GAPS IN THE EXPONENT SET OF PRIMITIVE MATRICES

BY

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1. Introduction and definitions

An n by n matrix A is *reducible* if there exists a permutation matrix P such that $P^{-1}AP = \begin{vmatrix} B & 0 \\ D & C \end{vmatrix}$ where B and C are square matrices and 0 is a zero matrix. A matrix A is positive if every entry is positive, and is *non-negative* if every entry is zero or positive. An n by n non-negative irreducible matrix A is *primitive* if there exists an integer $t \geq 0$ such that A^t is positive. In this paper a non-negative irreducible primitive matrix will be called simply a primitive matrix. Let $\gamma(A)$ be the least integer with the property that A^t is positive for $t \geq \gamma(A)$. Wielandt [10] has stated that $\gamma(A) \leq (n-1)^2 + 1$. Proofs of this theorem have been given by Holladay and Varga [4] and Perkins [7].

In [3], $\gamma(A)$ has been called the *exponent* of the primitive matrix A. Let S be the set of all exponents of n by n primitive matrices. The main result of this paper concerns gaps in this exponent set S. Explicitly, if n is odd, there is no primitive matrix A for which

$$n^2 - 3n + 4 < \gamma(A) < (n-1)^2 \quad \text{or} \quad n^2 - 4n + 6 < \gamma(A) < n^2 - 3n + 2.$$

If n is even, there is no primitive matrix A for which

$$n^2 - 4n + 6 < \gamma(A) < (n-1)^2.$$

A *directed graph* consists of a *vertex set* $V = (1, 2, 3, \cdots, n)$ and a set of edges each of which is an ordered pair (i, j) of vertices. An edge (i, j) may also be called a *path of length 1* from vertex i to vertex j. If vertices $k_1, k_2, \cdots, k_{t-1}$ exist such that $(i, k_1), (k_1, k_2), \cdots, (k_{t-1}, j)$ are edges of the graph, then i is said to be connected to j by a *path of length t*. A directed graph is said to be strongly connected, if for any two vertices i, j of the vertex set with $i \neq j$, there is a path of some length connecting i to j. A *cycle* is a path which begins and ends with the same vertex. In such a path an edge may appear more than once. A *circuit* is a cycle of which no proper subgraph is a cycle. If the pair (i, i) is an edge, then this circuit of length 1 is called a loop and i is called a loop vertex. The greatest common divisor of the lengths of all cycles is equal to the greatest common divisor of the lengths of all circuits. A strongly connected directed graph D in which this greatest common divisor is 1 will be called *primitive*. The *exponent* $\gamma(D)$ of a primitive graph D is the least integer with the property that for every $t \geq \gamma(D)$ and every ordered pair of vertices, there is a directed path from the first vertex

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to the second of length t. A theorem of Schur asserts that a set of positive integers which is closed under addition contains all but a finite number of the multiples of its greatest common divisor. It is an easy consequence of this theorem that every primitive graph D has an exponent.

The *directed graph* D_A of an n by n matrix $A = (a_{ij})$ has vertex set $V = (1, 2, 3, \cdots, n)$ and the ordered pair (i, j) is an edge of D_A if and only if $a_{ij} \neq 0$. It is well known [3], [5], for a non-negative matrix A, that A is primitive if and only if the graph D_A is primitive. Moreover, the exponent $\gamma(A)$ is equal to the exponent $\gamma(D_A)$. Two directed graphs are *isomorphic* if there is a 1 to 1 correspondence between vertices which preserves edges. For two n by n matrices A and B, there exists a permutation matrix P such that A and $P^{-1}BP$ have the same zero entries if and only if the graphs D_A and D_B are isomorphic.

If D is a directed graph with vertex set $V = (1, 2, 3, \cdots, n)$ then the t^{th} *power of* D, denoted by D^t is the directed graph with the same vertex set V, such that the ordered pair (i, j) is an edge of D^t if and only if there is a path in D from vertex i to vertex j of length t. Thus $\gamma(D)$ is the smallest power of D which is a complete graph with n loops.

2. Theorems on the exponent of a primitive graph

In this section the theorems on the gaps in the exponent set of n by n primitive matrices are established.

Theorem 1 is a generalisation of Wielandt's result. The method of proof is essentially that of Holladay and Varga [2] but the conclusion, which gives an upper bound for $\gamma(A)$ in terms of the length of the shortest circuit in D_A is stronger. The theorem is based on a few preliminary remarks.

Remark 1. If D is a primitive graph then D^t is primitive for all $t > 0$.

Remark 2. If D is a primitive graph, then there exists for every vertex i an integer h with the property that for every vertex j there is a path from i to j of length h.

The least such integer h, denoted by h_i is called the *reach* of vertex i.

Remark 3. Let D be a primitive graph and let h_i be the reach of vertex i. If $p \geq h_i$ then there exists a path from i to any vertex j of length p.

Proof. Since D is strongly connected there is at least one vertex k of D such that (k, j) is an edge of D. Thus there is a path of length $h_i + 1$ from i to j for every j. The proof follows by induction.

Remark 4. If D is a primitive graph then $\gamma(D) = \text{Max}\,[h_1, h_2, \cdots, h_n]$.

Remark 5. If D strongly connected and i is a loop vertex then $h_i \leq n - 1$.

Proof. There is a path from i to j of length $q_{ij} \leq n - 1$. Combining this with $n - 1 - q_{ij}$ loops, we have a path from i to j of length $n - 1$.

THEOREM 1. *If D is a primitive graph and if s is the length of the shortest circuit in D then $\gamma(D) \leq n + s(n - 2)$. In other words, if A is a primitive matrix, and if s is the length of the shortest circuit in the directed graph D_A then $\gamma(A) \leq n + s(n - 2)$.*

Proof. Since D is primitive, D^s is primitive by Remark 1.

Since D has a circuit of length s, D^s has at least s loop vertices. Thus for any vertex i of D, there is a path in D of length $p_i \leq n - s$ from i to some vertex k of D which is a loop vertex in D^s.

Since k is a loop vertex in D^s, there exists, by Remark 5, for any vertex j, a path of D^s from k to j of length exactly $n - 1$. Thus there is, for any vertex j, a path in D from k to j of length $(n - 1)s$.

Combining these paths we have a path from i to any vertex j of length exactly $p_i + (n - 1)s$. It follows that $h_i \leq p_i + (n - 1)s$. Thus

$$\gamma(D = \text{Max}\,[h_1\ h_2, \cdots, h_n] \leq n - s + (n - 1)s = n + s(n - 2).$$

Since the greatest common divisor of the lengths of the circuits in a primitive graph is 1, it follows that $s \leq n - 1$. Thus

$$\gamma(A) \leq n + (n - 1)(n - 2) = (n - 1)^2 + 1.$$

Theorem 1 may be generalised as follows.

THEOREM 2. *Let D be a primitive graph with vertex set V and let Y be any subset of V. For each $k \in V$ let $h_k^{(q)}$ denote the reach of vertex k in D^q. Let p_{ik} be the length of the shortest path in D from vertex i to a vertex k of Y. Then*

$$\gamma(D) \leq \text{Max}_{i \in V}\, \text{Min}_{k \in Y}\{p_{ik} + h_k^{(q)} q\}.$$

Proof. We have $h_i \leq p_{ik} + h_k^{(q)} q$ for all $k \in V$ and hence we have $h_i \leq p_{ik} + h_k^{(q)} q$ for all $k \in Y$. Thus $h_i \leq \text{Min}_{k \in Y}\{p_{ik} + h_k^{(q)} q\}$. Since $\gamma(D) = \text{Max}_{i \in V}(h_i)$, the result follows.

Let X be the set of vertices of D each of which is in some circuit of length q. Theorem 2 is most useful when Y is a subset of X and when q is the length of the shortest circuit.

The following corollary is useful.

COROLLARY 1. *Let $h = \text{Max}_{k \in Y}\{h_k^{(q)}\}$ and let t be the length of the longest path in D required to get from any vertex i to some vertex k of Y. Then*

$$\gamma(D) \leq t + hq.$$

Proof. We have $t = \text{Max}_{i \in V}\{\text{Min}_{k \in Y}\, p_{ik}\}$. Thus

$$\gamma(D) \leq \text{Max}_{i \in V}\, \text{Min}_{k \in Y}\{p_{ik} + h_k^{(q)} q\} \leq \text{Max}_{i \in V}\, \text{Min}_{k \in Y}\{p_{ik} + hq\} = t + hq.$$

The number t in this corollary is $\leq n - |Y|$ where $|Y|$ is the cardinality of Y.

Let $p_1, p_2, \cdots, p_u$ be relatively prime and let $F(p_1, p_2, \cdots, p_u)$ denote the largest integer which is not expressible in the form $a_1 p_1 + a_2 p_2 + \cdots + a_u p_u$ where a_r is a non-negative integer for $r = 1, 2, \cdots, u$. This function F has been discussed by Bateman [1], Brauer and Seelbinder [2], Johnson [4], and Roberts [8]. It is well known, if m and n are relatively prime, that $F(m, n) = mn - m - n$. Roberts has shown, if $a_j = a_0 + jd$, $j = 0, 1, \cdots, s$, $a_0 \geq 2$, then

$$F(a_0, a_1, \cdots, a_s) = \left(\left[\frac{a_0 - 2}{s}\right] + 1\right) a_0 + (d - 1)(a_0 - 1) - 1,$$

where as usual $[x]$ denotes the greatest integer $\leq x$. The proof of this result has been simplified by Bateman. Johnson has given an ingenious algorithm which can be used to find F in the case of three variables. At the end of this paper two graphical methods for computing such F functions are described.

Let D be a primitive graph in which every circuit is of length p_1, p_2 $\cdots$, or p_u. For any ordered pair $(i; j)$ of vertices, a non-negative integer r_{ij} is defined as follows. If $i = j$ and if for $s = 1, 2, \cdots, u$ there is a circuit through vertex i of length p_s then $r_{ij} = 0$; otherwise r_{ij} is the length of the shortest path from i to j which has at least one vertex on some circuit of length p_s for $s = 1, 2, \cdots, u$. Let $r = \text{Max}(r_{ij})$ taken over all ordered pairs $(i; j)$.

THEOREM 3. *If D is a primitive graph then*

$$\gamma(D) \leq F(p_1, p_2, \cdots, p_u) + 1 + r.$$

Proof. For any set of non-negative integers $a_1, a_2, \cdots, a_u$ and any ordered pair $(i; j)$ of vertices, there is a path from vertex i to vertex j of length

$$r_{ij} + a_1 p_1 + a_2 p_2 + \cdots + a_u p_u .$$

Thus there is a path from vertex i to vertex j of length

$$F(p_1, p_2, \cdots, p_u) + r_{ij} + N$$

for every $N \geq 1$. Choosing $N = 1 + r - r_{ij}$, we have a path from vertex i to vertex j of length

$$F(p_1, p_2, \cdots, p_u) + 1 + r,$$

so that

$$h_i \leq F(p_1, p_2, \cdots, p_u) + 1 + r.$$

Thus

$$\gamma(D) = \text{Max}_{i \in V}\{h_i\} \leq F(p_1, p_2, \cdots, p_u) + 1 + r.$$

An ordered pair $(k; 1)$ of vertices in a primitive graph D is said to have the *unique path property* if every path from vertex k to vertex l which has length $\geq r_{kl}$ consists of some path α of length r_{kl} augmented by a number of circuits each of which has a vertex in common with α. (Note that the word "unique" in this definition refers to the length of the path α rather than to the path α itself.)

THEOREM 4. *If D is a primitive graph in which the ordered pair of vertices $(k; l)$ has the unique path property, then*

$$F(p_1, p_2, \cdots, p_u) + 1 + r_{kl} \leq \gamma(D).$$

Proof. There is no path from vertex k to vertex l of length

$$w = F(p_1, p_2, \cdots, p_u) + r_{kl}$$

for such a path would imply the existence of non-negative $a_1, a_2, \cdots, a_u$ with

$$F(p_1, p_2, \cdots, p_u) = a_1 p_1 + a_2 p_2 + \cdots + a_u p_u .$$

Thus by Remark 3, we have $F(p_1, p_2, \cdots, p_u) + r_{kl} < h_k$. Since $h_k \leq \text{Max}_{i \in V}\{h_i\} = \gamma(D)$, the result follows.

The following corollaries to Theorems 3 and 4 are immediate.

COROLLARY 2. *If in Theorem 4, $r_{kl} = r$ then*

$$h_k = \gamma(D) = F(p_1, p_2, \cdots, p_u) + 1 + r.$$

COROLLARY 3. *If in Theorem 4, $r_{ij} < r_{kl} = r$ for all ordered pairs $(i; j)$ other than $(k; l)$ then the graph $D^{\gamma(D)-1}$ is complete except for the missing edge (k, l).*

Theorems 3 and 4 may be generalised as follows. The definition of r_{ij} may be weakened by defining r_{ij} to be the length of the shortest path from vertex i to vertex j which has at least one vertex in common with a circuit of each of the lengths $p_{i_1}, p_{i_2}, p_{i_3}, \cdots, p_{i_v}$ (some subset of the circuit lengths) with $F(p_{i_1}, p_{i_2}, \cdots, p_{i_v} = F(p_1, p_2, \cdots, p_u)$. The unique path property, may be replaced by the weaker property for the ordered pair $(k; l)$ of vertices that if there is a path from vertex k to vertex l of length $w \geq r_{kl}$ then there exist non-negative integers $a_1, a_2, \cdots, a_u$ such that

$$w = r_{kl} + a_1 p_1 + a_2 p_2 + \cdots + a_u p_u .$$

(It is a simple matter to show that this property is indeed weaker). Theorems 3 and 4 and Corollaries 2 and 3 are valid if these weaker definitions are used.

THEOREM 5. *If s and n are relatively prime $(s < n)$, there exists a primitive graph D with n vertices and $n + 1$ edges for which $\gamma(D) = n + s(n - 2)$.*

Proof. The graph D with the $n + 1$ edges $(1, 2)(2, 3), \cdots (n - 1, n)(n, 1)$ and $(s, 1)$ has a circuit of length s, and a circuit of length n. Since s and n are relatively prime, D is primitive. The ordered pair $(s + 1; n)$ has the unique path property, with $r_{s+1,n} = 2n - s - 1$. Moreover, $r_{s+1,n} = r$. By Corollary 2,

$$\gamma(D) = F(n, s) + 1 + r = ns - s - n + 1 + 2n - s - 1 = n + s(n - 2).$$

In this graph, we have $r_{ij} < r$ for every ordered pair other than $(s + 1; n)$. By Corollary 3, it follows, if s and n are relatively prime $(s < n)$, then there exists an n by n matrix A of zeros and ones, with exactly $n + 1$ ones, such that $A^{n+s(n-2)} > 0$ and $A^{n+s(n-2)-1}$ has exactly one zero entry.

THEOREM 6. *Apart from isomorphism, there is exactly one primitive graph D on n vertices for which $\gamma(D) = (n - 1)^2 + 1$, and exactly one for which $\gamma(D) = (n - 1)^2$. These are the only graphs for which the length of the shortest circuit is $n - 1$.*

Proof. If $s < n - 1$, then from Theorem 1,

$$\gamma(D) \leq n + (n-2)^2 = n^2 - 3n + 4.$$

If $s = n - 1$, then, since the greatest common divisor of the lengths of the circuits is 1, the graph must have a circuit of length $n - 1$ and another of length n. Thus the graph D must have as a subgraph, a graph which is isomorphic to the graph of Theorem 4 with $s = n - 1$. Denote this graph by E. The graph E is isomorphic to the graph of the matrix used by Wielandt [11] to show that his result was best possible. There are two cases to consider.

Case (i). $D = E$. From Theorem 5, we have

$$\gamma(E) = n + (n-1)(n-2) = (n-1)^2 + 1.$$

Case (ii). E is a proper subgraph of D. The only edge which can be added to E without introducing a circuit of length less than $n - 1$ is the edge $(n, 2)$ or the edge $(n - 2, n)$. Since the resulting graphs are isomorphic, it is sufficient to consider the first case.

The ordered pair $(1; n)$ has the unique path property with $r_{1,n} = n - 1$. Moreover $r_{1,n} = r$. By Corollary 2,

$$\begin{aligned}\gamma(D) &= F(n, n-1) + 1 + r \\ &= n(n-1) - n - (n-1) + 1 + n - 1 = (n-1)^2.\end{aligned}$$

COROLLARY 4. *If A is an n by n primitive matrix and if*

$$\gamma(A) = (n-1)^2 + 1,$$

then there exists a permutation matrix P such that $P^{-1}AP$ has the same zero entries as

$$\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{vmatrix}.$$

If A is an n by n primitive matrix and if $\gamma(A) = (n-1)^2$ then there exists a permutation matrix P such that $P^{-1}AP$ has the same zero entries as

$$\begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{vmatrix}.$$

Such an explicit matrix formulation will not be given for the remaining results in this paper.

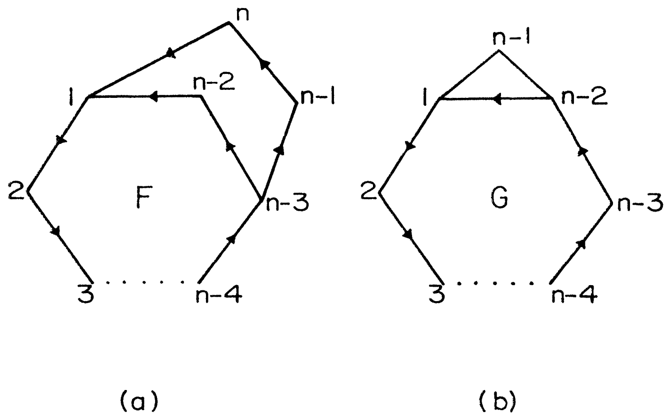

FIGURE 1

THEOREM 7. *If n is even* $(n > 4)$, *then* (a) *there is no primitive graph D such that*

$$n^2 - 4n + 6 < \gamma(D) < (n-1)^2,$$

and (b) *there are, apart from isomorphism, exactly* 3 *or exactly* 4 *primitive graphs D with* $\gamma(D) = n^2 - 4n + 6$, *according as n is or is not a multiple of* 3.

Proof. From Theorem 6, if $\gamma(D) < (n-1)^2$, we have $s \leq n-2$. For $s \leq n-3$, by Theorem 1,

$$\gamma(D) \leq n + (n-3)(n-2) = n^2 - 4n + 6.$$

If $s = n-2$, since n and $n-2$ are not relatively prime, a primitive graph D must have circuits of length $n-2$ and $n-1$. Beginning with a circuit of length $n-2$, a circuit of length $n-1$ must either involve both of the remaining vertices or one of the remaining vertices. It follows, that D must have as subgraph, a graph which is isomorphic to one of the graphs F in Figure 1(a) or G in Figure 1(b). There are two cases in the proof of (a).

Case (i). F is a subgraph of D. In F the ordered pair $(n-1; n)$ has the unique path property and $r_{n-1,n} = n = r$. By Corollary 2,

$$\gamma(D) \leq \gamma(F) = F(n-1, n-2) + 1 + r = n^2 - 4n + 6.$$

Case (ii). G is a subgraph of D. G is a primitive graph with $n-1$ vertices of the same type as the graph E of Theorem 6. Thus

$$\gamma(G) = (n-2)^2 + 1 = n^2 - 4n + 5.$$

There must be at least one edge (n, i_1) of D, $i_1 \neq n$, and at least one edge (i_2, n) of D, $i_2 \neq n$. We may assume $i_1 \neq i_2$, since otherwise we have a circuit of length 2 which is less than s for $n > 4$. Let H be the subgraph of D consisting of G together with the two edges (n, i_1) and (i_2, n). We show that $\gamma(H) \leq n^2 - 4n + 6$. If $j \neq n$, there is a path from i_1 to j in G of length exactly $n^2 - 4n + 5$ and adjoining (n, i_1) we have a path from n to j of length exactly $n^2 - 4n + 6$ in H. If $i \neq n$, there is a path from i to i_2 in G of length exactly $n^2 - 4n + 5$ and adjoining (i_2, n) there is a path of length exactly $n^2 - 4n + 6$ from i to n in H. For a path from n to n we use the fact that at least one of i_2 and $i_2 \neq n - 1$. If $i_1 \neq n - 1$, there exists a vertex n_1 of G such that (n_1, i_1) is the only edge in G out of n_1. There is a path from n_1 to i_2 in G of length exactly $n^2 - 4n + 5$. Replacing (n_1, i_1) by (n, i_1) and adjoining (i_2, n) yields a path from n to n in H of length $n^2 - 4n + 6$. Similarly if $i_2 \neq n - 1$, there exists a vertex n_2 of G such that (i_2, n_2) is the only edge into n_2. There is a path from i_1 to n_2 in G of length exactly $n^2 - 4n + 5$. Adjoining (n, i_1) and replacing (i_2, n_2) by (i_2, n) yields a path from n to n of length $n^2 - 4n + 6$ in H. We have

$$\gamma(D) \leq \gamma(H) \leq n^2 - 4n + 6.$$

If $i_1 \neq n - 1$ and $i_2 \neq n - 1$ it is easy to see, using the replacement edges (n_1, i_1) and (i_2, n_2) that $\gamma(D) \leq \gamma(H) \leq n^2 - 4n + 5$.

It remains to prove (b). If D has a circuit of length n in addition to those of length $n - 2$ and $n - 1$, and if F is a subgraph of D, then the r_{ij} for a pair of vertices in D is less than or equal to the r_{ij} for the same pair in F, since the additional condition that the path must have a vertex on a circuit of length n is satisfied for any path. If G is a subgraph of D, then r for D is less than or equal to $r + 1$ for G. For F and G we have $r = n$ and $n - 1$ respectively. Thus for D, we have $r \leq n$. By Corollary 2, and the result of Roberts [9], referred to earlier, we have

$$\begin{aligned}\gamma(D) &\leq F(n, n-1, n-2) + 1 + r \\ &\leq \left[\frac{n-2}{2}\right](n-2) - 1 + 1 + n < n^2 - 4n + 6.\end{aligned}$$

Cases in which D has a circuit of length n have been disposed of.

Now consider the case in which F is a subgraph of D. The only edges which can be added to F without introducing a circuit of length n, or of length $< n - 2$ or reducing the number r (either of which reduces the exponent), are the edges $(n - 2, n)$ and $(n - 1, n - 2)$. In each case the ordered pair $(n - 1; n)$ has the unique path property and $r_{n-1,n} = n = r$, so that $\gamma(D) = n^2 - 4n + 6$. If $(n - 2, n)$ is added the resulting graph is denoted by F_1, if $(n - 1, n - 2)$ by F_2. If both edges are added there is a circuit of length n.

There remains the case in which G is a subgraph of D. In the proof of (a) it was noted that, if $\gamma(D) = n^2 - 4n + 6$ then, either $i_1 = n - 1$ or $i_2 = n - 1$, but not both. If $i_1 = n - 1$, then $i_2 = n - 3$, for if $i_2 = n - 2$ we have a cycle of length n and if $i_2 = n - 4$, we have $r = r_{nn} = n - 1$, and in other cases, we have a circuit of length less than $n - 2$. With $i_1 = n - 1$ and $i_2 = n - 3$, the resulting graph is isomorphic to F_1. The case in which

$i_2 = n - 1$ is handled in a similar way. In this case we find that i_1 must be 2 and we get a graph isomorphic with F_2.

If $s = n - 3$ and the graph has a circuit of length $n - 1$ or $n - 2$, then using Corollary 2, it follows that $\gamma(D) < n^2 - 4n + 6$. In other cases, since D is primitive, n is not a multiple of 3. Moreover, D has as a subgraph the graph K of Theorem 5 with $s = n - 3$. In K, the ordered pair $(n - 2; n)$ has the unique path property and $r_{n-2,n} = n + 2 = r$. Thus $\gamma(K) = n^2 - 4n + 6$. Since no edge can be added to K without introducing a cycle of length other than n and $n - 3$, or reducing the number r, we have $K = D$.

THEOREM 8. *If n is odd* $(n > 3)$ *then* (a) *there is no primitive graph D such that* $n^2 - 3n + 4 < \gamma(D) < (n - 1)^2$, *and* (b) *apart from isomorphism there is exactly one primitive graph D with* $\gamma(D) = n^2 - 3n + 4$, *and exactly one primitive graph D with* $\gamma(D) = n^2 - 3n + 3$, *and exactly two primitive graphs D with* $\gamma(D) = n^2 - 3n + 2$, *and* (c) *there is no primitive graph D such that* $n^2 - 4n + 6 < \gamma(D) < n^2 - 3n + 2$, *and* (d) *apart from isomorphisms, there are exactly* 3 *or exactly* 4 *primitive graphs D with* $\gamma(D) = n^2 - 4n + 6$ *according as n is or is not a multiple of* 3.

Proof. If $\gamma(D) < (n - 1)^2$, then $s \leq n - 2$ and hence, by Theorem 1, $\gamma(D) \leq n^2 - 3n + 4$. If $s \leq n - 2$, and if the graph has a circuit of length $n - 1$, then $\gamma(D) \leq n^2 - 4n + 6$. This follows, because the proof of this result given in Theorem 7 for n even holds also for n odd. Thus parts (a) and (d) of Theorem 8 are established.

The only other graphs which need be classified are those in which the circuits are of lengths $n - 2$ and n. Any such graph must have as a subgraph the graph L of Theorem 5 in which $s = n - 2$. In L, the pair $(n; n - 1)$ has the unique path property with

$$r_{n,n-1} = n + 1 = r \quad \text{and} \quad \gamma(L) = F(n, n - 2) + 1 + r = n^2 - 3n + 4.$$

In the graph L_1, formed by adding to L the edge $(n - 1, 2)$ the ordered pair $(n; n)$ has the unique path property and $r_{nn} = n = r$. By Corollary 2, $\gamma(L_1) = n^2 - 3n + 3$.

In the graph L_2, formed by adding to L the edge $(n, 3)$, the ordered pair $(1; n)$ has the unique path property with $r_{1n} = n - 1 = r$. Thus

$$\gamma(L_2) = n^2 - 3n + 2.$$

In the graph L_3 formed by adding to L the edges $(n, 3)$ and $(n - 1, 2)$, the ordered pair $(1; n)$ has the unique path property and $r_{1n} = n - 1 = r$. Thus $\gamma(L_3) = n^2 - 3n + 2$.

There are alternative edges which may be added to L without introducing circuits of other lengths, but, in each case, the resulting graph is isomorphic to L_1, L_2 or L_3. This completes the proof of Theorem 8.

For any $s < n$, let $a(s)$ and $b(s)$ be the minimum and maximum of the set of exponents of all primitive graphs in which the shortest circuit has length

s. Theorem 1 implies that $b(s) \leq n + s(n - 2)$. Theorem 9 leads to an upper bound for $a(s)$. In Theorem 9 we require the following definition. A graph with m vertices is complete with respect to the ordering $v_1, v_2, \cdots, v_m$ if the ordered pair (v_i, v_j) is an edge if and only if v_i precedes v_j in the ordering.

THEOREM 9. *For any n and s, $s < n$, there exists a primitive graph M_1 in which the shortest circuit has length s, and*

$$\gamma(M_1) = s\left[\frac{n-2}{n-s}\right] + 1 + s,$$

and a primitive graph M_2 in which the shortest circuit has length s, and

$$\gamma(M_2) = \left[\frac{n-2}{n-s}\right] s + s.$$

Proof. Let N be the graph with $n - s + 2$ vertices $s, s + 1, s + 2, \cdots, n - 1, n, 1$ which is complete with respect to this ordering. If the edges $(1, 2)(2, 3)(3, 4)\cdots(s - 1, s)$ are added to N, denote the resulting graph by M_1. In M_1, the pair $(n; s + 1)$ has the unique path property with $r_{n,s+1} = s + 1 = r$ and circuits of lengths $s, s + 1, s + 2, \cdots, n - 1, n$. Thus

$$\gamma(M_1) = F(n, n-1, n-2, \cdots, s) + 1 + r = \left[\frac{n-2}{n-s}\right] s + 1 + s.$$

Now let M_2 be the graph obtained from M_1 by adding the edges $(i, 2)$ for $i = s + 1, s + 2, \cdots, n$. In M_2, the pair $(1; s + 1)$ has the unique path property with $r_{1,s+1} = s = r$ and circuits of lengths $s, s + 1, \cdots, n$. Thus

$$\gamma(M_2) = F(n, n-1, \cdots, s) + 1 + r = \left[\frac{n-2}{n-s}\right] s + s.$$

COROLLARY 5. *If $a(s)$ is the minimum of the set of exponents of all primitive graphs with shortest circuit of length s, then*

$$a(s) \leq \gamma(M_2) = \left[\frac{n-2}{n-s}\right] s + s.$$

THEOREM 10. *If D is a primitive graph with n vertices and if w is a positive integer then*

(a) $(h_i^{(w)} - 1)w < h_i \leq h_i^{(w)}w$, *for* $i = 1, 2, \cdots, n$, *and*

(b) $w\gamma(D^w) - w < \gamma(D) \leq w\gamma(D^w)$.

Proof. Part (a) follows from the definition of h_i and $h_i^{(s)}$. Since $\gamma(D) = \mathrm{Max}_{i \epsilon V}\{h_i\}$ and $\gamma(D^s) = \mathrm{Max}_{i \epsilon V}\{h_i^{(s)}\}$, we have (b).

COROLLARY 6. *Let D be a primitive graph in which the ordered pair of vertices $(k; l)$ has the unique path property with $r_{kl} = r$. If $p_1, p_2, \cdots, p_u$ are the lengths of the circuits in D then, for every positive integer w,*

$$w\gamma(D^w) - (1 + r) - w < F(p_1, p_2, \cdots, p_u) \le w\gamma(D^w) - (1 + r).$$

This follows since $F(p_1, p_2, \cdots, p_u) + 1 + r = \gamma(D)$.

3. A Connection with number theory

We now illustrate the graphical methods referred to earlier for computing F functions. Since the computation of the function F is still an untractable problem in number theory and since the graphical methods can be applied in different ways, we give two proofs of the result that

$$F(n, n - 1, n - 2, \cdots, s) = s\left[\frac{n - 2}{n - s}\right] - 1.$$

In the first proof, the formula which we use is the formula $\gamma(D) \le t + hq$ of Corollary 1, applied to the graph M_1 of Theorem 9.

In the graph M_1^s, consider the vertices arranged in cyclic order $1, 2, 3, \cdots, n$, as in the circuit C of length n in M_1. In M_1^s, the vertex 1 is the first member of the edges $(1, 1)(1, n)(1, n - 1), \cdots, (1, s + 1)$ and the vertex 2 is the first member of the edges $(2, 2)(2, 1)(2, n), \cdots, (2, s + 2)$. In fact each of the vertices $1, 2, \cdots, s$ is edge connected to itself and to the $n - s$ edges which precede it in the circuit C. The vertex $s + 1$ is edge connected to the $n - s$ previous edges, the vertex $s + 2$ to $n - s - 1$ previous edges beginning with $s, \cdots$ and finally, vertex $n - 1$ is edge connected to s and $s - 1$ and n is edge connected only to s. In Corollary 1 take $Y = \{1, 2, \cdots, s\}$. It is a simple matter to see that

$$h_i^{(s)} = \left[\frac{n - 2}{n - s}\right] + 1 \qquad \text{for } i \in Y.$$

Thus

$$h = \left[\frac{n - 2}{n - s}\right] + 1.$$

Also $t = 1$. We have $\gamma(M_1) \le t + hq = 1 + sh$ where

$$h = h_1^{(s)} = h_2^{(s)} = \cdots = h_s^{(s)} = \left[\frac{n - 2}{n - s}\right] + 1.$$

We have $h_n^{(s)} = h_s^{(s)} + 1 = 1 + h$, since the only path of length s for vertex n terminates in vertex s. Also $h_n^{(s)} \le \text{Max}_{i \in V}(h_i^{(s)}) = \gamma(M_1^s)$. Thus by Theorem 10 part (a), $sh_n^{(s)} - s < \gamma(M_1)$. But $\gamma(M_1) \le 1 + sh$, so that $s(h + 1) - s < \gamma(M_1) \le 1 + sh$. Thus $\gamma(M_1) = 1 + sh$. Since

$$F(n, n - 1, \cdots, s) + 1 + r = \gamma(M_1)$$

we have

$$F(n, n - 1, \cdots, s) = \left[\frac{n - 2}{n - s}\right] s - 1.$$

To sum up the relationship between the F function and the exponent which is revealed in this first proof, note that if $F(p_1, p_2, \cdots, p_u)$ is known and if there is an ordered pair $(k; l)$ with the unique path property such that $r_{kl} = r$, then Corollary 2 may be used to find $\gamma(D)$. Conversely, if $F(p_1, p_2, \cdots, p_u)$ is unknown, it may be possible to construct an appropriate graph D and use Corollary 1 and Theorems 3 and 4, to get

$$F(p_1, p_2, \cdots, p_u) + 1 + r_{kl} \leq \gamma(D) \leq t + hq$$

which gives an upper bound for F. Furthermore, Corollary 6, gives upper and lower bounds for F and as we have just seen these can actually yield exact values.

We proceed now to the second proof. In M_1, the pair $(n; s + 1)$ has the unique path property with $r_{n,s+1} = r = s + 1$. By Corollary 2,

$$h_n = F(n, n - 1, \cdots, s) + s + 2 = \gamma(M_1).$$

Also, as in the previous proof,

$$h_i^{(s)} = \left[\frac{n-2}{n-s}\right] + 1, \qquad \text{for } i = 1, 2, \cdots, s.$$

By Theorem 10, $h_1^{(s)}s - s < h_1 \leq h_1^{(s)}s$ and $(h_n^{(s)} - 1)s < h_n < h_n^{(s)}s$. Combining these with $h_n^{(s)} = h_1^{(s)} + 1$ and $h_n = h_1 + 1$ we obtain $h_1 = sh_1^{(s)}$ and $h_n = sh_1^{(s)} + 1$ and from these the value of

$$F(n, n - 1, \cdots, s) = s\left[\frac{n-2}{n-s}\right] - 1.$$

It may be worth mentioning that the computation of the more general result of Roberts, namely, the formula for $F(n, n - d, n - 2d, \cdots, n - kd)$ can be carried out in the same way.

We conclude this paper with a theorem which is suggested by the second proof of the formula for $F(n, n - 1, \cdots, s)$. Then two examples are given each of which translates the number theory problem of finding an F function into the problem of locating a certain vertex in a primitive graph D, together with the reach of this vertex in a power of D. Finally, a few F functions found by this method are listed.

The *out-valence* of a vertex i of a directed graph is the number of edges which have i as first member.

THEOREM 11. *Let* $(1, 2)(2, 3) \cdots (f, f + 1)$ *be a path of length* f *in a primitive graph* D *in which each of the* f *vertices* $1, 2, 3, \cdots, f$ *has out-valence* 1. *Let* w *be an integer*, $0 < w < f + 1$.

Then (a) $h_{i+1} = h_i - 1$ *for* $i = 1, 2, \cdots, f$,

(b) $h_{i+w}^{(w)} = h_i^{(w)} - 1$ *for* $i = 1, 2, \cdots, f + 1 - w$,

and (c) $h_{i+1}^{(w)} \leq h_i^{(w)}$ *for* $i = 1, 2, \cdots, f$.

Moreover, if W *is a set of* w *vertices* $(0 < w < f + 1)$ *which are consecutive* *in the path from* 1 *to* $f + 1$, *then there exists a unique vertex* $g \in W$ *such that* $h_g = h_g^{(w)}w$.

Proof. Let j be any vertex of D and let p be any positive integer. For $i = 1, 2, \cdots, f + 1 - w$ there is a 1 to 1 correspondence between paths from vertex i to vertex j of length pw and paths from vertex $i + w$ to vertex j of length $(p - 1)w$. This proves (b). Putting $w = 1$ we have (a).

By Theorem 10 we have

$$(h_i^{(w)} - 1)w < h_i \leq h_i^{(w)}w, \quad \text{and} \quad (h_{i+1}^{(w)} - 1)w < h_{i+1} \leq h_{i+1}^{(w)}w.$$

Since $h_{i+1} = h_i - 1$, we have $(h_{i+1}^{(w)} - 1)w + 1 < h_i \leq h_i^{(w)}w$. Thus $h_{i+1}^{(w)} \leq h_i^{(w)}$.

Let $i_1, i_2, \cdots, i_w$ be w consecutive vertices in the path from 1 to $f + 1$ and let $W = (i_1, i_2, \cdots, i_w)$.

Now suppose $h_i^{(w)} = h_{i_w}^{(w)}$. By Theorem 10, $(h_{i_w}^{(w)} - 1)w < h_{i_w} \leq h_{i_w}^{(w)}w$ and $w(h_{i_1}^{(w)} - 1) < h_{i_1} \leq h_{i_1}^{(w)}w$. By (a), $h_{i_1} = h_{i_w} + w - 1$. Combining these, we have $(h_{i_1}^{(w)} - 1)w + (w - 1) < h_{i_1} \leq h_{i_1}^{(w)}w$. Thus $h_{i_1} = h_{i_1}^{(w)}w$. By (a), it follows that $h_i < h_i^{(w)}w$ for $i = i_2, i_3, \cdots, i_w$. Thus the vertex g is i_1.

If $h_{i_1}^{(w)} \neq h_{i_w}^{(w)}$, it follows from (b) and (c) that $h_{i_1}^{(w)} = h_{i_w}^{(w)} + 1$ and that there exists a unique vertex g in W such that $h_{g-1}^{(w)} = h_g^{(w)} + 1$. Using Theorem 10 relative to vertices $g - 1$ and g, and using $h_g = h_{g-1} - 1$, it follows that $h_g = h_g^{(w)}w$. Furthermore $h_i > h_g^{(w)}w$ if vertex i precedes vertex g in the path from i to $f + 1$ and $h_i < h_g^{(w)}w$ if g precedes i in this path. This completes the proof.

Corollary 7. *In the previous theorem let the circuit lengths in the graph* D *be* $p_1, p_2, \cdots, p_u$. *Suppose there exists a vertex* k *such that the ordered pair* $(1; k)$ *has the unique path property with* $r_{1k} = r$. *Let* g *be defined as follows. If* $h_{i_1}^{(w)} = h_{i_w}^{(w)}$ *then* $g = 1$. *If* $h_{i_1}^{(w)} \neq h_{i_w}^{(w)}$ *then* g *is the unique vertex of* W *such that* $h_{g-1}^{(w)} = h_g^{(w)} + 1$. *If* v *is the length of the unique path from vertex* 1 *to vertex* g,

$$h_1 = \gamma(D) = h_g + v = h_g^{(w)}w + v = 1 + r + F(p_1, p_2, \cdots, p_u).$$

Thus $F(p_1, p_2, \cdots, p_u) = h_g^{(w)}w + v - r - 1$.

In the following examples the problem of finding an F function is replaced by the problem of finding the vertex g of Corollary 7 and the reach of vertex g in D^s. This may be done in various ways using different primitive graphs for the same F.

Example 1. Consider $F(s, n - k, n - k + 1, \cdots, n)$ where $k \geq 1$ and $s < n - k$. Let D be the graph defined as follows. D consists of the circuit $(1, 2)(2, 3)\cdots(n, 1)$ together with edges $(s, 1)$ and those edges necessary to make the subgraph with vertices $s, s + 1, s + 2, \cdots, s + k + 1$ complete with respect to that ordering. The circuit lengths are $s, n - k, n - k + 1,$

$\cdots, n$. In the path from $s + k$ to s every vertex except the last has out-valence 1. This path has $n - k + 1$ vertices and hence $f = n - k$. Moreover $s < f + 1$. Now use Corollary 7 with $w = s$ and $W = (1, 2, \cdots, s)$. The ordered pair of vertices $(s + k; s + 1)$ has the unique path property with $r_{s+k,s+1} = r = n - k + 1$. If $g \epsilon W$ is the unique vertex defined in corollary 7 then $v = n - (s + k) + g$. Thus

$$F(s, n - k, n - k + 1, \cdots, n) = h_g^{(s)}s + v - r - 1 = h_g^{(s)}s - s + g - 2.$$

Example 2. Consider $F(s, s + 1, s + 2, \cdots, s + u, n)$ with $u \geq 1$ and $s + u < n$. Let D consist of the circuit $(1, 2), (2, 3), \cdots, (n, 1)$ augmented by the edges $(s, 1)$ $(s + 1, 1)$ $(s + 2, 1), \cdots, (s + u, 1)$. In the path from $s + u + 1$ to s every vertex except s has out-valence 1. The path has $n - u = f + 1$ vertices with $f + 1 > s$. Now use Corollary 7 with $w = s$ and $W = (1, 2, \cdots, s)$. The pair $(s + u + 1; s + u + 1)$ has the unique path property with $r_{s+u+1,s+u+1} = r = n$. If $g \epsilon W$ is the unique vertex defined in the corollary then $v = n - (s + u + 1) + g$. Thus

$$F(s, s + 1, s + 2, \cdots, s + u, n) = h_g^{(s)}s + v - r - 1 = h_g^{(s)}s - s - u + g - 2.$$

The following are samples of results obtained by the authors using these graphical methods.

$$F(n, n + 1, n + 2, n + 4) = \left[\frac{n}{4}\right](n + 1) + \left[\frac{n + 1}{4}\right] + 2\left[\frac{n + 2}{4}\right] - 1$$

$$F(n, n + 1, n + 2, n + 5) = n\left[\frac{n + 1}{5}\right] + \left[\frac{n}{5}\right] + \left[\frac{n + 1}{5}\right] + \left[\frac{n + 2}{5}\right] + 2\left[\frac{n + 3}{5}\right] - 1$$

$$F(n, n + 1, n + 2, n + 6) = n\left[\frac{n}{6}\right] + 2\left[\frac{n}{6}\right] + 2\left[\frac{n + 1}{6}\right] + 5\left[\frac{n + 2}{6}\right] + \left[\frac{n + 3}{6}\right] + \left[\frac{n + 4}{6}\right] + \left[\frac{n + 5}{6}\right] - 1.$$

These results apparently cannot be obtained by a direct application of other methods.

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