# ENGEL CONDITIONS AND DIRECT DECOMPOSITIONS INTO GROUPS OF COPRIME ORDER

BY

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## 1. Introduction

In the present note we shall be concerned exclusively with finite groups possessing the following property:

(A) *If $x$ is a primary $\pi$-element and if $y$ is a primary $\pi'$-element of a finite group, then $x^{(i)} \circ y = 1$ and $y^{(i)} \circ x = 1$ for almost every positive integer $i$.*

Here and in what follows $\pi$ and $\pi'$ denote fixed complementary sets of primes. For the terminology the reader is referred to Section 2.

Every direct product of a $\pi$-group and a $\pi'$-group meets requirement (A) and it appears to be an open question whether the converse holds true too. Thus a proof of the following weaker result may be of interest.

THEOREM. *The following properties of the finite group $G$ and the set $\pi$ of primes are equivalent:*

(I) *$G$ is a direct product of a $\pi$-group and a $\pi'$-group.*

(II) *$x^{(3)} \circ y = y^{(3)} \circ x = 1$ for every primary $\pi$-element $x$ and every primary $\pi'$-element $y$.*

(III)
- (a) *$x^{(i)} \circ y = y^{(i)} \circ x = 1$ for every primary $\pi$-element $x$, every primary $\pi'$-element $y$ and almost all $i$.*
- (b) *If the simple factor $F$ of $G$ is neither a $\pi$-group nor a $\pi'$-group, then elements of order 2 in a Sylow subgroup of $F$ commute.*

The proof of the theorem will be carried out in Section 3.

*Remark.* As is well known, the group $G$ is called $\pi$-homogeneous, if elements of $G$ do not induce $\pi'$-automorphisms in $\pi$-subgroups of $G$. There is a conjecture, for which no proof or counter-example seems to exist, that a group is the direct product of a $\pi$-group and a $\pi'$-group, provided it is $\pi$-homogeneous as well as $\pi'$-homogeneous (cf. [2, Satz, p. 454]). Therefore it should be remarked, that an easy application of Satz LR of [5, *p.* 239] and Theorem 1 of [1, p. 38] shows that $\pi$- and $\pi'$-homogeneity is a consequence of property (A).

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## 2. Definitions and notations

The group $G$ (element $g$) is a $\pi$-group ($\pi$-element), if the order $o(G)$ of $G$ ($o(g)$ of $g$) is divisible by primes in $\pi$ only.

The group element $g$ is called primary, if the order $o(g)$ of $g$ is a prime power.

The group $G$ is termed $\pi$-soluble, if every minimal normal subgroup of every epimorphic image of $G$ is either a $\pi$-group or a $\pi'$-group.

$G_\pi$ denotes the set of all $\pi$-elements of the group $G$.

The group $G$ is termed $\pi$-closed, if $G_\pi$ is a subgroup of $G$.

$N(A, G)$ = normalizer of the subset $A$ of $G$ in $G$.

$C(A, G)$ = centralizer of the subset $A$ of $G$ in $G$.

Factor of the group $G$ = epimorphic image of a subgroup of $G$.

$x \circ y = x^{-1}y^{-1}xy, \quad x^{(0)} \circ y = y, \quad x^{(i+1)} \circ y = x \circ (x^{(i)} \circ y)$.

## 3. Proof of the theorem

If the group $G$ is the direct product of a $\pi$-group and a $\pi'$-group, then all $\pi$-elements commute with all $\pi'$-elements and simple factors of $G$ are either $\pi$-groups or $\pi'$-groups. This shows that the conditions (II) and (III) are consequences of (I).

If $G$ satisfies the condition (II), then it follows immediately from a result of H. Heineken [4; Lemma 3, p. 685], that every $\pi$-element of $G$ commutes with every $\pi'$-element of $G$. Therefore $G$ is the direct product of a $\pi$-group and a $\pi'$-group and we have shown the equivalence of the conditions (I) and (II).

Now we proceed to prove that (III) implies (I). If this were false, then there would exist among the groups of finite order a group $G$ of minimal order with the following properties:

(1) $G$ meets requirement (III).

(2) $G$ is not the direct product of a $\pi$-group and a $\pi'$-group.

Obviously, the property (III) of a group is inherited by subgroups and epimorphic images. Hence we have the following from the minimality of $G$:

(3) Every proper subgroup and every proper epimorphic image of $G$ is the direct product of a $\pi$-group and a $\pi'$-group.

Assume now by way of contradiction the nonsimplicity of $G$. Then there exists a minimal normal subgroup $K$ of $G$ with $1 \subset K \subset G$. Because $o(K) < o(G)$, application of (3) yields that $K$ is the direct product of a $\pi$-group $K_\pi$ and a $\pi'$-group $K_{\pi'}$. As $K_\pi$ and $K_{\pi'}$ are characteristic subgroups of the normal subgroup $K$ of $G$ they are normal in $G$. Hence $K = K_\pi$ or $K = K_{\pi'}$ because of the minimality of $K$. Since the property (III) is symmetric in $\pi$ and $\pi'$, we may assume without loss of generality that $K = K_\pi$ is a $\pi$-group. From $K \neq 1$ and (3) we deduce that $(G/K)_\pi$ is a subgroup of $G/K$. But $K$ is a $\pi$-group and it follows that $(G/K)_\pi = G_\pi/K$. Thus we have shown that $G_\pi$ is a subgroup of $G$, and therefore $G$ is $\pi$-closed. Since $\pi$-closed groups are certainly $\pi$-soluble it follows from [5; Korollar 11 and 11′, pp. 246 and 247] that $\pi$-closure and properties (1) and (2) are contradictory properties. Hence we have the following:

(4) $G$ is simple and nonabelian.

Application of the Theorem of W. Feit and J. G. Thompson [3] yields the existence of an element $a$ of order 2 in $G$. Without loss of generality let 2 be a prime in $\pi$. Denote by $\Theta$ the characteristic set of all $p$-Sylow subgroups of $G$ for all $p \in \pi'$. It is a consequence of (2), that $\Theta$ consists not only of the unit group 1. If $P$ is an element of $\Theta$ denote by $a^P$ the set of all $a^x = x^{-1}ax$ with $x \in P$. It follows from (1) that $a^{(i)} \circ x = 1$ for every $x$ in $P$ and almost every positive integer $i$. We shall show that $a \circ x$ is a 2-element. To do this it clearly sufficies to prove the equation

$$a^{(i)} \circ x = (a \circ x)^{2i-1}.$$

The equation is obviously true for $i = 1$. For the inductive proof we assume the validity of the equation for all positive integers $i \leqq n$. Then

$$\begin{aligned} a^{(n+1)} \circ x &= a \circ (a^{(n)} \circ x) = a \circ (a \circ x)^{2n-1} \\ &= a(x \circ a)^{2n-1} a (a \circ x)^{2n-1} \\ &= (a \circ x)^{2n-1} a a (a \circ x)^{2n-1} = (a \circ x)^{2n}, \end{aligned}$$

which completes the inductive argument and shows that $a \circ x$ is of order a power of 2. Since $a \in N(\{a \circ x\}, G)$, the subgroup $\{a, a \circ x\} = \{a, a^x\}$ of $G$ is a 2-group. Application of (III, b), (2), and (4) yields $a \circ a^x = 1$. Since $a^x \circ a^y = 1$ holds for all elements $x$ and $y$ of $P$ if and only if $a \circ a^{yx^{-1}} = 1$, we have shown that every element of $a^P$ commutes with every element of $a^P$. Hence $\{a^P\} = W$ is an abelian 2-group with $P \subseteq N(W, G)$. From (1) and [5; Korollar 4, p. 242] it follows that the $(\pi' \cup 2)$-group $PW$ is the direct product of $P$ and $W$. Thus the element $a$ commutes with all elements of $P$. Since $G$ is simple by (4), it is generated by the subgroups $P$ in the characteristic set $\Theta$. Hence $a$ belongs to the center of $G$ contradicting (4). This contradiction we have derived from the assumption that (I) is not a consequence of (III). Therefore (III) implies (I) and the theorem is proved.

## References

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