ON GROUPS ADMITTING FIXED POINT FREE ABELIAN OPERATOR GROUPS[1]

BY

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1. Introduction

This paper concerns the existence of bounds for nilpotent length in groups, G, admitting a fixed point free group of automorphisms, V. Among results in this direction we may list the following:

Group of Operators V	Bound for Nilpotent Length	Investigators
cyclic of prime order	1	J. G. Thompson [6]
cyclic, order 4	2	D. Gorenstein and I. N. Herstein [4]
the 4-group	2	S. Bauman [2]

The results in this paper may be summarized in the following theorem.

THEOREM. *Let G be a group admitting the abelian group, V, as a fixed point free group of operators.*

(i) *If $|V| = n = p_1^{a_1} \cdots p_s^{a_s}$, is prime to $|G|$, and G is solvable, then G has nilpotent length at most $\sum a_i = \psi(n)$, the total number of primes dividing n, provided*

(a) *$|G|$ is not divisible by primes, q, such that $q^k + 1 = d$ for some integer k and some divisor, d, of the exponent, e, of the abelian group, V.*

(ii) *If V is cyclic of order p^k, and G is π-solvable, then G has π-length at most $[(k+1)/2]$, provided*

(b) *p is not a Fermat prime when $|G|$ is even, and that $|G|$ is not divisible by Mersenne primes $<2^k$ when $p = 2$.*

From (i), $[(\psi(n)+1)/2]$ is a bound for π-length whenever G is solvable, but in (ii) G is only π-solvable. Both results follow from two technical theorems, Theorem 3.1 and 4.1, which may be regarded as the main theorems of the paper. Theorem 3.1 treats the case that V is cyclic of prime power order, and leads to (ii). Theorem 4.1 is a kind of analogue of Theorem 3.1, for the case that V is not cyclic of prime power order; it requires more special hypotheses, and its proof is more difficult. Section 5 shows that the bound in (i) is best possible (and is a lower bound for the exceptional cases).

There are good reasons to believe the bound doubles in the presence of the exceptional case, but this is rather difficult to show, especially when V is not cyclic of prime power order.

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2. Preliminary results and notation

Let V be a group of operators acting on a group G. The following subgroups of G are of interest:

$$G_V = \{g : g \in G, v(g) = g \quad \text{for all} \quad v \in V\}$$

$$(V, G) = \text{the subgroup generated by} \quad \{v(g)g^{-1} : v \in V, g \in G\}$$

$$\phi^V(G) = \text{the intersection of all maximal } V\text{-invariant subgroups of } G.$$

If $W \lhd V$, G_W, (W, G) and $\phi^W(G)$ are all V-invariant.

The following basic results are used repeatedly in this study. Throughout, V is assumed to have order prime to $|G|$.

LEMMA 2.1. *$\phi^V(G) = \phi(G)$ whenever G is nilpotent.*

LEMMA 2.2. *If $H \subseteq G_V$ and $N = N_G(H)$ then $N = N_V C_N(H)$.*

LEMMA 2.3. *If H is normal and V-invariant, V acts on G/H in a natural way, and $(G/H)_V = G_V H/H$.*

LEMMA 2.4. *If N is normal and V-invariant, $(V, G) \subseteq N$ if and only if G/N is fixed elementwise by V.*

LEMMA 2.5. *At least one q-Sylow subgroup is V-invariant for every prime q dividing $|G|$.*

LEMMA 2.6. *$(V, (V, G)) = (V, G)$ and is normal in G.*

Many of these results follow readily from one another and almost all can be proved by means of a theorem of G. Glauberman [3] which generalizes a result of Wielandt [7]. Lemmas 2.3 and 2.5 can be found explicitly in Wielandt [7] and Lemma 2.2 in Alperin [1].

3. Fixed point free automorphisms of prime power order

The main theorem of this section plays the role of a technical lemma for the proof of Theorem 3.3. The reductions follow closely, those given in the important work of P. Hall and G. Higman [5], and may be regarded as a partial analysis of their situation for the case that the characteristic of the underlying field is prime to the order of the automorphism acting on G.

THEOREM 3.1. *Let V be a group of operators acting on a group G. Suppose that V is cyclic of order p^k and that G has order prime to p (p is a prime number). Let H be the semidirect product GV and let A be a faithful KH-module where K is a splitting field for all subgroups of H and where the characteristic of K does not divide H. If the generator, v, of V, acts in fixed point free manner on A, then $w = v^{p^{k-1}}$ centralizes G, provided neither of the following exceptional cases occur:*

(a) *G has a non-abelian 2-sylow subgroup and p is a Fermat prime and* $k = 1$,

(b) $p = 2$ and $|G|$ *is divisible by a Mersenne prime $q < 2^k$ for which G has a non-abelian q-Sylow subgroup.*

Proof. (By induction on $|G| + \dim_K A + |V|$.) Since the characteristic of K does not divide $|H|$ the module A may be decomposed into its irreducible KH-components, $A_1, \cdots, A_t$, each affording a representation, ρ_i, with kernel K_i, $(i = 1, 2, \cdots, t)$. Since A is faithful, the K_i have trivial meet. If w acts trivially on any A_i, w centralizes $G/G \cap K_i$. If w does not act trivially on A_k, $K_i \subseteq G$, and A_i is a faithful VG/K_i-module. If either (a) or (b) held for the groups V and G/K_i they would hold also for V and G. Finally, V is fixed point free on each A_i and char K does not divide $|V| \cdot [G : K_i]$. Since $|G/K_i| + \dim_K A_i + |V| < |G| + \dim_K A + |V|$ if $t > 1$, induction yields that w centralizes G/K_i. Thus in any case, whether V acts faithfully on A_i or not, w centralizes $G/K_i \cap G$, if $t > 1$. Thus by Lemma 2.4

$$(w, G) \subseteq \cap_{i=1}^{t} K_i = 1,$$

whence w fixes G elementwise, our conclusion. Thus we may suppose $t = 1$, so A is an irreducible KH-module.

Now let L be any proper V-invariant subgroup of G, form the group $H_0 = LV$, and regard A as a KH_0-module by restriction. V acts fixed point free on A and the conditions (a) or (b) cannot arise for L since they would then hold for G. Thus, V, L, $H_0 = LV$ and A together satisfy the conditions of the theorem, and since $|L| + \dim_K A + |V| < |G| + \dim_K A + |V|$, we may apply induction to obtain that w fixes L elementwise. Thus every proper v-invariant subgroup of G may be assumed to be fixed by w.

If L_1 and L_2 are distinct maximal V-invariant subgroups of G, $L_i \subseteq G_w$, $i = 1, 2$ so $G = \{L_1, L_2\} \subseteq G_w$, whence G is fixed elementwise by w. Thus without loss of generality we may suppose that G has a unique maximal V-invariant subgroup which contains every proper V-invariant subgroup of G and is fixed elementwise by $w = v^{p^{k-1}}$. Evidently, this unique subgroup is $\phi^V(G)$ itself.

Now suppose G is not a prime power group. Since V is soluble and has order prime to $|G|$, by the result of Wielandt (Lemma 2.5), at least one q-Sylow subgroup of G is V-invariant, for every q dividing $|G|$. Such a group is proper, and so lies in the unique maximal V-invariant subgroup $\phi^V(G)$. Then $\phi^V(G)$ has order at least as large as $|G|$, which is impossible since $\phi^V(G)$ is proper. Thus G is a q-group. Then, by Lemma 2.1, since G is nilpotent, $\phi^V(G) = \phi(G)$ is the unique maximal V-invariant subgroup of G. Thus $G/\phi(G)$ is an elementary abelian q-group admitting v irreducibly. Since V is abelian, $(G/\phi(G))_w$ is a V-invariant subgroup of $G/\phi(G)$ and so is either $G/\phi(G)$ or is trivial. In the former case, by Lemma 2.3, $G_w \phi(G) = G$ so $G_w = G$, and we are done. We next show that the remaining alternative $(G/\phi(G))_w = E$, leads to a contradiction.

Thus we have that $\phi(G) = G_w$. Since $\phi(G)$ is normal in G, by Lemma 2.2, $G = C_G(\phi(G))G_w = C_G(\phi(G))\phi(G)$ and so $G = C_G(\phi(G))$, whence $\phi(G) \subseteq Z(G)$.

Since A is an irreducible KH-module and $G \lhd H$, A may be decomposed into a sum of its homogeneous KG-components, $A = B_1 \oplus \cdots \oplus B_s$, where $vB_i = B_{i+1}$, $i \leq s - 1$ and $vB_s = B_1$ so $s = p^h$, $h \leq k$. Now $u = v^{p^h}$ leaves each B_i invariant. If $a \in B_1$ is fixed by u, then

$$(1) \qquad \sum_{j=0}^{p^h-1} v^j a$$

is an element of A fixed by v and hence is zero. Since each $v^j a \in B_{j+1}$, $j \leq s - 1$, the quantities appearing in (1) are linearly independent and so each must be zero. Thus $a = 0$ and so $h < k$ and $u = v^{p^h}$ acts in fixed point free manner on each B_i. Let D_i be the kernel of the representation of G afforded by B_i. Then the groups $U = \{u\}$, $G_i = G/D_i$, and B_i satisfy the conditions of our theorem (note that (a) and (b) cannot hold for G/D_i). If $s > 1$, then $|G_i| + \dim_K B_i + |U| < |G| + \dim_K A + |V|$ and induction yields the result that an element of order p in U centralizes G/D_i, $i = 1, \cdots, s = p^h$. But since V is cyclic of prime power order, this element of order p generates $\{w\}$. Thus by Lemma 2.4, $(\{w\}, G) \subseteq \bigcap_{i=1}^{s} D_i = \{1\}$ and we are done. Thus we must suppose that $s = 1$ so that A is an homogeneous KG-module.

In this case $\rho \simeq X \otimes Y$ where ρ is the representation of H afforded by A, X and Y are irreducible projective representations of H, $X(g) = I$ for all $g \in G$, and Y has the same degree as that of an irreducible KG-submodule of A. Thus X can be viewed as an irreducible projective representation of V. Since V is cyclic, X has degree 1 and so A must have the same dimension as one of its irreducible KG-submodules. Thus A remains irreducible when viewed as a KG-module.

We established earlier that $\phi(G) = \phi^V(G) = G_w \subseteq Z(G)$. Since $Z(G)$ is V-invariant and contains the unique maximal V-invariant subgroup of G, either $Z(G) = G$ or $Z(G) = \phi(G)$. In the former case G is abelian, and the irreducible KG-module A is therefore one-dimensional. Then, if ρ denotes the representation of H afforded by A, $\rho(v)$ commutes with $\rho(x)$ for all $x \in G$, whence v centralizes G, since A is faithful. Certainly in that case $w = v^{p^{k-1}}$ centralizes A. Thus we may suppose that $Z(G) = \phi(G)$. Now by Clifford's Theorem, all $KZ(G)$ submodules of A are conjugate (under the action of G alone) and since $Z(G)$ is the center of G, they are equivalent. Thus the abelian group, $Z(G)$ is represented on A by left scalar multiplication by elements of K. Thus $Z(G)$ is cyclic and the matrix $\rho(v)$ commutes with the scalar matrices $\rho(z)$ for all $z \in Z(G)$. Since ρ represents $H = GV$ faithfully, $Z(G) \subseteq G_V \subseteq G_w = \phi(G) = Z(G)$. Also, we may assume $G = (v, G)$ since $G/\phi(G)$ admits v irreducibly and v does not act trivially on it. Thus every element of G is a product of elements of the form $v(x)x^{-1}$, for various elements x in G. Now the Frattini subgroup of G is generated by G' and the q-th powers of elements of G. If it can then be shown that the q-th power of any element lies in G' we shall have that $G' = \phi(G)$. This is an easy consequence of the fact that every element of G is a product of elements of the form $v(x)x^{-1}$ and that $\phi(G)$ is fixed elementwise by v. Let y be an arbitrary element of G. Then

$$y = (v(x_1)x_1^{-1})\cdot(v(x_2)x_2^{-1}) \cdots (v(x_t)x_t^{-1}).$$

Then, modulo G',

$$\begin{aligned} y^q &\equiv (v(x_1)x_1^{-1})^q \cdots (v(x_t)x_t^{-1})^q \\ &\equiv (v(x_1)^q(x_1^{-1})^q) \cdots (v(x_t)^q(x_t^{-1})^q) \\ &\equiv (v(x_1^q)(x_1^{-1})^q) \cdots (v(x_t^q)(x_t^{-1})^q) \\ &\equiv (x_1^q(x_1^{-1})^q) \cdots (x_t^q\cdot(x_t^{-1})^q) \\ &\equiv 1 \mod G' \end{aligned}$$

since every q-th power lies in $\phi(G)$ and hence is fixed by v. Thus $y^q \in G'$ and so $G' = \phi(G)$. Now every non-trivial element in G' has order q since $G' = Z(G)$ is abelian, and for any generator, (x, y) we have

$$(2) \qquad \begin{aligned} (x, y)^q &= (x, y) \cdots (x, y)(x, y) \qquad \text{(to } q \text{ factors)} \\ &= (x, y)^{x^{q-1}}(x, y)^{x^{q-2}} \cdots (x, y)^x(x, y) \end{aligned}$$

since $(x, y) \in Z(G)$. By applying the identity $(ab, c) = (a, c)^b(b, c)$ successively on the two right most terms in (2), $q - 1$ times, we obtain $(x, y)^q = (x^q, y) = 1$ since x^q lies in the center. Thus all of the generators of the abelian group G' have order q. Consequently, since $G' = Z(G)$ is cyclic, this group has order q.

Thus G is an extra special q-group of order q^{2d+1}, and $\rho|_G$ is irreducible of degree q^d. Consequently, the enveloping algebra $\mathfrak{A}$ for this representation is a full matrix algebra of dimension q^{2d}. Let ρ^* denote the matrix representation of the group algebra, KG induced by ρ. Then since ρ^* is a ring homomorphism, $\mathfrak{A} \simeq \rho^*(KG) \simeq KG/\ker \rho^*$. Set $J = KG(z - \theta\cdot 1)$, the two sided principal ideal in KG generated by $z - \theta\cdot 1$ where 1 denotes the identity of G, z is a fixed generator of $Z(G)$ and θ is the q-th root of unity such that $\rho(z) = \theta I$. Now

$$\rho^*(z - \theta\cdot 1) = \rho(z) - \theta\rho(1) = \theta I - \theta I = 0$$

so

$$(3) \qquad J \subseteq \ker \rho^*.$$

It is easily shown that all elements of G which as elements of KG lie in cosets of the form

$$\xi\cdot 1 + J, \qquad \xi \in K$$

belong to $Z(G)$. More generally, if L is any collection of cosets of J for which L/J is a one-dimensional subspace of KG/J and if L contains the group element, x, it even contains the entire coset $xZ(G)$. Moreover, if $y \in G$ and $y \in 1$ then $xZ(G) = yZ(G)$. Using these facts, it is not difficult to show that if $y_1, \cdots, y_{q^{2d}}$ are a complete set of coset representatives of $Z(G)$ in G, then the cosets $y_i + J$, $i = 1, 2, \cdots, q^{2d}$, form a K-basis of the factor algebra

KG/J. Thus $\operatorname{codim}_K J = \dim_K(KG/J) = q^{2d}$ and so, from (3) and the fact that $\dim_K \mathfrak{A} = q^{2d}$ we have $\ker \rho^* = J = KG(z - \theta \cdot 1)$ whence

$$(4) \qquad KG/KG(z - \theta \cdot 1).$$

It is easily shown that the representatives $y_1, \cdots, y_{q^{2d}}$ which are used to form a K-basis for KG/J can be chosen so that they are invariant (as a set) under the automorphism v. The permutation induced on the y_i then corresponds exactly to that induced by v on the elements of $G/Z(G)$. Relative to the v-invariant basis $y_i + J$ of KG/J, v is represented by a permutation matrix, and the v-orbits in $y_i + J$ can be utilized (by forming sums on each orbit) to construct a K-basis for the centralizer of v in KG/J. The number of such orbits is, of course, the number of orbits v produces on the elements of $G/Z(G)$. Since $G/Z(G)$ is an irreducible v-module, each orbit distinct from the identity element of $G/Z(G)$ has length p^k. Thus we have

$$(5) \qquad \dim_K(\mathfrak{C}_{KG/J}(v)) = 1 + (q^{2d} - 1)/p^k.$$

But the isomorphism in (4) is a v-isomorphism, and so the centralizer of v in $\mathfrak{A}$ is isomorphic to that in KG/J.

The next step is the determination of the dimension of $\mathfrak{C}_{\mathfrak{A}}(v)$. First, since the module, A, is completely reducible as a V-space, it is the sum of one-dimensional V-spaces—that is, the matrix $\rho(v)$ can be put in diagonal form by means of a change of basis in A. Thus, without loss of generality, we may write

$$\rho(v) = \operatorname{diag}(\lambda_1, \cdots, \lambda_1; \lambda_2, \cdots \lambda_2; \cdots; \lambda_{p^k}, \cdots, \lambda_{p^k})$$

where λ_1, λ_2, etc. denote the p^k distinct roots of $x^{p^k} - 1 = 0$ in K, and where it may happen that some roots do not appear in $\rho(v)$. Let a_i denote the number of times the root λ_i appears in the diagonal matrix $\rho(v)$. Then

$$(6) \qquad \textstyle\sum_{i=1}^{p^k} a_i = q^d, \quad \text{the degree of } \rho.$$

What are the matrices in $\mathfrak{A}$ which commute with $\rho(v)$? If X is a matrix of the enveloping algebra which commutes with the diagonal matrix $\rho(v)$ as given above, X can be partitioned into blocks in the following manner:

$$(7) \qquad X = \begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & \cdots \\ X_{21} & X_{22} & & \ddots & \\ X_{31} & X_{32} & \cdots & & X_{ij} \\ \vdots & & & & \end{bmatrix}$$

where each X_{ij} is an a_j by a_i matrix (of course if either a_i or a_j is zero, the block does not actually appear). The condition that X be commutative with $\rho(v)$ is that

$$(8) \qquad \lambda_i I_{a_i} X_{ij} = X_{ij} \lambda_j I_{a_j}$$

or simply

$$X_{ij} = \lambda_i^{-1} \lambda_j X_{ij}$$

where I_{a_i} denotes the a_i by a_i identity matrix. Thus if $i \neq j$, then $\lambda_i \neq \lambda_j$, since the roots were chosen distinct, and so X_{ij} is forced to be the zero matrix. If $i = j$, *any* X_{ij} satisfies (8). The K-dimension of the algebra of such matrices X which commute with $\rho(v)$ is thus that for the algebra of all matrices of the form

$$X = \text{diag}\,(X_{11}, \cdots, X_{p^k,p^k}),$$

X_{ii} any a_i by a_i matrix, namely the sum of the squares, $\sum a_i^2$. Thus, from (5) we have

$$\sum_{i=1}^{p^k} a_i^2 = (q^{2d} - 1)/p^k + 1 \tag{9}$$

Thus (6) and (9) comprise two conditions of the a_i's.

Now suppose one of the a_i was zero. Then (6) reduces to a sum of $p^k - 1$ terms which add up to q^d. Omitting the term fixed to be zero, say $a_1 = 0$, what are the possible values of

$$\sum_{i=2}^{p^k} a_i^2 \text{ ?} \tag{10}$$

If we think of the a_i, $(1 \leq i \leq p^k)$ as ranging over the reals, the form (10) achieves its minimum, given the constant sum in (6), when each of the a_i are equal $(i > 1)$. In this case we have $a_1 = 0$ and $a_i = q^d/(p^k - 1)$, the mean value of the a_i's. In this case, the minimum value of (10) is

$$\sum_{i=2}^{p^k} \left(\frac{q^d}{p^k - 1}\right)^2 = (p^k - 1)\left(\frac{q^d}{p^k - 1}\right)^2 = q^{2d}/(p^k - 1).$$

But this quantity actually exceeds the right hand side of (9) unless $q^d \leq p^k - 1$.

Replacing p^k by $q^d + 1$ in the right hand side of (9), we obtain the inequality

$$q^d = \sum a_i \leq \sum a_i^2 \leq q^d \tag{11}$$

Thus each a_i is zero or 1, and

$$\sum a_i^2 = (q^{2d} - 1)/p^k + 1 = q^d \tag{12}$$

now implies

$$p^k = q^d + 1. \tag{13}$$

In this case either $p = 2$, $d = 1$ and q is a Mersenne prime, or $q = 2$, $k = 1$, and p is a Fermat prime, or the equation is $9 = 2^3 + 1$ (the only case in which both exponents can differ from unity). But in all of these cases, the forbidden conditions (a) or (b) must hold for $|G|$ and p. Thus, under the conditions of the theorem, a contradiction is obtained, thus denying the supposition that at least one a_i is zero. Consequently, every one of the p^k distinct roots of $x^{p^k} - 1 = 0$ in K must appear in the diagonal form of the matrix $\rho(v)$. In particular, this means that v acts with minimal polynomial $x^{p^k} - 1$ on the module A, and that the eigen-value $\lambda_1 = 1$ is present. As a consequence, v fixes a non-trivial subspace of A elementwise. But this is contrary to the hypothesis that v acts in fixed point free manner on A, which completes the proof of the theorem.

COROLLARY 3.2. *In Theorem* 3.1, *the condition that K be a splitting field for all subgroups of $H = GV$ may be dropped.*

Proof. Let A be the KH-module of the theorem, where K is any field whose characteristic does not divide H. Since H is finite, there exists a finite extension, L, of K which is a splitting field for all subgroups of H. Everything is now a matter of observing that the hypotheses of the theorem hold for the module $A^L = A \otimes_K L$. A^L is a left LH-module where L is a splitting field for all subgroups of H. As a KH-module, A^L is isomorphic to the sum of $[L:K]$ copies of A. By hypothesis, the generator, v, of V, acts in fixed point free manner on A. Then

$$\begin{aligned}(A^L)_V = (A \dotplus \cdots \dotplus A)_V &= A_V \dotplus \cdots \dotplus A_V \\ &= 0 \dotplus \cdots \dotplus 0,\end{aligned}$$

the trivial module. Hence V is also fixed point free on A^L. The divisibility conditions on p and $|G|$ carry over automatically, and char L = char K does not divide $|H|$. Finally, A^L is faithful as a KG-module, since A is. Thus the hypotheses of the theorem hold with A^L in place of A and L in place of K. By Theorem 3.1, $v^{p^{k-1}}$ centralizes G and the corollary is proved.

At this stage it is possible to prove the existence of a bound on the nilpotent length of solvable groups admitting a fixed point free automorphism of prime power order, but this result will be contained in the more general treatment of the next section. For the moment, we can make use of the fact that G need not be solvable in Theorem 3.1, in proving

THEOREM 3.3. *Let G be a π-solvable group having no normal π'-groups. Suppose G admits a fixed point free automorphism, v, of order p^k, where p is a prime, and either*
(a) *p is odd and G has a trivial or abelian 2-Sylow subgroup,*
(b) *G has a non-abelian 2-Sylow subgroup and p is not a Fermat prime,*
(c) *$p = 2$ and G has abelian q-Sylow subgroups for all Mersenne primes, q, dividing $|G|$, for which $q < 2^k$.*
Then the number of distinct terms n_u appearing in the upper π-series for G (including E and G) does not exceed $k + 1$.

Proof. *Case* I. $k = 1$. Here G is nilpotent, and since $O_{\pi'}(G) = E$, G is a π-group with the upper π-series: $E \lhd G$. Thus $n_u = 2$, $k + 1 = 1 + 1 = 2$.

Case II. $k > 1$. Without loss of generality we may suppose that there exist at least three terms in the upper π-series for G, and accordingly we consider the first three terms of that series:

$$(14)\qquad E = O_{\pi'}(G) \lhd O_{\pi}(G) \lhd O_{\pi\pi'}(G),$$

since $O_{\pi'\pi}(G) = O_{\pi}(G)$ and $O_{\pi'\pi\pi'}(G) = O_{\pi\pi'}(G)$. (We shall later make use of the fact that the three members of (14) are also the first three terms of the upper π'-series for G.) Let v be the automorphism of order p^k and set $w = v^{p^{k-1}}$ an element of order p. We shall show that w fixes $O_{\pi\pi'}(G)/O_{\pi}(G)$ elementwise.

Now either $O_{\pi}(G)$ or $O_{\pi\pi'}(G)/O_{\pi}(G)$ has odd order, and by the recent result of Thompson and Feit, one of the two is solvable. In this case, the Schur-Zassenhaus Theorem tells us that $O_{\pi\pi'}(G)$ splits over $O_{\pi}(G)$. Thus $O_{\pi\pi'}(G) = AB$, where $A = O_{\pi}(G)$ and B is a complement of A. Moreover, all such complements are conjugate in AB and so, since the number of these is prime to p, at least one of these conjugate π-complements is V-invariant. Thus without loss of generality, we may choose B to be V-invariant. Then BV is a group of automorphisms acting on A (by conjugation in the case of elements of B) and having order prime to $| A |$. Consequently, by Lemma 2.5, for every prime r dividing A, there exists an r-Sylow subgroup of A, say A_r, which is BV-invariant. Then $F_r = A_r/\phi(A_r)$ is a $GF(r)BV$-module affording the representation, α_r, and is fixed point free under the action of V. Now since B has order prime to r, the characteristic of $GF(r)$, and $| V |$ and $| B |$ together satisfy the divisibility conditions (a), (b) and (c), by Corollary 3.2 w fixed $B/(B \cap \ker \alpha_r)$ elementwise. Each of the groups $B \cap \ker \alpha_r$ is a normal V-invariant subgroup of B. Now select x an element of

$$(15)\qquad \textstyle\bigcap_{r \mid |A|} (B \cap \ker \alpha_r).$$

Then x is an element of B centralizing each F_r. Then by Lemma 2.3

$$A_r = C_{A_r}(x)\phi(A_r)$$

since x, being a member of B, has order prime to $| A |$ and hence prime to r. Thus $A_r = C_{A_r}(x)$. But if x centralizes one member of each set of conjugate r-Sylow subgroups of A, it must centralize all of A. On the other hand, $A = O_{\pi}(G)$ contains its own centralizer in G and so $x \in A \cap B = E$. Thus the intersection (15) is trivial.

Now since w fixed $B/B \cap \ker \alpha_r$ elementwise, by Lemma 2.4 $(w, B) \subseteq B \cap \ker \alpha_r$ for every r and so $(w, B) = E$, that is, w fixes B elementwise. Because of the V-isomorphism $B \simeq O_{\pi\pi'}(G)/O_{\pi}(G)$, we have that w fixes $O_{\pi\pi'}(G)/O_{\pi}(G)$ elementwise, as we set out to prove.

But now $BA/A \lhd G/A$, $BA/A \subseteq (G/A)_w$ and BA/A contains its own centralizer in G/A, since this is $O_{\pi'\pi\pi'}(G)/O_{\pi'\pi}(G)$. Thus, by Lemma 2.2

$$G/A = (G/A)_w C_{G/A}(BA/A) = (G/A)_w .$$

That is, w fixes $G/O_{\pi}(G)$, elementwise. Thus $G/O_{\pi}(G)$ admits the cyclic group $V/\{w\}$ of order p^{k-1} as a fixed point free group of automorphisms. Since this group has no normal π-groups, and it and $V/\{w\}$ satisfy the divisibility conditions (a)–(c), by induction, the number of distinct terms appearing in the upper π'-series does not exceed k, i.e.,

$$n_u^{(\pi')}(G/O_\pi(G)) \leq (k-1)+1 = k. \tag{16}$$

But this series (or at least the preimages in G of the terms of this series) consists of the successive terms of the upper π-series for G, beginning with $O_\pi(G)$, taken modulo $O_\pi(G)$. As a consequence

$$n_u^{(\pi')}(G/O_\pi(G)) = n_u - 1$$

and so, from (16), $n_u \leq k+1$.

COROLLARY 3.4. *Let G be a π-solvable group admitting a fixed point free automorphism, v, of order p^k, where $|G|$ and p satisfy the divisibility conditions of* (a), (b), *and* (c) *of Theorem* 3.3. *Then if G has π-length l,*

$$l \leq \left[\frac{k+1}{2}\right].$$

Proof. If G has π-length l, so does $G/O_{\pi'}(G)$ and the latter has $2l$ or $2l+1$ distinct terms in its upper π-series according as the series terminates in a π or π'-factor. In any event $n_u(G/O_{\pi'}(G)) \geq 2l$. But since $G/O_{\pi'}(G)$ has no normal π'-groups and satisfies the conditions (a), (b) and (c), by Theorem 3.3.1, $2l \leq n_u(G/O_{\pi'}(G)) \leq k+1$, whence, considering that l is an integer,

$$l \leq \left[\frac{k+1}{2}\right].$$

4. Groups admitting a fixed point free abelian group of operators

This section extends the results of Section 3 to the case where V is an abelian group of operators. As in Section 3, there is a technical preliminary theorem cast in the language of linear groups. Although this theorem resembles Theorem 3.1 in that it asserts that certain factor groups of G are fixed elementwise by a subgroup of V, it differs in several important respects. Evidently if V acts in fixed point free manner on a faithful G-module A, it is not true in general that some subgroup $W \lhd V$ fixes G elementwise. Examples illustrating this fact are easily constructed for the case that V is elementary abelian of order p^2. For this reason, the hypotheses are altered to require that A be a homogeneous KGV-module, where K is a splitting field for every subgroup of GV. Again, since the condition of homogeneity must be demonstrated before it is possible to apply induction on KMV-submodules of A, where M is a V-invariant subgroup of G, the requirement that G be solvable is included to facilitate this argument. Finally, the proof provides a reduction to the case that G is extra special be a route different from that of Theorem 3.1, which, like "Theorem C" of Hall and Higman, results from having a group minimal with respect to being normalized but not centralized by a given fixed element. In this case, there is no such element available.

DEFINITION. Let V be an arbitrary abelian group having exponent m (i.e. $v^m = 1$ for every v in V and moreover, V contains an element of this order). A prime number, q, is called *exceptional* if $q^f = d - 1$ for some divisor, d, of m, and some integer f. The set of exceptional primes depends only on m, and is denoted Π_m, or Π_V.

Π_m is always a finite set. Given a divisor, d, then q, if it exists, is unique, and so $|\Pi_m|$ is bounded by the number of divisors of m. If m is odd, Π_m is either empty or contains the prime 2 only. If m is an odd prime power, then Π_m is empty unless m is a power of a Fermat prime. If m is a power of 2, Π_m consists of Mersenne primes only, unless $m < 4$, in which case it is empty.

Note that if m' divides m,

$$\Pi_{m'} \subseteq \Pi_m .$$

In particular, since the exponent of a factor group, V/W, divides the exponent of V, we have

$$\Pi_{V/W} \subseteq \Pi_V .$$

Similarly, for subgroups: If $W \subseteq V$,

$$\Pi_W \subseteq \Pi_V .$$

THEOREM 4.1. *Let V be an abelian group of operators acting on a solvable group G of order prime to $|V|$, and suppose that $|G|$ is not divisible by an exceptional prime q belonging to $\Pi_e = \Pi_V$ where e is the exponent of V. Form the semi-direct product $H = GV$ and let A be a faithful KH-module, where K is a splitting field for all subgroups of H and which has characteristic not dividing $|V|$. Suppose further that*

(i) *A is a sum of equivalent indecomposable KH-modules,*

(ii) *V acts in fixed point free manner on the elements of A in this representation,*

(iii) *G has no normal p-groups, where $p =$ char K. (If char $K = 0$, this requirement can be ignored.)*

Then there exists a non-trivial subgroup $W \lhd V$ such that W fixes G elementwise.

Proof. (By induction on $|G| + \dim_K A + |V|$) Decompose A into equivalent indecomposable KH-modules, $A_1, A_2, \cdots, A_t$. Since A is faithful and each of the A_i's are equivalent, each is faithful. Moreover, each is fixed point free under V since A is. Then, $H = GV$ and A_1 satisfy the conditions of the theorem, and if $t > 1$, $\dim_K(A_1) < \dim_K(A)$ and we obtain our result by induction. Thus we may suppose $t = 1$, so A is an indecomposable KH-module.

If char $K = 0$, A is irreducible. In this paragraph, we shall achieve a reduction to the case that A is irreducible when char $K = p$. Let us suppose, then, that A is not irreducible. Then we can find a maximal KH-submodule, B, in A. Then the factor module A/B affords a representation, β, of H, which when restricted to G has a kernel $L = G \cap \ker \beta$. Since $L \lhd H$, L is a normal V-invariant subgroup of G, and so $O_p(L) \subseteq O_p(G) = E$, i.e. L has no normal p-groups. Then setting $L_0 = O_{p'}(L)$, we see that L_0 is V-invariant, and, being characteristic in L, is normal in G. Moreover, L_0 can be regarded as a group of operators of order prime to p, acting on the (additive) elementary abelian group, A. Since $L_0 \subseteq \ker \beta$, in the operator notation of Section 2, we have

$$A/B = (A/B)_{L_0} = (A)_{L_0} B/B$$

by Lemma 2.3. Since $B \neq A$, $(A)_{L_0} \neq E$. On the other hand $(A)_{L_0} \neq A$, since A is faithful. Thus, using the complete reducibility of A as L_0-module,

$$A = (A)_{L_0} \oplus (L_0, A)$$

where neither component is trivial. Since $L_0 \lhd H$, each component is H-invariant, that is, they are KH-modules, contradicting the indecomposability of A. Thus we may suppose A is an irreducible KH-module.

We may now decompose A into its homogeneous KG-components, $B_1, \cdots, B_s$, which are permuted transitively by V according to some permutation representation, π, of V, of degree s. Since V is abelian, π is a right regular permutation representation for some factor group of V. That is, $s = \text{degree } \pi = [V:\ker \pi]$. Thus if $\ker \pi = E$, V permutes the components in a V-orbit of length V, and this would yield

$$m_0 = \sum_{v \in V} v(m)$$

as a non-trivial fixed point of A for any m lying in a single component. But this contradicts the hypothesis that V is fixed point free on A. Thus we must have $\ker \pi \neq E$.

Now suppose ket $\pi \neq V$. Then, since π is permutation isomorphic to the right regular permutation representation of $V/\ker \pi$, $\ker \pi$ is the stability group in V for each B_i. Moreover, $\ker \pi$ is fixed point free on each B_i, for if $b_1 \in B_1$ were fixed elementwise by $\ker \pi$, again

$$m_{0'} = \sum_{v \in S} v(b_1),$$

where S is a complete system of distinct coset representative of $\ker \pi$ in V, would be a non-trivial point of A, fixed by V, contrary to hypothesis. Thus $\ker \pi$ is fixed point free on each B_i. Let each B_i afford the representation β_i of $(\ker \pi)G$. Now $\Pi_{(\ker \pi)} \subseteq \Pi_V$ and $[G:\ker \beta_i \cap G]$ is not divisible by primes belonging to Π_V. Moreover, we can also see that $G/\ker \beta_i \cap G$ has no normal p-groups because of the following argument: It follows that since the *normal* subgroup $\ker \pi$ is the stability group for B_i, each B_i is a faithful *irreducible* $K(\ker \pi)(G/\ker \beta_i \cap G)$-module. If

$$T_i = O_p(\ker \pi(G/\ker \beta_i \cap G)),$$

then B_i is a sum of conjugate irreducible T_i-modules. But T_i is a p-group and, assuming char $K = p$, we see that any irreducible T_i-module in B_i is trivial. This means $T_i = \ker \beta_i/\ker \beta_i = E$. Thus $G/\ker \beta_i \cap G$ has no normal p-groups. (For the case that char $K = 0$, the condition that $G/\ker \beta_i \cap G$ has no normal p-group does not have to be verified.) Since we have assumed that $V \neq \ker \pi$, $B_i \neq A$ and so $\dim_K(B_i) < \dim_K(A)$, we are now in a position to apply induction on the irreducible $K(\ker \pi)(G/\ker \beta_i \cap G)$-module, B_i. For $i = 1$ we obtain that there exists a non-trivial element $v \in \ker \pi$ (recall that $\ker \pi \neq E$) fixing $G/\ker \beta_i \cap G$ elementwise. Thus, by Lemma 2.4, $(v, G) \subseteq \ker \beta_1$. But with appropriate indexing of the v's belonging to S, the system of distinct coset representatives of $\ker \pi$ in V, we may write $B_i = v_i(B_1)$ and $\ker \beta_i \cap G = v_i(\ker \beta_1 \cap G)$, $v_i \in S$. Since V is abelian, $\{v\} \triangleleft V$ and so $v_i(v, G) = (v, G)$. Thus $(v, G) \subseteq \ker \beta_i$ for $i = 1, 2, \cdots, s$. Since A is faithful

$$G \cap (\textstyle\bigcap_{i=1}^{s} \ker \beta_i) = E$$

and so

$$(v, G) = E,$$

that is, v fixes G elementwise, which was to be shown.

Thus we may rule out the case $\ker \pi \neq V$. We thus are left with $s = 1$, and A, a *homogeneous* KG-module.

Since G is solvable, there exists a maximal normal V-invariant subgroup, M, necessarily containing G', and G/M is an elementary abelian group which may be regarded as a V-module over the field of q elements (q is some prime dividing G, and for the moment, the possibility that $q = p$ is not excluded). Because of the maximality of M, G/M acts as an irreducible V-module. Since V is abelian, this irreducible V-module has kernel, $W \triangleleft V$, such that V/W is cyclic.

Since A is a homogeneous KG-module, and $M \triangleleft VG$, we may decompose A into its homogeneous KM-components, $C_1, C_2, \cdots, C_k$. These will be permuted by $H/M = GV/M$, and since A is homogeneous as a KG-module, these components will be permuted transitively by the elements of G/M alone. Let x be an element of G/M and suppose $x(C_i) = C_i$, $i = 1, 2, \cdots, k$. Then for any $v \in V$, $v(C_i) = C_j = v(x(C_i)) = v(x)(C_j)$, for $i = 1, 2, \cdots, k$, or equivalently, $j = 1, \cdots, k$. Thus $v(x)$ is also an element leaving each C_i invariant, and thus we see that the elements of G/M leaving each C_i invariant, form a V-invariant subgroup of G/M. Thus this subgroup is either E or G/M itself since M is maximal V-invariant. That is, either $k = 1$, or $k = [G:M]$. We shall rule out the latter case.

Suppose $k = [G:M]$. Then with appropriate indexing of the elements

x of G/M, we may write

$$C_i = x_i(C_1). \tag{17}$$

The permutations of the C_i are the images of some permutation representation of the group $V(G/M)$, of degree $[G:M]$. This representation is therefore permutation-isomorphic to one obtained by multiplication of the left cosets of some subgroup $U \subset V(G/M)$ of index $[G:M]$, by elements of $V(G/M)$. But V has order prime to G/M and clearly $|U| = |V|$ so U is a q-complement in the solvable group $V(G/M)$. Consequently U is a conjugate of V in $V(G/M)$ and so the permutation representation obtained by left multiplication of the left cosets of U is itself, permutation isomorphic to one obtained by multiplication on the left cosets of V in $V(G/M)$. The latter is then (by transitivity) permutation-isomorphic to that induced by $V(G/M)$ on the C_i's. But in the former, the elements of V leave the coset, V, fixed. As a result, we have proved that V leaves some component, say C_1, invariant.

Now select $w \in W$, the subgroup of V leaving G/M fixed elementwise. (It is entirely possible that $W = E$.) Then from (17) we have

$$w(C_i) = w(x_i C_1) = w(x_i)w(C_1) = x_i C_1 = C_i$$

since $w(x_i) = x_i$ and C_1 is V-invariant. Thus W leaves each C_i invariant. Moreover

$$v(C_i) = v(x_i C_1) = v(x_i)C_1 \tag{18}$$

so that the elements of V permute the C_i's in exactly the same manner in which the elements of V permute the elements, x_i, of G/M. Now G/M represents V/W irreducibly and faithfully, where V/W is cyclic of order n.

Since G/M is an irreducible V/W-module, all V-orbits produced on the non-identity elements of G/M have length n. Thus by (18) and the remark which follows it, all but one of the C_i's are permuted in cycles of length n (C_1 is V-invariant). If x_i is a non-identity element of G/M, then $x_i(C_1) = C_i$ belongs to a V-orbit of length n. If $n > 1$, $W \neq E$ since otherwise, for any $c_i \in C_i$,

$$\sum_{v \in V} v(c_i)$$

is a non-trivial point of A fixed by V, contrary to hypothesis. But then, if $W \neq E$, W is the subgroup of V leaving C_i invariant (since C_i belongs to a V-orbit of length n). If $n = 1$, of course $W = V$, and stabilizes each C_j, $j = 1, \cdots, k$. In any event, W is fixed point free on C_i, for if $c_i \in C_i$ is fixed by w, then for any system, R, of coset representatives of W in V,

$$c = \sum_{v_j \in R} v_j(c_i)$$

would be a non-trivial point of A fixed by V. Thus W acts in fixed point free manner on each WM-module, C_i, $i \neq 1$. Since WM necessarily coincides with the stability group of C_i in $H = VG$, C_i is an indecomposable

WM-module. Let μ_i be the restriction of the representation of WM, on C_i to M. Since C_i is a sum of equivalent irreducible M/ker μ_i-modules, $M/\ker \mu_i$ has no normal p-groups. Since $k > 1$,

$$|W| + \dim_K C_i + |M/\ker \mu_i| < |V| + \dim_K A + |G|,$$

and we may apply induction to obtain that a non-trivial subgroup $W_0 \lhd W$ fixes $M/\ker \mu_i$ elementwise. Since G/M is fixed by W, the modules C_j, $j = 1, \cdots, k$ are all conjugate by elements of G_w. Since these elements are themselves centralized by W_0, W_0 centralizes each $M/\ker \mu_j$, $j = 1, \cdots, k$. Thus

$$(W_0, M) \subseteq \bigcap \ker \mu_j = E$$

and so W_0 fixes M elementwise. Since W centralizes G/M, $G = G_w M$, $M \subseteq G_{w_0}$ and $G_w \subseteq G_{w_0}$ imply $G = G_{w_0}$, our conclusion.

Thus we must suppose that $k = 1$ so that A is a homogeneous KM-module. But $O_p(G) = E$ implies $O_p(M) = E$ and certainly A is a faithful KM-module, fixed point free under V. Moreover, $|M|$ is not divisible by a prime belonging to Π_m since $|G|$ is not.

Since $|V| + \dim_K A + |M| < |V| + \dim_K A + |G|$, we may apply induction. Thus M is fixed elementwise by some element $v \in V$. Now if G/M is also fixed by v we are done. Thus, since V is abelian, we have $(G/M)_v \neq G/M$ and hence $(G/M)_v = M/M$ so $M = G_v$. Since $M \lhd G$, by Lemma 2.2, $MC_G(M) = G$. In fact, if $C_G(M)_q$ is a V-invariant q-Sylow subgroup of $C_G(M)$, then $MC_G(M)_q = G$. Select the subgroup Q minimal with respect to being a V-invariant subgroup of $C_G(M)_q$ such that $MQ = G$. Then since M centralizes Q, we have $Q \lhd G$. Also, then $Z(Q)$ is centralized by both Q and M and so $Z(Q) \subseteq Z(G)$. Since A is a homogeneous KG-module where K is a splitting field for every subgroup of GV, $Z(G)$ is not only non-trivial, but is represented by scalar multiplication on A. Thus the elements of V commute with those of $Z(G)$ and so $Z(Q) \subseteq G_V \subseteq G_v = M$. Since M is properly contained in G, and $Z(Q) \subseteq M$, it is evident that if $M \neq E$, Q is non-abelian. We can even show that Q is extra special. Let D be any proper V-invariant subgroup of Q. Then, by the minimality of Q, and the maximality of M, $DM = M$ so $D \subseteq M$. Since Q centralizes M, it centralizes D, whence $D \subseteq Z(Q)$. Thus $Z(Q)$ is the unique maximal V-invariant subgroup of Q and so $Z(Q) = \phi^V(Q) = \phi(Q)$. $Z(Q)$ is cyclic, since it is abelian and possesses a faithful irreducible representation (this follows from the fact that all $KZ(Q)$-modules in A are equivalent). But since $Q \lhd G$ and is centralized by M, where $MQ = G$, by the Clifford theorems, A is also a homogeneous KQ-module, since the homogeneous Q-components must be permuted by elements in G (transitively) which centralize Q. Now W fixes G/M (which is V-isomorphic to $Q/\phi(Q)$) elementwise, and since W has order prime to Q, W must fix all of Q elementwise. (Note that in the case $k = 1$, with which we are now concerned, the possibility that $W = E$ is presented anew.) Clearly we cannot have $W = V$ since $(G)_v = M$.

Thus $W \neq V$. Now if W were fixed point free on A we could observe that $\Pi_W \subseteq \Pi_V$, that $W \neq E$, and obtain our result immediately by induction on $|W| < |V|$. Thus we must have that $(A)_W \neq E$. But since W fixes Q elementwise, and V is abelian, $(A)_W$ is both a V and a Q-module,—i.e. it is a KQV-module. Since all irreducible KQ-modules in A are equivalent, it is in fact a faithful KQV-module. Moreover, it is fixed point free under V. Also, since V is prime to Q and char K, and since $q \notin \Pi_V$, we may apply induction once more, provided $\dim_K((A)_W) + |Q| < \dim_K A + |G|$ to obtain that some non-trivial subgroup $U/W \lhd V/W$ fixes Q elementwise (for W acts trivially on *both* Q and $(A)_W$). In that case, U properly contains W and fixes Q elementwise. But this is a contradiction since W was *defined* to be the group fixing $Q/\phi(Q) \simeq G/M$ elementwise. Thus $\dim_K(A_W) + |Q| = \dim_K A + |G|$. This means $(A)_W = A$ and $Q = G$. In this case, W fixes all of G and so $W = E$.

By Clifford's Theorem on homogeneous modules, we may write the representation, ρ, of $H = QV$ afforded by A in the form $\rho = Y \otimes X$ where Y and X are irreducible projective representations of V and Q respectively. The restriction of X to Q is equivalent to an irreducible representation of Q and X has degree equal to that of an irreducible KQ-component of ρ. Since V is cyclic, Y has degree 1 and so the degree of ρ coincides with the degree of an irreducible KQ-component of ρ. Thus A is already an irreducible KQ-module.

Now we have A an irreducible KQ-module which at the same time is an irreducible KQV-module, where V is cyclic of order n and Q is an extra special q-group. Since $G/\phi(G)$ is an irreducible V-space, $[G:\phi(G)] = q^e$ where e is the exponent of q mod n, and e is even. This is exactly the situation encountered in Theorem 3.1 except that v, the generator of V, has order n rather than prime power order As before, we can compute the K-dimension of the centralizer of v in the enveloping algebra obtained from the representation of GV afforded by A and also in the factor algebra $KG/KG(z - \theta\cdot 1)$ where z is a generator of the center of G, and θ is a primitive q-th root in K. These algebras are V-isomorphic and have dimension $q^e = q^{2k}$; they are full matrix algebras. Since char K does not divide n, the matrix representing v satisfies $x^n = 1$ and its minimal polynomial on A is an irreducible divisor of $x^n = 1$, we have that the latter has all its roots distinct, whence the matrix for v can be put in diagonal form. Letting a_i denote the number of times the n-th root λ_i appears in the diagonal form for $\rho(v)$, we obtain

$$\sum_{i=1}^{n} a_i^2 = (q^{2k} - 1)/n + 1 \tag{19}$$

for the dimension of the centralizer and

$$\sum_{i=1}^{n} a_i = q^k \tag{20}$$

for the degree of the representation afforded by A. If one of the a_i were zero, then, given (20), the form on the left hand side in (19) achieves its minimum when the remaining a_i's are all equal. In order that (19) and (20) have a solution in integers when one a_i is zero it is necessary that this minimum be less than the right side of (19). This condition yields $q^k + 1 \leq n$. On the other hand, since the a_i are integers, we always have the left side of (19) $\geq$ the left side of (20). This means $q^k + 1 \geq n$. Thus (19) and (20) both hold, given that one of the a_i's is zero, only if $q^k + 1 = n$. But in this case q belongs to $\Pi_n = \Pi_V$, contrary to our suppositions on $|G|$. Thus no a_i is zero, and so v acts on A with minimum polynomial $x^n - 1$ and has $x = 1$ as one of its characteristic roots. Consequently V does not act fixed point free on A, contradicting our hypothesis. This completes the proof of the theorem.

COROLLARY 4.2. *In Theorem* 4.1, *the module A may be taken to be an indecomposable KH-module over an arbitrary field K. (If A is irreducible, the requirement that $O_p(G) = E$, may be dropped.)*

Proof. If A is irreducible and by hypothesis, faithful, all irreducible $KO_p(G)$-submodules of A are not only conjugate, but each is trivial since $\operatorname{char} K = p$. Then $O_p(G) \subseteq \ker \alpha = E$ where α is the representation afforded by A.

If char K does not divide $|G|$, A is already irreducible. Otherwise, if A is not irreducible we may select a maximal submodule B in A, and consider the irreducible KH-factor module, A/B. Clearly, since $O_p(H) = E$, H is faithful on A/B and our result then follows by induction on $\dim_K(A/B)$. Thus we may take A to be an irreducible KH-module. Let L be a finite extension of K which is a splitting field for all subgroups of H extending K, and let C be an irreducible LH-submodule of $A \otimes_K L$. Then as KH-modules,

$$C \simeq A \dotplus A \dotplus \cdots \dotplus A, \tag{21}$$

where the number, t, of copies of A appearing in the external direct sum divides $[L:K]$.

Clearly C is now an irreducible LH-module, char L = char K = p and $O_p(H) = E$. V acts on C in fixed point free manner, for if $c \in C$ were fixed by V, then by (21) we may write $c = a_1 + \cdots + a_t$ where a_i lies in the i-th copy of A in the external direct sum (21). Then

$$v(c) = v(a_1) = + \cdots + v(a_t) = a_1 + \cdots + a_t$$

so $a_i \in v(a_i)$ for some $a_i \neq 0$, because of the "directness" of the sum (21). But this is impossible since A is fixed point free under V.

C, H and V now satisfy the conditions of Theorem 4.1 and so some $W \lhd V$ fixes G.

This result can now be used to produce a bound for nilpotent length in groups admitting a fixed point free abelian group of automorphisms. If $n = p_1^{a_1} \cdots p_t^{a_t}$, let

$$\psi(n) = \sum_{i=1}^{t} a_i$$

the total number of primes appearing in the factorization of n. Let $F(G)$

denote the Fitting subgroup of G. The subgroups, $F_k(G)$ are defined inductively by

$$F_{k+1}(G)/F_k(G) = F(G/F_k(G))$$

and $F_1(G) = F(G)$. If G is a solvable group, the series

$$E \subset F_1(G) \subset F_2(G) \subset \cdots \subset F_n(G) = G$$

is called the *upper Fitting series* of G and the invariant $n = n(G)$ is called the *nilpotent length* of G.

THEOREM 4.3. *Let G be a solvable group with nilpotent length n, admitting a fixed point free abelian group of operators, V, of order prime to $|G|$. If $|G|$ is not divisible by primes belonging to Π_m, where m is the exponent of V, then*

$$n \leq \psi |V|. \tag{22}$$

Proof. First, if V is cyclic of prime order, by the theorem of Thompson [6], G is nilpotent, so $G = F(G)$ and $n(G) = 1 = \psi |V|$. We may suppose, then, that $|V|$ is not a prime.

Let M_1 and M_2 be minimal normal V-invariant subgroups of G, and suppose $M_1 \neq M_2$. Then G/M_i, $i = 1, 2$, admit V in fixed point free manner and have orders not divisible by primes belonging to Π_m. Applying induction

$$n(G/M_i) \leq \psi |V|, \qquad i = 1, 2. \tag{23}$$

Since M_1 and M_2 intersect trivially, the inequality (22) follows immediately.

Thus we may suppose that distinct minimal normal V-invariant subgroups of G cannot be found; that is, $M = M_i$ is the unique minimal normal V-invariant subgroup of G. Then, since G is solvable, M is an elementary abelian p-group for some p dividing $|G|$, and

$$M \subseteq O_p(G). \tag{24}$$

But since M is unique $O_{p'}(G) = E$, and so the Fitting subgroup is given by

$$F(G) = O_p(G) \tag{25}$$

Thus

$$n(G/O_p(G)) = n(G/F(G)) = n - 1. \tag{26}$$

Suppose $\phi(O_p(G)) = \phi(F(G)) \neq E$. Then $F(G)/\phi(F(G))$ is a self-centralizing maximal normal p-group in $G/\phi(F(G))$ and thus coincides with the Fitting subgroup of $G/\phi(F(G))$. A simple induction argument shows that if $B_j/\phi(F(G))$ and F_j are the $(j+1)$-st members of the upper Fitting series for the groups $G/\phi(F(G))$ and G respectively, then

$$B_j = F_j, \qquad j = 1, 2, \cdots, n$$

so

$$n(G/\phi(F(G)) = n(G) = n. \tag{27}$$

But $G/\phi(F(G))$ and V together satisfy the conditions of the theorem and since $[G:\phi(F(G))] < |G|$, induction and (27) yield $n \leq \psi|V|$, our result.

Thus we may suppose $O_p(G)$ is elementary abelian. Since $M \subseteq O_p(G)$, and M is a unique minimal normal V-invariant subgroup of G, $O_p(G)$ is an indecomposable GV-module over the field of p elements. Furthermore, its associated kernel in G is simply $O_p(G)$. Finally, since V is abelian, of order prime to both p and $[G:O_p(G)]$, acting fixed point free on the module $O_p(G)$, and since $G/O_p(G)$ is solvable and has no normal p-groups and does not have its order divisible by primes belonging to Π_m, we may apply Corollary 4.2 to obtain that there exists a non-trivial subgroup $W \subset V$ such that W fixes $G/O_p(G)$ elementwise. (Note that $W \neq V$, if V is fixed point free on $G/O_p(G)$. This is possible since $|V|$ is not a prime.) Under these conditions, $G/O_p(G)$ is a group admitting V/W as a fixed point free group of operators. Moreover, since the exponent m' of V/W divides the exponent of V, namely m, then $\Pi_{m'} \subseteq \Pi_m$. As $G/O_p(G)$ is not divisible by primes belonging to Π_m, it is certainly not divisible by primes belonging to $\Pi_{m'}$. Finally, V/W has order prime to $[G:O_p(G)]$. Thus $G/O_p(G)$ and V/W satisfy the conditions of our theorem and so, by induction (since $[G:O_p(G)] < |G|$)

$$n(G/O_p(G)) \leq \psi[V:W].$$

But $\psi[V:W] \leq \psi|V| - 1$ since $W \neq E$. We also have by (26)

$$n(G/O_p(G)) = n(G/F(G)) = n - 1.$$

Thus $n - 1 = n(G/O_p(G)) \leq \psi[V:W] \leq \psi|V| - 1$, whence $n \leq \psi|V|$ as was to be shown.

5. The bound is best possible

The following theorem is self-explanatory.

THEOREM 5.1. *Let V be a cyclic group of order p^n, where p is a prime. Then there exists a solvable group, G, having nilpotent length n which admits V as a fixed point free group of automorphisms.*

Proof. Let $n = 1$. Then if G is cyclic of order q, where $q \equiv 1 \bmod p$, and v, the generator of V, acts on G by the rule $g^v = g^\alpha$ where the integer α has exponent p modulo q. Clearly G is solvable, fixed point free under V and has nilpotent length 1.

Now suppose G_k is a solvable, with nilpotent length k, and that G_k is fixed point free under the action of V_k, which is cyclic of order p^k. Let q_k be a prime number not dividing $|G_k|$ such that $q_k \equiv 1 \bmod p$. Let α_k be an integer less than q_k having exponent $p \bmod q_k$, and let F denote the field of q_k elements. Let M be a faithful FG_k-module. Let V_{k+1} be the cyclic group of order p^{k+1} and suppose v_{k+1} generates V_{k+1}. V_{k+1} acts on G_k in the following way: First U, the subgroup of order p in V_{k+1}, centralizes G_k, and V_{k+1} induces on G_k the group of automorphisms $V_{k+1}/U \simeq V_k$. Set

$$M_{k+1} = \oplus_{j=1}^{p^k} M'_j$$

where $M'_1 \simeq M$ as FG_k-modules, and the modules $M'_1, M'_2, \cdots, M'_{p^k}$ are conjugate modules affording representations ρ_j, $j = 1, \cdots, p^k$ defined by

$$\rho_{j+1}(g) = \rho_1(v_{k+1}^{-j}(g)).$$

If v_{k+1} permutes the modules in the cycle $(M_1, \cdots, M_{p^k})$, and $v_{k+1}^{p^k}$ acts as scalar multiplication by α_k, M_{k+1} becomes a faithful $FG_k\, V_{k+1}$-module.

Now set $G_{k+1} = M_{k+1}\, G_k$. Then, since M_{k+1} was faithful, M_{k+1} is a self centralizing normal q_k-Sylow subgroup of G_{k+1}, whence

$$M_{k+1} = O_{q_k}(G_{k+1}) = F(G_{k+1}),$$

the Fitting subgroup of G_{k+1}. Then G_{k+1} has nilpotent length $k + 1$. Now V_{k+1} acts fixed point free on G_{k+1}/M_{k+1} since $V_{k+1}/U = V_k$ is fixed point free on G_k. Also V_{k+1} is fixed point free on M_{k+1} since its subgroup U is fixed point free. Since $|\, V_{k+1} \,|$ is prime to $|\, G_{k+1} \,|$, V_{k+1} is fixed point free on G_{k+1}.

Added in proof. Since this paper was submitted two other papers dealing with this problem have appeared. In [4′], F. Hoffman obtained a character-theoretic proof of a special case of Corollary 3.4 which gives the bound for nilpotent length when the group of automorphisms is cyclic of prime power order. In a recent paper by J. Thompson [6′], a rather large bound has been obtained without assuming that the group of automorphisms is abelian.

References

1. J. Alperin, *System normalizers and Carter subgroups*, Amer. Math. Soc. Notices, vol. 10 (1963), p. 251.
2. S. Bauman, *The Klein group as a fixed point free automorphism group*, Ph.D. thesis, University of Illinois, 1962.
3. G. Glauberman, *Finite groups and operator groups of relatively prime order*, Amer Math. Soc. Notices, vol. 10 (1963), p. 251.
4. D. Gorenstein and I. N. Herstein, *Finite groups admitting a fixed point free automorphism of order* 4, Amer. J. Math., vol. 83 (1961), pp. 71–78.
4′. F. Hoffman, *Nilpotent height of finite groups admitting fixed point free automorphisms*, Math. Zeitschrift, vol. 85 (1964), pp. 260–267.
5. P. Hall and G. Higman, *On the p-length of p-solvable groups and reduction theorems for the Burnside problem*, Proc. London Math. Soc. (3), vol. 6 (1956), pp. 1–42.
6. J. G. Thompson, *Finite groups with fixed point free automorphisms of prime order*, Proc. Nat. Acad. Sci. U. S. A., vol. 45 (1959), pp. 578–581.
6′. J. G. Thompson, *Automorphisms of solvable groups*, J. Algebra, vol. 1 (1964), pp. 259–267.
7. H. Wielandt, *Beziehungen zwischen den Fixpunktzahlen von Automorphisgruppen einer endlichen Gruppe*. Math. Zeitschrift, vol. 73 (1960), pp. 146–158.

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