A MAPPING OF REGRESSIVE ISOLS[1]

BY

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1. Introduction

We assume familiarity with the principal definitions and results of [1] and [7]. Denote the set of all non-negative integers by ε, the collection of all isols by Λ, and the collection of all regressive isols by Λ_R. If f is a function, we use the notation ρf and δf to denote the range and domain of f respectively. For a combinatorial function f, the notation f_Λ is employed to denote its canonical extension to Λ. In [1], it was shown that if f is a recursive, combinatorial function and $T \in \Lambda_R - \varepsilon$, then $\sum_T f_n = (s_f)_\Lambda(T)$, where s_f is the partial sum function of f. The main result of [7] states that for f strictly increasing, recursive, combinatorial and $T \in \Lambda_R - \varepsilon$, $\Phi_f(f_\Lambda(T)) = T$. One of the purposes of this paper is to extend both of these results, the first to the class of recursive functions and the second to the class of strictly increasing, recursive functions. The principal result obtained states that for f strictly increasing, recursive and $T \in \Lambda_R - \varepsilon$, $\Phi_f(T) = \bar{f}_\Lambda(T)$, where $\bar{f}(n) = (\mu y)[f(y) \geq n]$.

2. The Generalized sum

In this section, we define and study an infinite series of integers, summed with respect to a regressive isol T. This sum is called a star-sum. It is shown that if the terms of the series are given by a recursive function, then the star-sum and the sum defined in [1] are equivalent.

Let f and g be recursive, combinatorial functions. It is well known that for $X \in \Lambda$,

(1) $f_\Lambda(X) + g_\Lambda(X) = (f+g)_\Lambda(X)$,
(2) $f_\Lambda(X) \cdot g_\Lambda(X) = (f \cdot g)_\Lambda(X)$.

PROPOSITION 1. *Let f and g be recursive, combinatorial functions. Then for $T \in \Lambda_R$,*

$$\sum_T f_n + \sum_T g_n = \sum_T (f_n + g_n).$$

Proof. Denote the partial sum functions of f and g as defined in [1] by s_f and s_g respectively. Then s_f and s_g are also recursive, combinatorial. For $T \in \Lambda_R$, we have by Theorem 2 of [1],

$$\sum_T f_n = (s_f)_\Lambda(T), \qquad \sum_T g_n = (s_g)_\Lambda(T).$$

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Since $s_f + s_g = s_{f+g}$, and $f + g$ is recursive, combinatorial, the result follows by (1) and another application of Theorem 2 of [1].

In the following, $m(i, k)$ denotes the maximum of the two non-negative integers i, k.

PROPOSITION 2. *Let f and g be recursive, combinatorial functions. Then the function*

$$h_n = \sum_{m(i,k)=n} f_i g_k$$

is also recursive and combinatorial. Moreover, for $T \in \Lambda_R$

$$\left(\sum_T f_n\right)\left(\sum_T g_n\right) = \sum_T h_n .$$

Proof. As in Proposition 1, $\sum_T f_n = (s_f)_\Lambda(T)$ and $\sum_T g_n = (s_g)_\Lambda(T)$. Hence by (2),

$$\left(\sum_T f_n\right)\left(\sum_T g_n\right) = (s_f \cdot s_g)_\Lambda(T).$$

It is readily verified however, that for all n

$$s_f(n) \cdot s_g(n) = s_h(n).$$

Since $s_f \cdot s_g$ is recursive, combinatorial, it follows that h is recursive, combinatorial and

$$\left(\sum_T f_n\right)\left(\sum_T g_n\right) = (s_h)_\Lambda(T) = \sum_T h_n .$$

By a number-theoretic function, we mean any function defined on the non-negative integers, having integral values. Every number-theoretic function f can be written as the difference of the two combinatorial functions f^+ and f^-, called the positive and negative parts of f. We call a number-theoretic function recursive if the functions f^+ and f^- are both recursive. For a recursive, number-theoretic function f, we make use of the canonical extension to Λ defined in [4]:

$$f_\Lambda(X) = f_\Lambda^+(X) - f_\Lambda^-(X).$$

It follows that the extension of a recursive, number-theoretic function maps Λ into Λ^*, the ring of isolic integers. It is easily shown that if f and g are recursive, number-theoretic functions, then for $X \in \Lambda$,

$$f_\Lambda(X) + g_\Lambda(X) = (f + g)_\Lambda(X),$$

$$f_\Lambda(X) \cdot g_\Lambda(X) = (f \cdot g)_\Lambda(X).$$

DEFINITION. Let f be a recursive, number-theoretic function. For $T \in \Lambda_R$,

$$\sum_T^* f_n = \sum_T f_n^+ - \sum_T f_n^- .$$

This sum is referred to as the *star-sum*. We note that for every recursive, number-theoretic function f and every regressive isol T, $\sum_T^* f_n \in \Lambda^*$. Clearly, if f is recursive, combinatorial, $\sum_T^* f_n = \sum_T f_n$.

The following two propositions are proved by decomposing the functions involved into their positive and negative parts and then applying Propositions 1 and 2. Their proofs will be omitted.

PROPOSITION 3. *Let f and g be recursive, number-theoretic functions. Then so is $f + g$ and for all $T \in \Lambda_R$,*

$$\sum_T^* f_n + \sum_T^* g_n = \sum_T^* (f_n + g_n).$$

PROPOSITION 4. *Let f and g be recursive, number-theoretic functions. Then the function*

$$h_n = \sum_{m(i,k)=n} f_i \, g_k$$

is recursive, number-theoretic. Moreover, for $T \in \Lambda_R$,

$$\left(\sum_T^* f_n\right)\left(\sum_T^* g_n\right) = \sum_T^* h_n .$$

The next result is obtained immediately from Proposition 3.

PROPOSITION 5. *Let f and g be recursive, number-theoretic functions. Then so is $f - g$ and for $T \in \Lambda_R$,*

$$\sum_T^* f_n - \sum_T^* g_n = \sum_T^* (f_n - g_n).$$

THEOREM 1. *Let f be a recursive, number-theoretic function. Then for all $T \in \Lambda_R$,*

$$\sum_T^* f_n = (s_f)_\Lambda(T),$$

where s_f is the partial sum function of f.

Proof. For all n, we have

$$s_{f^+}(n) - s_{f^-}(n) = s_{f^+ - f^-}(n) = s_f(n).$$

Hence, if $T \in \Lambda_R$,

$$(s_{f^+})_\Lambda(T) - (s_{f^-})_\Lambda(T) = (s_f)_\Lambda(T).$$

Since,

$$\sum_T^* f_n = \sum_T f_n^+ - \sum_T f_n^- = (s_{f^+})_\Lambda(T) - (s_{f^-})_\Lambda(T),$$

it follows that

$$\sum_T^* f_n = (s_f)_\Lambda(T).$$

In the next theorem, we give a representation of the canonical extension of a recursive, number-theoretic function as a star-sum of integers. This representation is then utilized to show that for f recursive, the star-sum and the sum defined in [1] agree.

THEOREM 2. *Let f be any recursive, number-theoretic function. Then for all $T \in \Lambda_R$,*

$$f_\Lambda(T) = f_0 + \sum_T^* \Delta f_n ,$$

where Δ is the usual finite difference operator.

Proof. The function Δf takes on values

$$f(1) - f(0), f(2) - f(1), f(3) - f(2), \cdots .$$

Clearly, since f is recursive, number-theoretic, so is Δf. Thus, by Theorem 1, $\sum_T^* \Delta f_n = (s_{\Delta f})_\Lambda(T)$. The function $s_{\Delta f}$ however, takes values

$$0, f(1) - f(0), f(2) - f(0), f(3) - f(0), \cdots .$$

Hence

$$(s_{\Delta f})_\Lambda = (f(n) - f(0))_\Lambda = f_\Lambda - f(0).$$

It follows that for $T \in \Lambda_R$,

$$\textstyle\sum_T^* \Delta f_n = f_\Lambda(T) - f(0).$$

THEOREM 3. *Let f be a recursive function and let $T \in \Lambda_R$. Then*

$$\textstyle\sum_T^* f_n = \sum_T f_n .$$

Proof. In case T is finite, the result is clear. Let T be infinite. Let t_n be a regressive function such that $\rho t_n \in T$. By the definition of the star-sum,

$$\textstyle\sum_T^* f_n = \sum_T f_n^+ - \sum_T f_n^- ,$$

where

$$\textstyle\sum_T f_n^+ = \text{Req} \bigcup_{n=0}^\infty j(t_n , \nu(f_n^+))$$

and

$$\textstyle\sum_T f_n^- = \text{Req} \bigcup_{n=0}^\infty j(t_n , \nu(f_n^-)).$$

It therefore suffices to show

$$(1) \quad \text{Req} \bigcup_{n=0}^\infty j(t_n , \nu(f_n)) + \text{Req} \bigcup_{n=0}^\infty j(t_n , \nu(f_n^-)) = \text{Req} \bigcup_{n=0}^\infty j(t_n , \nu(f_n^+)).$$

To prove (1), we note that: since f is recursive,

$$(2) \qquad \bigcup_{n=0}^\infty j(t_n , \nu(f_n)) \mid \bigcup_{n=0}^\infty j(t_n , \nu(f_n^+) - \nu(f_n)),$$

and moreover,

$$(3) \quad \bigcup_{n=0}^\infty j(t_n , \nu(f_n)) + \bigcup_{n=0}^\infty j(t_n , \nu(f_n^+) - \nu(f_n)) = \bigcup_{n=0}^\infty j(t_n , \nu(f_n^+)).$$

Hence, it suffices to prove

$$(4) \qquad \bigcup_{n=0}^\infty j(t_n , \nu(f_n^-)) \simeq \bigcup_{n=0}^\infty j(t_n , \nu(f_n^+) - \nu(f_n)).$$

Denote the left hand side of (4) by α and the right hand side of (4) by β. Let $p(x)$ be a regressing function of the regressive function t_n. Furthermore, let

$$f(x) = j(k(x),\ l(x) + f_p{}^* k(x)),$$

$$g(x) = j(k(x), l(x) \dot{-} f_p{}^* k(x)).$$

The function $f(x)$ is partial recursive, 1-1 on α and it maps α onto β. The function $g(x)$ is partial recursive, 1-1 on β and it maps β onto α. Moreover for $x \in \alpha$, $gf(x) = x$. An application of Proposition 1 of [1] completes the proof.

The corollaries below follow immediately from the preceding results.

COROLLARY 1. *Let f and g be recursive functions. For* $T \in \Lambda_R$,

$$\sum_T f_n + \sum_T g_n = \sum_T (f_n + g_n).$$

COROLLARY 2. *Let f and g be recursive. Then for* $T \in \Lambda_R$,

$$(\sum_T f_n)(\sum_T g_n) = \sum_T h_n,$$

where h is defined as in Proposition 4.

COROLLARY 3. *Let f be an increasing, recursive, number-theoretic function. Then for* $T \in \Lambda_R$,

$$f_\Lambda(T) = f_0 + \sum_T \Delta f_n.$$

COROLLARY 4. *Let f be a recursive, number-theoretic function. If* $f = g - h$ *where g and h are recursive functions, then for* $T \in \Lambda_R$,

$$\sum_T^* f_n = \sum_T g_n - \sum_T h_n.$$

COROLLARY 5. *Let f be a recursive function. Then for* $T \in \Lambda_R$,

$$\sum_T f_n = (s_f)_\Lambda(T),$$

where s_f *is the partial sum function of f.*

3. The mapping Φ_f

We recall the definition of the mapping Φ_f as given in [7].

DEFINITION. Let f be a one to one function from ε into ε and let $T \in \Lambda_R - \varepsilon$. Then

$$\Phi_f(T) = \text{Req } \rho t_{f(n)},$$

where t_n is any regressive function ranging over a set in T.

Then Φ_f is a well defined mapping from $\Lambda_R - \varepsilon$ into $\Lambda - \varepsilon$. Moreover, if f is strictly increasing and recursive, $\Phi_f(T) \in \Lambda_R$. The main result of [7] states that if f is a strictly increasing, recursive, combinatorial function and $T \in \Lambda_R - \varepsilon$, then $\Phi_f(f_\Lambda(T)) = T$. We proceed to extend this theorem along two different lines, both of which yield it as an immediate corollary.

LEMMA. *Let f be a recursive function. Then for* $T \in \Lambda_R$, $\sum_T f_n \in \Lambda_R$.

Proof. For T finite, $\sum_T f_n$ is also finite and hence is a member of Λ_R. If T is infinite, then $\sum_T f_n = \text{Req } \bigcup_{n=0}^{\infty} j(t_n, \nu(f_n))$. Since f is recursive and t_n is regressive, it is clear that the function u_n which takes on successive values

$$j(t_0, 0), j(t_0, 1), \cdots, j(t_0, f_0 - 1), j(t_1, 0), j(t_1, 1), \cdots, j(t_1, f_1 - 1), \cdots$$

is regressive. It is assumed that those values $j(t_n, k)$ for which f_n is zero are omitted in the above enumeration. Since the range of u_n is a member of $\sum_T f_n$, it follows that $\sum_T f_n \in \Lambda_R$.

THEOREM 4. *Let f be strictly increasing and recursive. Then for $T \in \Lambda_R - \varepsilon$,*
(a) $f_\Lambda(T) \in \Lambda_R$,
(b) $\Phi_f(f_\Lambda(T)) = T$.

Proof. (a) By Corollary 3 of Theorem 3, it is seen that $f_\Lambda(T)$ has the representation

$$f_\Lambda(T) = f_0 + \sum_T \Delta f_n .$$

Since f is increasing and recursive, Δf is a recursive function. By the lemma therefore, $\sum_T \Delta f_n \in \Lambda_R$. Hence $f_\Lambda(T) \in \Lambda_R$.

(b) Let u_n be a regressive function ranging over a set in $T + 1$. Denote by v_n, the function which takes on values of the array

$$\begin{array}{ccc} j(u_0, 0) & \cdots & j(u_0, f_0 - 1) \\ j(u_1, 0) & \cdots & j(u_1, f_1 - f_0 - 1) \\ j(u_2, 0) & \cdots & j(u_2, f_2 - f_1 - 1) \\ \vdots & & \vdots \end{array}$$

reading from left to right in each row and from the top row down. If $f_0 = 0$, it is understood that the first row is to be deleted. Since f is recursive and u is regressive, it is clear that the function v is regressive. Since $\rho u_{n+1} \in T$, it follows that

$$\bigcup_{n=0}^{\infty} j(u_{n+1}, \nu(\Delta f_n)) \in \sum_T \Delta f_n .$$

Hence,

$$\rho v_n \in f_0 + \sum_T \Delta f_n = f_\Lambda(T).$$

Therefore,

$$\Phi_f(T) = \mathrm{Req}\,(j(u_1, 0), j(u_2, 0), \cdots) = T.$$

In the following theorem we make use of the well known canonical enumeration $\{\rho_n\}$ of the class of all finite subsets of ε together with the recursive function $r(n) =$ cardinality ρ_n. A lemma due to Dekker, whose proof appears in [7] states that if t_n is a regressive function and

$$t'_n = e_{n0} \cdot 2^{t(0)} + \cdots + e_{nn} \cdot 2^{t(n)},$$

where $e_{n0}, \cdots, e_{nn}$ is the sequence of zeros and ones such that

$$n = e_{n0} \cdot 2^0 + \cdots + e_{nn} \cdot 2^n,$$

then t'_n is also regressive. Moreover,

$$t'(2^n) = 2^{t(n)}, \quad \rho_{t'(n)} = t(\rho_n) \quad \text{and} \quad \rho t' \in 2^T.$$

THEOREM 5. *Let f be a strictly increasing, recursive, combinatorial function with $\{c_j\}$ as its sequence of combinatorial coefficients. Define for each $k > 0$,*

$$a_k(n) = \textit{the principal function of} \quad \{x \mid r(x) = k\},$$

$$b_k(n) = \sum_{i=0}^{a_k(n)-1} c_{r(i)} .$$

Then we have for every number k such that both k and c_k are positive,
(a) *$b_k(n)$ is a strictly increasing function of n,*
(b) *for $T \in \Lambda_R - \varepsilon$,*

$$\Phi_{b_k}(f_\Lambda(T)) = \binom{T}{k}.$$

Proof. (a) For each $k > 0$, $a_k(n)$ is strictly increasing, since it is the principal function of some infinite set. Assume for a fixed k that $c_k > 0$. By definition,

$$b_k(n + 1) = \sum_{i=0}^{a_k(n+1)-1} c_{r(i)}$$

Hence

$$(1) \qquad b_k(n + 1) = \sum_{i=0}^{a_k(n)-1} c_{r(i)} + \sum_{i=a_k(n)}^{a_k(n+1)-1} c_{r(i)},$$

where the last sum is non-vacuous, since $a_k(n)$ is strictly increasing. Hence by (1),

$$(2) \qquad b_k(n + 1) = b_k(n) + c_{ra_k(n)} + (\text{non-negative terms, if any}).$$

But $c_{ra_k(n)} = c_k > 0$. We thus see by (2), that $b_k(n + 1) > b_k(n)$, and hence $b_k(n)$ is a strictly increasing function. Clearly, $b_k(n)$ is recursive for each k.

(b) Assuming the hypothesis, since $b_k(n)$ is strictly increasing, $\Phi_{b_k}(f_\Lambda(T))$ has meaning. Let $\tau \in T \in \Lambda_R - \varepsilon$, and assume that t_n is a regressive function ranging over τ. Put $g(n) = t'(n)$. By the above lemma, $\rho_{g(n)} = t(\rho_n)$. Hence, if n assumes successively the values

$$0, 1, 2, 3, 4, 5, 6, 7, \cdots,$$

$\rho_{g(n)}$ assumes successively the "values"

$$\sigma, (t_0), (t_1), (t_0, t_1), (t_2), (t_0, t_2), (t_1, t_2), (t_0, t_1, t_2), \cdots.$$

By definition we have

$$f_\Lambda(T) = \text{Req}\,\{j(x, y) \mid \rho_x \subset \tau, y < c_{r(x)}\}.$$

Since $g(n)$ ranges without repetitions over $\{n \mid \rho_n \subset \tau\}$, and $rg(x) = r(x)$, it follows that

$$(3) \qquad f_\Lambda(T) = \text{Req}\,\{j(g(x), y) \mid x \in \varepsilon, y < c_{r(x)}\}.$$

We shall use u_n to denote the function which for $0, 1, 2, \cdots$, takes on the values of the array,

$$\begin{array}{l} j(g(0), 0), \cdots, j(g(0), c_{r(0)} - 1) \\ j(g(1), 0), \cdots, j(g(1), c_{r(1)} - 1) \\ j(g(2), 0), \cdots, j(g(2), c_{r(2)} - 1) \\ \quad \vdots \qquad\qquad\qquad \vdots \end{array}$$

reading from left to right in each row, and from the top row down. It is understood that every row which starts with $j(g(k), 0)$, for some k with

$c_{r(k)} = 0$, is to be deleted. The function $g(n) = t'(n)$ is regressive by the above lemma. Since c_i is recursive, it readily follows that u_n is a regressive function. In view of (3) we have $\rho u_n \in f_\Lambda(T)$. It therefore suffices to prove that for $k > 0$, with $c_k > 0$, $\rho u b_k(n) \in \binom{T}{k}$. We recall that

$$\binom{T}{k} = \text{Req } \{x \mid \rho_x \subset \tau \text{ and } r(x) = k\}.$$

Hence

$$(4) \qquad \binom{T}{k} = \text{Req } \{g(x) \mid r(x) = k\}.$$

Since $b_k(n) = \sum_{i=0}^{a_k(n)-1} c_{r(i)}$, we have

$$\begin{aligned} b_k(0) &= c_{r(0)} + \cdots + c_{r(a_k(0)-1)}, \\ b_k(1) &= c_{r(0)} + \cdots + c_{r(a_k(1)-1)}, \\ b_k(2) &= c_{r(0)} + \cdots + c_{r(a_k(2)-1)}, \\ &\vdots \qquad \vdots \qquad\qquad \vdots \quad . \end{aligned}$$

Since $c_k > 0$, we can be assured that the rows of the array used to define u_n which begin with $j(ga_k(0), 0), j(ga_k(1), 0), \cdots$ are not deleted. Hence

$$ub_k(0) = j(ga_k(0), 0), \qquad ub_k(1) = j(ga_k(1), 0), \cdots .$$

We conclude that $ub_k(n) \simeq ga_k(n)$. By a further transformation of (4) however, we see that

$$\binom{T}{k} = \text{Req } \{g(x) \mid r(x) = k\} = \text{Req } \rho g a_k(n).$$

Hence

$$\rho u b_k(n) \in \binom{T}{k}.$$

4. The representation of Φ_f as an extension

Let f be a strictly increasing recursive function. We wish to prove that the mapping Φ_f has an interpretation as the canonical extension of a particular recursive function $\bar{f}$ to Λ.

DEFINITION. For f strictly increasing and recursive, $\bar{f}(n) = (\mu y)[f(y) \geq n]$.

Clearly, $\bar{f}$ is recursive since it is everywhere defined and partial recursive. Let $g(n) = c_{\rho f}(n)$ where $c_{\rho f}$ is the characteristic function of the range of f. Then g is also recursive.

LEMMA. *Let f be a strictly increasing, recursive function. Let $\bar{f}$ and g be defined as above. Then $s_g(n) = \bar{f}(n)$, where s_g is the partial sum function of g.*

Proof. We proceed by induction on n.
$n = 0$.

$$\bar{f}(0) = (\mu y)[f(y) \geq 0] = 0.$$

$$s_g(0) = 0, \quad \text{by definition of the partial sum function.}$$

Assume $s_g(k) = \bar{f}(k)$.

$$s_g(k+1) = s_g(k) + g(k),$$

$$\begin{aligned}\bar{f}(k+1) = (\mu y)[f(y) \geq k+1] &= \bar{f}(k) + 0 \quad \text{if} \quad f\bar{f}(k) \geq k+1 \\ &= \bar{f}(k) + 1 \quad \text{if} \quad f\bar{f}(k) = k.\end{aligned}$$

It only remains to show that

$$\begin{aligned}g(k) &= 0 \quad \text{if} \quad f\bar{f}(k) \geq k+1 \\ &= 1 \quad \text{if} \quad f\bar{f}(k) = k.\end{aligned}$$

But,

$$\begin{aligned}g(k) = c_{\rho f}(k) &= 0 \quad \text{if} \quad k \notin \rho f \Leftrightarrow f\bar{f}(k) \geq k+1 \\ &= 1 \quad \text{if} \quad k \in \rho f \Leftrightarrow f\bar{f}(k) = k.\end{aligned}$$

Hence $s_g(n) = \bar{f}(n)$ for all n.

THEOREM 6. *Let f be a strictly increasing, recursive function. For $T \in \Lambda_R - \varepsilon$, $\Phi_f(T) = \bar{f}(T)$.*

Proof. Let g be defined as above. It readily follows from the definition of $\sum_T g(n)$, that $\Phi_f(T) = \sum_T g(n)$. Applying Corollary 5 of Theorem 3, $\sum_T g(n) = (s_g)_\Lambda(T)$. Hence, by the preceding lemma, $\Phi_f(T) = \bar{f}_\Lambda(T)$.

5. Remarks

With the use of the star-sum and the mapping Φ, it becomes a relatively simple matter to prove the existence of non-trivial idempotents in Λ^*, the ring of isolic integers. Another proof is given in [2, Theorem 95]. It is shown there that there exists an infinite, regressive isol T such that neither $\Phi_{2n}(T) = \Phi_{2n+1}(T)$ nor $\Phi_{2n}(T) = \Phi_{2n+1}(T) + 1$ holds. This, together with the fact that $\Phi_{2n}(T) - \Phi_{2n+1}(T)$ is an idempotent element, leads to the existence of non-trivial idempotents in Λ^*. The second of these two results follows immediately upon consideration of the star-sum, $\sum_T^* (-1)^n$. From Corollary 4 of Theorem 3, it follows that,

$$\sum_T^* (-1)^n = \sum_T c_e(n) - \sum_T c_0(n),$$

where $c_e(n)$, $c_0(n)$ are the characteristic functions of the even numbers and odd numbers respectively. Clearly,

$$\sum_T c_e(n) = \Phi_{2n}(T), \qquad \sum_T c_0(n) = \Phi_{2n+1}(T).$$

From Proposition 4, we have

$$\left(\sum_T^* (-1)^n\right)\left(\sum_T^* (-1)^n\right) = \sum_T^* (-1)^n.$$

Hence, for every regressive isol T,

$$(\Phi_{2n}(T) - \Phi_{2n+1}(T))^2 = \Phi_{2n}(T) - \Phi_{2n+1}(T).$$

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