THE HOMOTOPY GROUPS OF THE SPACE OF HOMEOMORPHISMS OF A MULTIPLY PUNCTURED SURFACE

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1. Introduction

Let M^k denote the 2-manifold obtained from a compact connected surface M by deleting k points ($k = 0, 1, 2, \cdots$) from Core (M) and let $G(M^k)$ denote the identity component of the space of all homeomorphisms of M^k onto itself topologized by the compact-open topology.

We use the homeotopy (exact) sequence for M^k [1, Def. 4.12, p. 298] together with theorems of G. S. McCarty [1] and H. B. Griffiths [2] to obtain certain isomorphisms between the homotopy groups $\pi_n[G(M^k)]$ of $G(M^k)$ ($k = 0, 1, 2, \cdots$). Combining these isomorphisms with results obtained by M.-E. Hamstrom [3], [4] we establish the following theorem.

THEOREM. *Let M denote a compact orientable or non-orientable connected surface with or without boundary.*

(a) *If M has a non-abelian fundamental group and M is not a Klein bottle, then* $\pi_n[G(M)] \approx \pi_n[G(M^k)]$ ($n \geqq 0, k \geqq 0$).

(b) *If M is a closed orientable surface of genus g ($g \geqq 2$), then* $\pi_n[G(M^k)] = 0$ ($n \geqq 0, k \geqq 0$).

(c) *If M is a Klein bottle or a torus, then* $\pi_n[G(M^k)] = 0$ ($n \geqq 0, k \geqq 1$).

(d) *If M is a Moebius strip, then* $\pi_1[G(M^1)] \approx \pi_1[G(M^k)]$ *and*

$$\pi_n[G(M^k)] = 0 \qquad (n = 0, n \geqq 2\ k \geqq 1).$$

(e) *If M is an annulus, then* $\pi_n[G(M^1)] \approx \pi_n[G(M^k)]$ ($n \geqq 0, k \geqq 1$).

(f) *If M is a real projective plane, then* $\pi_1[G(M^2)] \approx \pi_1[G(M^k)]$ *and* $\pi_n[G(M^k)] = 0$ ($n = 0, n \geqq 2, k \geqq 2$).

(g) *If M is a disk, then* $\pi_n[G(M^k)] = 0$ ($n \geqq 0, k \geqq 2$).

(h) *If M is a 2-sphere, then* $\pi_n[G(M^k)] = 0$ ($n \geqq 0, k \geqq 3$).

Remark 1. With respect to the as yet unknown homotopy groups in the statements (a), (d), (e), and (f) of the theorem, it is expected that they will be explicitly determined using techniques currently being developed by M. -E. Hamstrom [4], [5].

Remark 2. We note that there are exactly eleven spaces $G(M^k)$ which are not included in the statement of the theorem and that each of these spaces has at least one non-zero homotopy group (cf. §3). This combined with what we presently know suggests the conjecture that all of the spaces $G(M^k)$ included in the statement of the theorem are homotopically trivial.

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We conclude by thanking both M.-E. Hamstrom and the referee for suggesting that the results of the originally submitted manuscript [6] be extended to include non-orientable surfaces.

2. Proof of the theorem

The space of all homeomorphisms $H(X)$ of a manifold X is a fiber bundle over Core (X) with fiber $H(X, x)$ the isotropy group of $H(X)$ at x ($x \in$ Core (X)). The homotopy sequence of this bundle is exact and is used to define the homeotopy (exact) sequence of X,

$$\cdots \to \pi_n[H(X, x)] \to \pi_n[H(X)] \to \pi_n(X) \to \cdots$$

(cf. [1, Def. 4.12, p. 298)].

We first prove, simultaneously, Theorem (a) (cf. §1) and the following lemma.

LEMMA 1. *If M is a Klein bottle, a torus, a Moebius strip, or an annulus, then* $\pi_n[G(M^1)] \approx \pi_n[G(M^k)]$ $(n \geqq 0, k \geqq 1)$.

Proof. For both Theorem (a) and Lemma 1, we have $\pi_n(M^k) = 0$ $(n \geqq 2, k \geqq 0)$. Thus, the homeotopy sequence for M^k implies

$$\pi_n[G(M^k)] \approx \pi_n[G(M^k, x)] \qquad (n \geqq 2, k \geqq 0) \tag{2.1}$$

where $G(M^k)$ and $G(M^k,x)$ are the identity components of $H(M^k)$ and $H(M^k,x)$ respectively. By [1, Th. 4.4, p. 300] we have

$$\pi_n[G(M^k, x)] \approx \pi_n[G(M^{k+1})] \qquad (n \geqq 1, k \geqq 0) \tag{2.2}$$

and by definition $\pi_0[G(M^k)] = 0$ $(k \geqq 0)$. These facts combined with (2.1) yield by induction

$$\pi_n[G(M)] \approx \pi_n[G(M^k)] \quad (n = 0, n \geqq 2, k \geqq 0) \tag{2.3}$$

H. B. Griffiths [2, Th. 4.4, p. 10] has shown that, if a surface which is not a Klein bottle has a non-abelian fundamental group π, then π has trivial center. Thus, for Theorem (a) we have $\pi_1(M^k)$ $(k \geqq 0)$ has trivial center and for Lemma 1 we have $\pi_1(M^k)$ $(k \geqq 1)$ has trivial center. The image of $\pi_1[G(M^k)]$ in $\pi_1(M^k)$ in the homeotopy sequence for M^k is central [1, Remark 5.24, p. 302] and as noted above $\pi_2(M^k) = 0$. Combining these facts with (2.2) we have for Theorem (a)

$$\pi_1[G(M^k)] \approx \pi_1[G(M^{k+1})] \qquad (k \geqq 0)$$

and by induction we obtain

$$\pi_1[G(M)] \approx \pi_1[G(M^k)] \qquad (k \geqq 0). \tag{2.4}$$

For Lemma 1 we have $\pi_1[G(M^k)] \approx \pi_1[G(M^{k+1})]$ $(k \geqq 1)$ and by induction we obtain

$$\pi_1[G(M^1)] \approx \pi_1[G(M^k)] \qquad (k \geqq 1). \tag{2.5}$$

Combining (2.3), (2.4), and (2.5) completes the proof of Theorem (a) and Lemma 1.

In [3] and [4] M.-E. Hamstrom has obtained the homotopy groups of certain spaces of homeomorphisms $G(X)$, with $G(X)$ topologized by the uniform convergence metric. In the cases considered X is a compact metric space, thus the topology on $G(X)$ coincides with the compact-open topology [7], thereby making her results valid in the compact-open topology.

In [4] it is announced that $\pi_n[G(M)] = 0$ $(n \geqq 0)$ if M is a compact orientable connected surface with two or more handles. Applying Theorem (a) we obtain Theorem (b).

Also announced in [4] was the result that, if M is a Klein bottle, then $\pi_n[G(M, x)] = 0$ $(n \geqq 0)$. In [3] it is proven that, if M is a torus, then $\pi_n[G(M)] \approx \pi_n(M)$ $(n \geqq 0)$. Applying Remark 3.31 [1, p. 296] it follows directly that in this case we also have $\pi_n[G(M, x)] = 0$ $(n \geqq 0)$. Recalling that $\pi_n[G(M, x)] \approx \pi_n[G(M^1)]$ $(n \geqq 0)$ and applying Lemma 1 we obtain Theorem (c).

Also given in [4] are the homotopy groups of $G(M)$, where M is a Moebius strip. In particular, $\pi_n[G(M)] = 0$ $(n \geqq 2)$. Substitution of this group into the homeotopy sequence for the Moebius strip yields $\pi_n[G(M^1)] = 0$ $(n \geqq 2)$. Thus, applying Lemma 1 we obtain Theorem (d).

Theorem (e) is contained in the statement of Lemma 1.

The following lemma can be proven in a manner completely analogous to the proof of Theorem (a), in this case noting that, if M is a disk or a real projective plane, then $\pi_1(M^k)$ $(k \geqq 2)$ has trivial center and $\pi_n(M^k) = 0$ $(n = 0, n \geqq 2, k \geqq 2)$.

LEMMA 2. *If M is a disk or a real projective plane, then*

$$\pi_n[G(M^2)] \approx \pi_n[G(M^k)] \qquad (n \geqq 0, k \geqq 2).$$

From [4] we have $\pi_n[G(M^1)] = 0$ $(n \geqq 2)$, where M^1 is a once punctured real projective plane. The homeotopy sequence for M^1 yields $\pi_n[G(M^2)] = 0$ $(n \geqq 2)$. Thus, applying Lemma 2 we obtain Theorem (f).

Let $H(x)$ and $H(X, x)$ be defined as in the beginning of this section. If $\pi_0[H(A, x)] \approx \pi_0[H(A)]$ and $\pi_0[H(B, y)] \approx \pi_0[H(B)]$ in the homeotopy sequence for A and B respectively, then

$$\pi_0[H(A \times B, (x, y))] \approx \pi_0[H(A \times B)]$$

in the homeotopy sequence for $A \times B$ [1 Remark 3.41, p 297]. We use this fact to show that, if M is a disk, then $\pi_n[G(M^2)] = 0$ $(n \geqq 0)$.

If A is a half open interval and B is a circle, then

$$\pi_0[H(X, x)] \approx \pi_0[H(X)] \qquad (X = A, B)$$

in the homeotopy sequences for A and B respectively. Thus,

$$\pi_0[H(A \times B, (x, y)] \approx \pi_0[H(A \times B)]$$

in the homeotopy sequence for $A \times B \equiv M^1$ a disk M with an interior point deleted. This implies that $\pi_1[G(M^1)]$ maps onto $\pi_1(M^1) \approx Z$ in this sequence. Since $\pi_1[G(M^1)] \approx Z$ (cf §3), this mapping is an isomorphism in this sequence. This implies $\pi_1[G(M^2)] = 0$. Furthermore, since $\pi_n[G(M^1)] = 0$ $(n \geqq 2)$ (cf. §3) and $\pi_n(M^1) = 0$ $(n \geqq 2)$, we have $\pi_n[G(M^2)] = 0$ $(n \geqq 2)$. Applying Lemma 2 we obtain Theorem (g).

A proof of Theorem (h), which is due to G. S. McCarty can be found in [1, p. 303].

This completes the proof of the theorem.

3. On the spaces $G(M^k)$ which are not included in the theorem

For M a compact orientable or non-orientable connected surface with or without boundary the following eleven spaces M^k are the only spaces for which $G(M^k)$ is not included in the theorem: a Klein bottle, torus, Moebius strip, annulus, disk, disk with one interior point deleted, real projective plane, real projective plane with one point deleted, and a 2-sphere with 0, 1, or 2 points deleted. This assertion follows from the fact that there is only a finite number of surfaces which have abelian fundamental groups [2, Th. 4.3, p. 10].

Not one of these eleven spaces $G(M^k)$ is homotopically trivial. For the homotopy groups of $G(M^k)$ where M^k is a Klein bottle, torus, Moebius strip, real projective plane, or a real projective plane with one point deleted see [3], [4] and when M^k is a disk or a 2-sphere with 0, 1, or 2 points deleted see [8], [9, p. 521], [1, pp. 302–303].

If M is a disk, the homeotopy sequence for M yields

$$\pi_n[G(M^1)] \approx \pi_n[G(M)] \qquad (n \geqq 0).$$

Since $\pi_n[G(M)] \approx \pi_n[G(S)] \approx \pi_n(S)(n \geqq 0)$, where S is a circle [8, p. 295], [1, p. 302] we have that $\pi_1[G(M^1)] \approx Z$ and $\pi_n[G(M^1)] = 0$ $(n \geqq 2)$. Thus, $G(M^1)$ is not homotopically trivial.

If M is an annulus, then $M \equiv I \times S$ where I is a closed interval and S is a circle. As in §2, we use Remark 3.41 [1, p. 297] to obtain

$$\pi_0[H(M, x)] \approx \pi_0[H(M)]$$

in the homeotopy sequence for M. This implies that $\pi_1[G(M)]$ maps onto $\pi_1(M) \approx Z$ in this sequence, thus $\pi_1[G(M)] \neq 0$.

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