coordinate map $g'_1 : Cl(St v_1) \rightarrow R^n$, obtaining h_1 , so that $h_1(g'_0)^{-1}$ is differentiable in a neighborhood of $g'_0(\sigma)$, as in the proof of 1.3. Let $h_v = g'_v$ otherwise.

The coordinate systems $(St \sigma, h_\sigma)$ automatically overlap differentially; if τ is not in c^* , then by construction of h , $(St \tau, h_\tau)$ overlaps all of these coordinate systems differentially, as desired.

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