ON THE ELEMENTARY THEORY OF DIOPHANTINE APPROXIMATION OVER THE RING OF ADELES I

BY

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Diophantine approximation, the study of approximations of real numbers by rational numbers, has been investigated extensively (see Cassels [2], Koksma [6], and the literature cited therein). To some extent it has been studied over the field of p-adic numbers using the p-adic valuation (see Mahler [10], Lutz [9], and the literature cited therein). There has been investigation of diophantine approximation over algebraic numbers fields, mostly in quadratic extensions of the field of rationals (see the literature cited in [2]; but see also [12], in which the geometry of numbers over arbitrary algebraic number fields is studied using the ordinary absolute values, and the version of Roth's theorem in [8]).

The purpose of this paper is to show that the ring of T-adeles of an algebraic number field is a natural realm in which to study diophantine approximations; that by doing so, one obtains a unified treatment, covering all algebraic number fields, using the ordinary absolute values and arbitrarily many p-adic absolute values. In this way, one obtains theorems, which when specialized to the case of the rational numbers and no p-adic absolute values, yield the classical theorems of diophantine approximation as found in [2], and when specialized to the case of the rational and one p-adic absolute value yield, after minor restatements, the theorems of [9]. A major advantage of this formulation is that it is easy to see what the analogues of many of the classical theorems are and how to generalize their proofs (the outlines remain the same, but many technical complications enter—in particular, the lack of a linear order).

It would also be possible to study diophantine approximation over the adele-ring of a function field of one variable, but in this case many of the results (especially in Sections 2 and 3) would be subsumed under slightly generalized versions of the homogeneous and inhomogeneous Riemann-Roch theorems.

The T-adele ring of the rational numbers, when T contains only finitely many valuations, is very closely related to the ring of g-adic numbers studied by Mahler [10], who has also studied diophantine approximation over algebraic number fields.[2]

In Section 1, we introduce notation and state the basic facts about adeles which will be used in the sequel. We refer the reader to [7] for proofs in the

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[2] *Added in proof.* The two papers of Mahler published in Acta Math., vol. 68 (1937), pp. 109–114 and J. Australian Math. Soc., vol. 4 (1964), pp. 425–448 are closely related to the present paper.

case $T = \Omega$; the proofs for $T \neq \Omega$ are easy modifications of these (note that our notation is different from that in [7]). Proofs for the facts we use about locally compact Abelian groups will be found in [5].

In Sections 2 and 3 we study the two basic theorems of Diophantine approximation—Minkowski's theorem and Kronecker's theorem. The T-adele formulation of Kronecker's theorem when reduced to the classical case gives a slight generalization, not involving approximation (mod 1), of the usual version.

In Section 4 we study uniform distribution and obtain the analogue of some of Weyl's classical theorems. In Section 5 we investigate the metrical theory and obtain analogues of some of Khintchine's classical theorems. In Sections 4 and 5 we have omitted the multidimensional versions of our theorems; they can be obtained by the usual procedure of adding subscripts and replacing elements by n-tuples of elements.

It is possible to study such topics as lattices, successive minima, transference theorems, etc., in the ring of T-adeles. These will be the subject of future papers.

1. Preliminaries

Let K be an algebraic number field of degree N over $\mathbf{Q}$, the field of rational numbers. A completion v of K is an embedding of K as a dense subset of a locally compact field K_v ; K_v is either a finite algebraic extension of a field of p-adic numbers $\mathbf{Q}_p$, in which case v is called *p-adic*; or $K_v = \mathbf{R}$ the field of real numbers, or $K_v = \mathbf{C}$ the field of complex numbers. If $K_v = \mathbf{R}$ or $\mathbf{C}$, v is called *infinite*, and *real* or *complex* according as $K_v = \mathbf{R}$ or $K_v = \mathbf{C}$. Two completions v and v' of K are *equivalent* if there is a continuous automorphism σ of K_v onto $K_{v'}$ with $\sigma v = v'$. From each class of equivalent completions we choose one, and denote the set of representatives so chosen by Ω. The subset of infinite completions in Ω will be denoted Ω_∞ . From now on, all completions of K will be assumed to be contained in Ω. We identify K with its image in K_v , so that $K \subset K_v$ for all $v \in \Omega$. If $\mathbf{Q}_\omega$ is a completion of $\mathbf{Q}$ and K_v is a finite algebraic extension of $\mathbf{Q}_\omega$, then v *divides* ω, written $v|\omega$. Put $N_v = [K_v:Q_\omega]$, then $\sum_{v|\omega} N_v = N$. In particular, if R_1 is the number of real completions of K and R_2 the number of complex completions of K, then $R_1 + 2R_2 = N$. Each K_v has, as an additive group, a Haar measure μ_v ; if $k \in K_v$, one defines the absolute value $|\ \ |_v$ by $\mu_v(kX) = |\,k\,|_v\, \mu_v(X)$, where X is any measurable subset of K_v with $0 < \mu_v(X) < \infty$. When $K = \mathbf{Q}$, the $|\ \ |_v$ are the ordinary absolute value or the p-adic absolute values. In this case, we shall denote by $|\ \ |_\infty$ (sometimes simply $|\ \ |$) the ordinary absolute value, and the corresponding completion by ∞; if $p \in \mathbf{Z}$, p a prime, $|\ \ |_p$ denotes the p-adic absolute value and p denotes the corresponding completion. For an algebraic number field K, the p-adic valuations are in 1-1 correspondence with the prime ideals. Suppose $\mathcal{P}$ is a prime ideal with v the corresponding valuati on If α is an algebraic integer in K, and $\mathcal{P}^r$ is the exact power of $\mathcal{P}$ which divides

α, then $|\alpha|_v = (N_m \mathcal{P})^{-r}$. Conversely, $\mathcal{P}$ is the set of algebraic integers $\alpha \in K$ for which $|\alpha|_v < 1$. Then the Artin product formula holds: if $k \in K$, $k \neq 0$, then $\prod_v |k|_v = 1$. If v is real, then $|\ \ |_v$ is the ordinary absolute value, while if v is complex, then $|\ \ |_v$ is the square of the ordinary absolute value. Put $H_v = 1$ if v is p-adic, $H_v = 2$ if v is real, and $H_v = 4$ if v is complex. Then for any $v \in \Omega$, $|a + b|_v \leq H_v \max(|a|_v, |b|_v)$ for all $a, b \in K_v$.

If v is p-adic, the set $\mathbf{O}_v = \{x \in K_v : |x|_v \leq 1\}$ is the *ring of integers* of K_v; it is the unique maximal compact subring of K_v. We assume that the Haar measure μ_v is normalized so that $\mu_v(\mathbf{O}_v) = 1$. If v is infinite, put $\mathbf{O}_v = K_v$ and choose μ_v to be ordinary Lebesgue measure if v is real, and twice ordinary Lebesgue measure if v is complex.

Now let T be any subset of Ω which contains Ω_∞. For each finite subset S of T put $\mathbf{A}_K^T(S) = \prod_{v \in S} K_v \times \prod_{v \in T \sim S} \mathbf{O}_v$. Each $\mathbf{A}_K^T(S)$ is a locally compact topological ring. We define the *T-adele ring* of K as $\mathbf{A}_K^T = \bigcup_S \mathbf{A}_K^T(S)$; where the union is over all finite subsets S of T. We take as a base for the open sets of $\mathbf{A}_K^T$, the open sets in each $\mathbf{A}_K^T(S)$, and then $\mathbf{A}_K^T$ becomes a locally compact topological ring. The elements $(a_v) \in \mathbf{A}_K^T$ are called T-adeles. We identify K with a subring of $\mathbf{A}_K^T$ by identifying $k \in K$ with the T-adele (k) which has every component equal to k. (Note: The ring of adeles as defined in [7] is the ring of Ω-adeles, $\mathbf{A}_K^\Omega$.)

We shall use *almost all* to mean "all but a finite number" and (*almost*) *everywhere* to mean "(almost) all $v \in T$".

For any T-adele a, "a_v" denotes the component of a in K_v. If $a = (a_v) \in \mathbf{A}_K^T$ we put $|a|_v = |a_v|_v$. Then $|a|_v \leq 1$ almost everywhere. The invertible elements of $\mathbf{A}_K^T$ form a subgroup $\mathbf{I}_K^T$, the group of *T-ideles*; if $i \in \mathbf{I}_K^T$, then $|i|_v \neq 0$ everywhere and $|i|_v = 1$ almost everywhere. If $a, b \in \mathbf{A}_K^T$, we write $a \leq b$ if $|a|_v \leq |b|_v$ everywhere, and $a < b$ if $a \leq b$ and $|a|_v < |b|_v$ for all $v \in \Omega_\infty$. We shall also use $<$ and $\leq$ to denote ordinary inequality. The meaning will be clear from the context. The sets

$$P(i, a) = \{x \in \mathbf{A}_K^T : x - a \leq i\}, \quad \text{and} \quad P'(i, a) = \{x \in \mathbf{A}_K^T : x - a < i\},$$

where $i \in \mathbf{I}_K^T$ and $a \in \mathbf{A}_K^T$ are called paralleletopes; they form a basis for the neighborhoods of a in $\mathbf{A}_K^T$. We define the *T-integers* K^T of $\mathbf{A}_K^T$ as

$$K^T = \{k \in K : |k|_v \leq 1, \forall v \notin T\};$$

more generally, if $i \in \mathbf{I}_K^\Omega$, put $K^T(i) = \{k \in K : |k|_v \leq |i|_v, \forall v \notin T\}$, thus $K^T = K^T(1)$. We have $K^\Omega = K$, while if $T = \Omega_\infty$, then K^T is the ring of algebraic integers of K; K^T is a Dedekind domain, and the sets $K^T(i)$ are K^T-modules, which are the nonzero fractional ideals of K^T.

If $a \in \mathbf{A}_K^T$, put $|a|_K^T = \prod_{v \in T} |a|_v$ (if the product diverges, put $|a|_K^T = 0$); then $|ab|_K^T = |a|_K^T |b|_K^T$; if $a \geq b$, then $|a|_K^T \geq |b|_K^T$. If $0 \neq a \in K^T$, then by Artin's product formula $|a|_K^T \geq 1$, in particular $|a|_K^\Omega = 1$.

Since $\mathbf{A}_K^T$ is a locally compact group it has a Haar measure μ_K^T. Suppose

X_v is a measurable subset of K_v with $X_v = \mathbf{O}_v$ almost everywhere. We assume that μ_K^T is normalized so that $\mu_K^T(\prod_v X_v) = \prod_v (\mu_v(X_v))$. Then

$$\mu_K^T\{x \in \mathbf{A}_K^T : x \leq a\} = C_K \mid a \mid_K^T$$

where $C_K = 2^{R_1+R_2}\pi^{R_2}$ does not depend upon T. Let $\xi_1, \xi_2, \cdots, \xi_N$ be an integral basis for the ring of algebraic integers of K. Put

$$D_K^\infty = \{a \in A_K^{\Omega_\infty} : a = \textstyle\sum_{i=1}^\infty u_i \xi_i, 0 \leq u_i < 1\};$$

then $D_K^T = D_K^\infty \times \prod_v \mathbf{O}_v$, where the product is over the p-adic completions $v \in T$, is a fundamental domain for A_K^T/K^T; D_K^T has compact closure and $\mu_K^T(D_K^T) = \mid d \mid^{1/2}$ where d is the discriminant of the field K. (By "fundamental domain", we always mean a fundamental domain of A_K^T/K^T.)

From now on, whenever convenient and no confusion will result, we will omit the subscript K and write $\mathbf{A}^T$, $\mid\ \mid^T$, instead of $\mathbf{A}_K^T$, $\mid\ \mid_K^T$, etc.

If $p \in \mathbf{Z}$, the ring of ordinary integers, is a prime and a is an element of $\mathbf{Q}_p$, then a can be written in the form $a = \sum_{r=-r_0}^\infty a_r p^r$ where the $a_r \in \mathbf{Z}$, and $0 \leq a_r \leq p - 1$. Define $f_p(a) = \sum_{r=-r_0}^{-1} a_r p^r$. If v is a valuation of K which divides the p-adic valuation of $\mathbf{Q}$, and $a \in K_v$, we define $\lambda_v(a) = f_p(\operatorname{tr}(a))$, where the trace is that of $K_v/\mathbf{Q}_p$. If $v \in \Omega_\infty$ and $a \in K_v$, we define $\lambda_v(a) = -\operatorname{tr}(a)$ where the trace is that of $K_v/\mathbf{R}$. Then if $a \in \mathbf{A}^T$,

$$\chi_1(a) = \exp\,(2\pi i \textstyle\sum_{v \in T} \lambda_v(a))$$

is a character of the additive group of $\mathbf{A}^T$; every character χ of $\mathbf{A}^T$ is of the form $\chi(a) = \chi_1(ba)$ for some $b \in \mathbf{A}^T$. Thus $\mathbf{A}^T$ is self-dual.

Put $\mathbf{D}^T = \{a \in \mathbf{A}^T : \chi_1(ab) = 1, \forall b \in K^T\}$; $\mathbf{D}^T$ is the annihilator of K^T; it is a fractional ideal of K^T, and if $T = \Omega_\infty$, is the inverse different of K.

Finally, we note that K^T is a discrete subset of $\mathbf{A}^T$ and that $\mathbf{A}^T/K^T$ is compact.

We will denote by H the T-idele with components H_v defined earlier. Since $R_1 + 2R_2 = N$, $\mid H \mid^T = \prod_{v \in T} \mid H \mid_v = 2^{R_1}4^{R_2} = 2^N$.

2. Minkowski's theorem

If G is a locally compact Abelian group, with Haar measure μ, and H a discrete subgroup with G/H compact, then μ induces a measure μ' on the factor group G/H by $\mu'(X/H) = \mu(X)$ where X is any measurable subset of G which does not contain two points whose difference lies in H. Since G/H is compact, $\mu'(G/H)$ is finite. If X is a measurable subset of G and $\mu(X) > \mu'(X/H)$, then the map $X \to X/H$ cannot be bijective. Minkowski's theorem asserts that any such set X must contain two points whose difference lies in H. We shall apply this to the case where $G = (\mathbf{A}^T)^m$ and $H = (K^T)^m$; we take the product measure $(\mu^T)^m$ on $(\mathbf{A}^T)^m$. The measure of $(\mathbf{A}^T)^m/(K^T)^m$ is clearly $\mid d \mid^{m/2}$.

The following theorem is the adele version of Minkowski's theorem on linear forms. Special cases of it with the constants not specified may be found in [9] and [10].

THEOREM 2.1. *Let* $L_i(x) = \sum_{j=1}^m a_{ij} x_j$, $i = 1, 2, \cdots, m$ *be* m *linear forms with coefficients* $a_{ij} \in \mathbf{A}^T$. *Suppose* $b_1, b_2, \cdots, b_n$ *are* T*-ideles satisfying*

$$\prod_{i=1}^m |b_i| > |d|^{m/2}(2/\pi)^{mR_2} |\det(a_{ij})|.$$

Then there exists $k = (k_1, k_2, \cdots, k_m) \in (K^T)^m$, $k \neq 0$, *with* $L_i(k) < b_i$, $i = 1, 2, \cdots, m$.

Proof . Put $b_i' = b_i/H$. (The T-idele H is defined in Section 1.) Let S be the open subset of $(\mathbf{A}^T)^m$ defined by the conditions $L_i(x) < b_i'$, $i = 1, 2, \cdots, m$. Then

$$\begin{aligned}(\mu^T)^m(S) &= C_K^m \prod_{i=1}^m |b_i'|/|\det(a_{ij})| \\ &= (\pi/2)^{R_2 m} \prod_{i=1}^m |b_i|/|\det(a_{ij})| \\ &> |d|^{m/2}.\end{aligned}$$

Thus, there exist $x, y \in S$ with $0 \neq x - y = k \in (K^T)^m$. Then

$$\begin{aligned}|L_i(k)|_v &\leq H_v \max(|L_i(x)|_v, |L_i(y)|_v) \\ &\leq H_v |b_i'|_v = |b_i|_v,\end{aligned}$$

with the second inequality strict when v is infinite. Thus $L_i(k) < b_i$.

COROLLARY 2.2. *Let* $L_i(x) = \sum_{j=1}^m a_{ij} x_j$, $i = 1, 2, \cdots, m$ *be* m *linear forms with coefficients* $a_{ij} \in \mathbf{A}^T$. *Suppose* $b_1, b_2, \cdots, b_m$ *are* T*-ideles satisfying*

$$\prod_{i=1}^m |b_i| \geq |d|^{m/2}(2/\pi)^{mR_2} |\det(a_{ij})|,$$

then there exists $k \in (K^T)^m$, $k \neq 0$, *with* $L_1(k) \leq b_1$, *and* $L_i(k) < b_i$, $i = 2, 3, \cdots, m$.

A T-adele a is *algebraic* if it satisfies a not-identically-zero polynomial equation with coefficients in K^T. The T-height H^T of two T-adeles x and y is defined as $H^T(x, y) = \prod_{v \in T} \max(|x|_v, |y|_v)$. The homogeneous version of Roth's theorem asserts that if T is finite, if λ is a real number >1, if c is a positive real number, and if α and β are algebraic T-adeles with not both α_v and $\beta_v = 0$ for any $v \in T$, then the solutions $x, y \in K^T$ to

$$0 < \prod_v |\alpha x + \beta y|_v \leq c(H^T(x, y))^{-\lambda} \tag{1}$$

have bounded height; hence only finitely many ratios x/y occur. This form of Roth's theorem may be obtained from the version in [8].

We wish to show that this result is best possible, with respect to λ, in a rather strong sense. Namely, that when $\lambda = 1$, there exist solutions $x, y \in K^T$ to (1) with arbitrarily large height $H^T(x, y)$, and that one can distribute the value of $\prod_{v \in T} |\alpha x + \beta y|_v$ among the various factors $|\alpha x + \beta y|_v$ in a preassigned fashion.

THEOREM 2.3. *Assume T is finite and let λ_v, $v \in T$, be positive real numbers satisfying $\sum_{v\in T}\lambda_v = 1$. Let α and β be T-adeles, such that for any $v \in T$, not both α_v and $\beta_v = 0$.* (i.e., $H^T(\alpha, \beta) \neq 0$). *Let c be a T-idele with $|c|^T > |d|(2/\pi)^{R_2}H^T(\alpha, \beta)$. Then there exist $x, y \in K^T$ which satisfy*

$$(2) \qquad |\alpha x + \beta y|_v \leq |c|_v H^T(x, y)^{-\lambda_v}$$

for all $v \in T$, with $H^T(x, y)$ arbitrarily large.

We first prove

LEMMA 2.4. *Let P_v, $v \in T$, be real numbers >1, and ε a real number >0. There exist arbitrarily large real numbers M and integers $r_v \in \mathbf{Z}$, $v \in T$, such that*

$$(3) \qquad (1 - \varepsilon)M^{\lambda_v} \leq P_v^{r_v} \leq (1 + \varepsilon)M^{\lambda_v}$$

for all p-adic $v \in T$.

Proof. Taking logarithms, we see that (3) is equivalent to requiring that $(\lambda_v/\log P_v)\log M$ be close to an integer for each p-adic $v \in T$. But, if $M = \exp(M')$, $M' \in \mathbf{Z}$, this is the same as requiring $M'(\lambda_v/\log P_v)$ be close to an integer. This is possible, for arbitrarily large M' (even with $M' \in \mathbf{Z}$) by the classical form of Minkowski's theorem on linear forms (Theorem 2.1 in the case $K = \mathbf{Q}$, $T = \{\infty\}$).

Proof of Theorem 2.3. By replacing α and β by αa and βa, if necessary, where a is an appropriately chosen T-idele, we may assume, without loss of generality, that $\max(|\alpha|_v, |\beta|_v) = 1$, for each $v \in T$; then $H^T(\alpha, \beta) = 1$. Choose a T-idele c' such that $c' < c$ and $|c'|^T > |d|(2/\pi)^{R_2}$. Let $\varepsilon' > 0$ satisfy

$$\sup_{v\in T}[(1 + \varepsilon)^t(1 - \varepsilon')^n]^{\lambda_v}/(1 - \varepsilon) = 1$$

where t is the cardinality of T, and $n = R_1 + R_2$ the number of infinite completions of K. Clearly, $\varepsilon' \to 0$ as $\varepsilon \to 0$. If v is p-adic, the possible values of $|\ |_v$ are rational numbers of the form P_v^m or 0, where P_v is a power of a prime $p \in \mathbf{Z}$ and $m \in \mathbf{Z}$. By Lemma 2.4, there exists a T-idele i such that

$$(1 - \varepsilon)M^{\lambda_v} \leq |i|_v \leq (1 + \varepsilon)M^{\lambda_v}$$

for all p-adic $v \in T$, and $|i|_v = (1 - \varepsilon')M^{\lambda_v}$ for all infinite $v \in T$, where M can be chosen arbitrarily large. Define T-adeles γ and δ by $\gamma_v = 1$ and $\delta_v = 0$ when $|\alpha|_v \leq |\beta|_v$, otherwise $\gamma_v = 0$ and $\delta_v = 1$. The linear inequalities

$$(4) \qquad \alpha x + \beta y \leq i^{-1}c', \qquad \gamma x + \delta y \leq i$$

have determinant 1; hence by the choice of c' and Corollary 2.2, there exist $x, y \in K^T$ satisfying (4). Let $v \in T$ be infinite. Suppose $\gamma_v = 1$, so that $|\alpha/\beta|_v \leq 1$ and $\delta_v = 0$. Then $|x|_v \leq |i|_v \leq (1 - \varepsilon')M^{\lambda_v}$. By the triangle inequality,

$$|y|_v \leq |\alpha x/\beta|_v + |i^{-1}c'/\beta|_v$$

when v is real, or

$$| y |_v^{1/2} \leq | \alpha x/\beta |_v^{1/2} + | i^{-1}c'/\beta |_v^{1/2}$$

when v is complex (since in this case $| \quad |_v$ is the square of the ordinary absolute value on $\mathbf{C}$). In either case, since $| i^{-1}c'/\beta |_v \to 0$ as $M \to \infty$, we have

$$| y |_v \leq (1 + \varepsilon)| \alpha x/\beta |_v \leq (1 + \varepsilon)(1 - \varepsilon')M^{\lambda_v}$$

for sufficiently large M. By the same argument when $\gamma_v = 0$, we obtain

$$\max (| x |_v , | y |_v) \leq (1 + \varepsilon)(1 - \varepsilon')M^{\lambda_v}.$$

When v is p-adic, a similar argument shows that $\max (| x |_v , | y |_v) \leq (1 + \varepsilon)M^{\lambda_v}$. Combining these results, we obtain

$$H^T(x, y) \leq (1 + \varepsilon)^t(1 - \varepsilon')^n M.$$

If $v \in T$ is p-adic,

$$\begin{aligned} | \alpha x + \beta y |_v &\leq | c' |_v/((1 - \varepsilon)M^{\lambda_v}) \\ &\leq \frac{| c' |_v ((1 + \varepsilon)^t(1 - \varepsilon')^n)^{\lambda_v}}{(1 - \varepsilon)H^T(x, y)^{\lambda_v}} \\ &\leq | c |_v H^T(x, y)^{-\lambda_v}. \end{aligned}$$

Similarly, if $v \in T$ is infinite,

$$\begin{aligned} | \alpha x + \beta y |_v &\leq | c' |_v/((1 - \varepsilon')M^{\lambda_v}) \\ &\leq \frac{| c' |_v ((1 + \varepsilon)^t(1 - \varepsilon')^n)^{\lambda_v}}{(1 - \varepsilon)(1 - \varepsilon')H^T(x, y)^{\lambda_v}} \\ &\leq | c |_v H^T(x, y)^{-\lambda_v} \end{aligned}$$

if ε is sufficiently small. If α and β are linearly independent over K^T, then infinitely many ratios x/y must occur, since there are solutions x, y to (4) with $i^{-1}c'$ arbitrarily small; hence $H^T(x, y)$ is unbounded [8]. If α and β are linearly dependent over K^T, the entire result is trivial.

If $x = (x_1 , x_2 , \cdots , x_n) \in (\mathbf{A}^T)^n$, put $H^T(x) = \prod_{v \in T} \max_i (| x_i |_v)$. Theorem 2.3 can be generalized to m forms in n variables, where $m < n$:

THEOREM 2.5. *Suppose $m < n$ are positive integers. Assume T is finite, and let λ_v, $v \in T$, be positive real numbers satisfying $\sum_v \lambda_v = n/m - 1$. Let a_{ij}, $i = 1, 2, \cdots , m, j = 1, 2, \cdots , n$ be T-adeles with $H^T(a_{ij}) \neq 0$. Then there exists a T-idele c (not depending on the a_{ij}) such that if $c_1 , c_2 , \cdots , c_m$ are T-ideles with $\prod_i | c_i | \geq | c |^T H^T(a_{ij})$, then the solutions*

$$x = (x_1 , x_2 , \cdots , x_n) \in (K^T)^n \quad \text{to} \quad | \textstyle\sum_j a_{ij} x_j |_v \leq | c_i |_v H^T(x)^{-\lambda_v},$$

for all $v \in T$ and $i = 1, 2, \cdots , m$, have unbounded height.

We omit the proof, since it is a straightforward generalization of the proof of Theorem 2.3.

The following results will be used later.

LEMMA 2.6. *There exists a real constant c depending only on the field K, such that every set of the form* $P(j, a) = \{x \in A^T : x - a \leq j\}$, *where a is a T-adele and j is a T-idele with* $|j|^T \geq c$, *contains a fundamental domain of* A^T/K^T.

Proof. Since any translate of a fundamental domain is a fundamental domain, we may assume that $a = 0$. Let i be a T-idele such that $P(i, 0)$ contains D^T, the fundamental domain defined in Section 1. Put

$$c = |d|^{1/2}(2/\pi)^{R_2}|i|^T.$$

Then, if j is a T-idele with $|j|^T \geq c$, there exists, by Corollary 2.2, $k \in K^T$, $k \neq 0$, such that $ki \leq j$. Then $kD^T \subset kP(i, 0) \subset P(j, 0)$. *If* $x \in \mathbf{A}^T$, let $y \in D^T$ be such that $y - x/k = h \in K^T$; y exists since D^T is a fundamental domain. Then $ky \in P(j, 0)$ and $ky - x = kh \in K^T$, so that ky is a representative of x in $P(j, 0)$. It follows that the image of $P(j, 0)$ in $\mathbf{A}^T/K^T$ is all of $\mathbf{A}^T/K^T$. Hence $P(j, 0)$ contains a fundamental domain.

If $i \in I^T$, put $M^T(i) = \{k \in K^T : k \leq i\}$.

LEMMA 2.7. *The cardinality of* $M^T(i)$ *satisfies*

$$\# M^T(i) = C_K|i|^T/|d|^{1/2} + O((|i|^T)^{1-1/N}),$$

as $|i|^T \to \infty$, *where* C_K *is defined in Section* 1.

Proof. It clearly suffices to prove this lemma when $T = \Omega$. In this case it is a restatement of Theorem 1 of [7, p. 70].

LEMMA 2.8. *There exists a constant c, depending only on the field K, such that for all* $i \in I^T$

$$\# M^T(i) - 1 \leq c|i|^T.$$

Proof. This is an immediate consequence of Lemma 2.7, when one observes that when $|i|^T < 1$, the only element of $M^T(i)$ is 0.

LEMMA 2.9. *For any real* $\delta > 0$ *there exists a real positive constant* c_δ *such that if i is any T-idele with* $|i|^T \geq \delta$, *then* $\# M^T(i) \leq c_\delta|i|^T$.

Proof. Clear.

3. Kronecker's theorem

Suppose $\theta_1, \theta_2, \cdots, \theta_m$ are real numbers, linearly independent from 1 over $\mathbf{Q}$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be arbitrary real numbers. Kronecker's theorem asserts that for every real $\varepsilon > 0$, there exists an integer m and integers $r_1, r_2, \cdots, r_n$ such that

$$|m\theta_i - \alpha_i - r_i| < \varepsilon, \qquad i = 1, 2, \cdots, n.$$

The following generalization is proved in [2].

THEOREM 3.1. *Let* $L_i(x) = \sum_{j=1}^{m} a_{ij} x_j$, $i = 1, 2, \cdots, n$ *be* n *linear forms with real coefficients. Let* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be real numbers.*

Then the following conditions are equivalent:

A. *If* $c_1, c_2, \cdots, c_n$ *are integers such that* $\sum_{i=1}^{n} c_i L_i(x)$ *has integral coefficients, when considered as a linear form in* $x_1, x_2, \cdots, x_m$, *then* $\sum_{i=1}^{n} c_i \alpha_i$ *is an integer.*

B. *For every* $\varepsilon > 0$, *there exist integers* $b_1, b_2, \cdots, b_m$ *and integers* $r_1, r_2, \cdots, r_n$ *such that if* $b = (b_1, b_2, \cdots, b_m)$ *then* $| L_i(b) - \alpha_i - r_i | < \varepsilon$, $i = 1, 2, \cdots, n$.

It is immediately clear that B implies A, and that the implication A implies B contains the above-mentioned Kronecker's theorem as a special case.

In this section we prove a similar theorem where the a_{ij} will be T-adeles. A corollary of this theorem in the special case $K = \mathbf{Q}$, $T = \{\infty\}$ will reduce to Theorem 3.1. This same corollary, when specialized to the case $K = \mathbf{Q}$, $T = \{\infty, p\}$, where p is a prime in $\mathbf{Z}$, reduces to the form of Kronecker's theorem given in [9]. We also obtain some standard approximation theorems of algebraic number theory as corollaries, and a standard theorem on solutions of linear diophantine equations. The proof of the theorem is related to some theorems of Hewitt and Zuckerman [5, p. 431, Th. 26.15].

THEOREM 3.2. *Let* $L_i(x) = \sum_{j=1}^{m} a_{ij} x_j$, $i = 1, 2, \cdots, n$ *be* n *linear forms with coefficients* $a_{ij} \in \mathbf{A}^T$. *Let* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be* T*-adeles. Then the following conditions are equivalent.*

A. *If* $c_1, c_2, \cdots, c_n$ *are* T*-adeles such that* $\sum_{i=1}^{n} c_i L_i(x)$ *has coefficients in* K^T, *when considered as a linear form in* $x_1, x_2, \cdots, x_m$, *then* $\sum_{i=1}^{n} c_i \alpha_i \in K^T$.

B. *For every* T*-idele* ε, *there exists* $b = (b_1, b_2, \cdots, b_m) \in (K^T)^m$ *such that* $L_i(b) - \alpha_i < \varepsilon$, $i = 1, 2, \cdots, n$.

Theorem 3.2, when specialized to the case $K = \mathbf{Q}$, $T = \{\infty\}$, leads to a generalization of Theorem 3.1, not involving approximation "mod 1".

We first prove

LEMMA 3.3. *Let* a_{ij}, α_i, *and* $L_i(x)$ *be as in the statement of Theorem* 3.2. *The following conditions are equivalent:*

A. *If* $c_1, c_2, \cdots, c_n$ *are* T*-adeles such that* $\sum_{i=1}^{n} c_i L_i(x)$ *has coefficients in* K^T, *when considered as a linear form in* $x_1, x_2, \cdots, x_m$, *then* $\sum_{i=1}^{n} c_i \alpha_i \in K^T$.

A′. *If* $c_1, c_2, \cdots, c_n$ *are* T*-adeles such that* $\sum_{i=1}^{n} c_i L_i(x)$ *has coefficients in* $\mathbf{D}^T$ ($\mathbf{D}^T$ *is defined in Section* 1), *when considered as a linear form in* $x_1, x_2, \cdots, x_m$, *then* $\sum_{i=1}^{n} c_i \alpha_i \in \mathbf{D}^T$.

Proof. Suppose condition A is satisfied, and $\sum_{i=1}^{n} c_i L_i(x)$ has coefficients in $\mathbf{D}^T$. Let $b \in K$ be such that $b\mathbf{D}^T \subset K^T$. Then $\sum_{i=1}^{n} c_i bL_i(x)$ has coefficients in K^T, hence by condition A, $b \sum_{i=1}^{n} c_i \alpha_i \in K^T$. Denote by $(\mathbf{D}^T)^{-1}$ the inverse ideal of $\mathbf{D}^T$ so that

$$\mathbf{D}^T(\mathbf{D}^T)^{-1} = K^T \quad \text{and} \quad (\mathbf{D}^T)^{-1} = \{b \in K : b\mathbf{D}^T \subset K^T\}.$$

We have just shown that $(\mathbf{D}^T)^{-1}\sum_{i=1}^n c_i\,\alpha_i \subset K^T$; but then $\sum_{i=1}^n c_i\,\alpha_i$ is in the ideal inverse to $(\mathbf{D}^T)^{-1}$, i.e. $\sum_{i=1}^n c_i\,\alpha_i \in \mathbf{D}^T$. The converse is proved in the same way.

Proof of Theorem 3.2. We observe first that every character χ on $(\mathbf{A}^T)^n$ can be written in the form

$$\chi(u_1, u_2, \cdots, u_n) = \chi_1(\textstyle\sum_{i=1}^n u_i\,v_i)$$

where $(v_1, v_2, \cdots, v_n) \in (\mathbf{A}^T)^n$ [5, p. 362] (χ_1 is defined in Section 1). Put

$$S = \{(L_1(x), L_2(x), \cdots, L_n(x)) : x \in (K^T)^m\}.$$

Let χ be a character on $(\mathbf{A}^T)^m$ such that $\chi(S) = \{1\}$. Then

$$\begin{aligned}\chi(L_1(x), \cdots, L_n(x)) &= \chi_1(\textstyle\sum_{i=1}^n c_i\,L_i(x))\\ &= \chi_1(\textstyle\sum_{i=1}^n c_i \sum_{j=1}^m a_{ij}\,x_j)\\ &= \chi_1(\textstyle\sum_{j=1}^m (\sum_{i=1}^n c_i\,a_{ij})x_j)\\ &= 1,\end{aligned}$$

for appropriate $c_1, c_2, \cdots, c_n \in \mathbf{A}^T$, and for all $x \in (K^T)^m$. Then, clearly, $\sum_{i=1}^n c_i\,a_{ij} \in \mathbf{D}^T$, $j = 1, 2, \cdots, m$; hence by condition A′ of Lemma 3.3, $\sum_{i=1}^n c_i\,\alpha_i \in \mathbf{D}^T$. Then $\chi(\alpha_1, \alpha_2, \cdots, \alpha_n) = 1$. Thus by the duality theorem for locally compact Abelian groups, $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ is in the closure of S. The proof that B implies A is clear.

Corollary 3.4. (The adele form of Theorem 3.1). *Let* $L_i(x) = \sum_{j=1}^m a_{ij}\,x_j$, $i = 1, 2, \cdots, n$, *be* n *linear forms with coefficients* $a_{ij} \in \mathbf{A}^T$. *Let* $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be* T-*adeles. The following conditions are equivalent.*

A. *If* $c_1, c_2, \cdots, c_n \in K^T$ *are such that* $\sum_{i=1}^n c_i\,L_i(x)$ *has coefficients in* K^T, *when considered as a linear form in* $x_1, x_2, \cdots, x_m$, *then* $\sum_{i=1}^n c_i\,\alpha_i \in K^T$.

B. *For every* T-*idele* ε, *there exist*

$$b = (b_1, b_2, \cdots, b_m) \in (K^T)^m \quad \textit{and} \quad r_1, r_2, \cdots, r_n \in K^T$$

such that $L_i(b) - r_i - \alpha_i < \varepsilon$, $i = 1, 2, \cdots, n$.

Proof. Put $L_i'(x) = L_i(x) - x_{m+i}$, $i = 1, 2, \cdots, n$, and apply Theorem 3.2 to the forms $L_i'(x)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$. If the form $\sum c_i\,L_i'(x)$ has coefficients in K^T, then the coefficient of x_{i+m} being c_i, we have $c_i \in K^T$. It follows that in applying Theorem 3.2, one may restrict attention to $c_1, c_2, \cdots, c_n \in K^T$. The rest of the proof is clear.

Corollary 3.5. (The very strong approximation theorem [11, p. 77]). *Let* v_0 *be a fixed completion in* I. *Let* α *be a* T-*adele and* ε *a* T-*idele. Then there exists* $b \in K^T$ *such that* $|\,b - \alpha\,|_v \leq |\,\varepsilon\,|_v$ *for all* $v \neq v_0$.

Proof. Let a be the T-adele with $a_v = 1$ if $v \neq v_0$, and $a_{v_0} = 0$. Let α' be the T-adele with $\alpha_v' = \alpha_v$ if $v \neq v_0$, and $\alpha_{v_0}' = 0$. If c is a T-adele such that $ca \in K^T$, then, since $(ca)_{v_0} = 0$, $ca = 0$; hence $c_v = 0$ if $v \neq v_0$. Thus $c\alpha' = 0$. Now apply Theorem 3.2 to the linear form ax.

COROLLARY 3.6 [1, p. 100]. *Let* $\sum_{j=1}^{m} a_{ij} x_j = \alpha_i$, $i = 1, 2, \cdots, n$ *be* n *consistent linear equations with coefficients* a_{ij} *and constant terms* $\alpha_i \in K^T$. *The necessary and sufficient condition that these equations should have a solution* $x_1, x_2, \cdots, x_m \in K^T$ *is that they have a solution* $x_1, x_2, \cdots, x_m \in \mathbf{O}_v$ *for all* $v \notin T$ ($\mathbf{O}_v$ *is defined in Section* 1).

Proof. Since the equations are consistent, they have a solution $y_1, y_2, \cdots, y_m \in K$. If $c_1, c_2, \cdots, c_n$ are T-adeles such that $\sum_{i=1}^{n} c_i \sum_{j=1}^{m} a_{ij} x_j$ has coefficients in K^T, then

$$\sum_{i=1}^{n} c_i \alpha_i = \sum_{j=1}^{m} y_j \sum_{i=1}^{n} c_i a_{ij} \in K.$$

Now if $v \notin T$, let $x_1, x_2, \cdots, x_m$ be a solution of the given equations in $\mathbf{O}_v$. Then

$$| \sum_{i=1}^{n} c_i \alpha_i |_v = | \sum_{j=1}^{m} x_j \sum_{i=1}^{n} c_i a_{ij} |_v \leq 1;$$

hence $\sum_{i=1}^{n} c_i \alpha_i \in K^T$. By Theorem 3.2, for every T-idele ε, there exists $x_1, x_2, \cdots, x_m \in K^T$ such that $\sum_{j=1}^{m} a_{ij} x_j - \alpha_i < \varepsilon$. But if $| \varepsilon |^T < 1$, then since $\sum_{j=1}^{m} a_{ij} x_j - \alpha_i \in K^T$, we have $\sum_{j=1}^{m} a_{ij} x_j = \alpha_i$, $i = 1, 2, \cdots, n$.

The following lemma will be used in Section 4.

LEMMA 3.7. *Let* θ *be a* T-*adele satisfying* $\theta \notin K$. *Then for any* T-*idele* ε, *there exists a* T-*idele* m, *such that for any* T-*adele* b, *the inequality*

$$x\theta + y - b < \varepsilon$$

has a solution $x, y \in K^T$, *with* $x \leq m$.

Proof. By Corollary 3.4, the range of $x\theta + y$ ($x, y \in K^T$) is dense in $\mathbf{A}^T$. We may assume without loss of generality that $b \in D^T$, (D^T is the fundamental domain for A^T/K^T defined in Section 1). The lemma now follows from the compactness of A^T/K^T.

4. Uniform distribution

Let f be a function with domain K^T and range $\mathbf{A}^T$; f is *uniformly distributed* if

$$\frac{\# [(f(M^T(j) + K^T) \cap P(\varepsilon, a)]}{\# M^T(j)} \to \frac{\mu^T P(\varepsilon, a)}{\mu^T D^T} \tag{1}$$

as $j \to \infty$ (here, and throughout this section, j runs through T-ideles and the limit is to be taken in the sense of nets under the direction induced by $\leq$), where a is any T-adele, ε any T-idele, $M^T(j)$ is defined in Section 2, and $\#$, as usual, indicates cardinality. By Lemma 2.7, (1) is the same as

$$\# [f(M^T(j) + K^T) \cap P(\varepsilon, a)]/| j |^T \to d_0 | \varepsilon |^T, \tag{2}$$

where $d_0 = (2^{R_1+R_2}\pi^{R_2})^2/| d |$. By a *step function*, we mean a finite linear combination (with complex coefficients) of characteristic functions of sets of the form $P(\varepsilon, a)$. Then the function f is uniformly distributed, if and only if, for all step functions s,

$$(3) \qquad \lim_{j\to\infty} \Big(\sum_{k\in K^T} \sum_{a\leq j} s(k + f(a))\Big)/\#M^T(j) \to \int_{A^T} s \, d\mu/|\, d\,|^{1/2},$$

where the inner sum is over those $a \in K^T$ satisfying $a \leq j$. It is easy to verify that the step functions are dense (in the uniform norm) in the complex-valued continuous functions on $\mathbf{A}^T$ which have compact support. Since, $\mathbf{A}^T$, as a topological space is normal, the characteristic function of a set of the form $P(\varepsilon, a)$ can be bounded from above and below by continuous functions with compact support, with the property that the integral over $\mathbf{A}^T$ of their difference can be made arbitrarily small. It follows that f is uniformly distributed if and only if (3) holds for all continuous functions s on $\mathbf{A}^T$ with compact support.

We say a complex-valued function g on $\mathbf{A}^T$ is *periodic* (mod K^T) if $g(x + k) = g(x)$ for all $k \in K^T$ and $x \in \mathbf{A}^T$. If s is a complex-valued continuous function with compact support, then

$$(4) \qquad g(x) = \textstyle\sum_{k\in K^T} s(x + k)$$

is continuous and periodic (mod K^T). Then (3) becomes

$$(5) \qquad \lim_{j\to\infty} \sum_{a} g(f(a))/\#M(j) \to \int_{D^T} g \, d\mu/|\, d\,|^{1/2}$$

where the summation is over those $a \in K^T$ satisfying $a \leq j$. If g is a complex-valued continuous function which is periodic mod K^T, then g can be written in the form (4) where s is continuous and has compact support (Proof: Let h be a non-negative continuous function with compact support which strictly includes D^T. Put $s(x) = h(x)g(x)/\sum_{t\in K^T} h(t + x))$; it follows that f is uniformly distributed if and only if (5) holds for all complex-valued continuous functions, periodic (mod K^T). Since the linear combinations of the characters of $\mathbf{A}^T$ which equal 1 on K^T are dense (in the uniform norm) in the space of continuous functions on $\mathbf{A}^T$ which are periodic (mod K^T), f is uniformly distributed if and only if (5) holds for every g of the form $g = \chi$ a character on $\mathbf{A}^T$ equal to 1 on K^T. If χ is the principal character (i.e. $\chi(\mathbf{A}^T) = \{1\}$) then (5) holds trivially. Since every character χ which equals 1 on K^T is of the form $\chi(x) = \chi_1(bx)$ with $b \in \mathbf{D}^T$, where χ_1 and $\mathbf{D}^T$ are defined in Section 1, and since any non-principal character χ with $\chi(K^T) = \{1\}$ satisfies $\int_{D^T} \chi(a)\, d\mu(a) = 0$, we obtain the analogue of Weyl's classical criterion for uniform distribution:

THEOREM 4.1. *The function f is uniformly distributed* (mod K^T) *if and only if, for every non-zero $b \in \mathbf{D}^T$,*

$$(6) \qquad \lim_{j\to\infty} \Big(\textstyle\sum_a \chi_1(bf(a))\Big)/|\, j\,|^T = 0$$

where the summation is over those $a \in K^T$ satisfying $a \leq j$.

By an *irrational T-adele* θ, we mean a T-adele $\theta \notin K$. If i and j are two T-ideles we shall let $i \oplus j$ denote a T-idele satisfying

$$(i \oplus j)_v = \begin{cases} \max (|i|_v, |j|_v) & \text{when } v \text{ is } p\text{-adic} \\ |i|_v + |j|_v & \text{when } v \text{ is real} \\ (|i|_v^{1/2} + |j|_v^{1/2})^2 & \text{when } v \text{ is complex.} \end{cases}$$

If $a, b \in A^T$ and $a \leq i, b \leq j$, then $a + b \leq i \oplus j$. Further for any T-idele j, $m^T(i \oplus j)/m^T(i) \to 1$ as $i \to \infty$, where $m^T(i) = \# M^T(i)$.

THEOREM 4.2. *If θ is an irrational T-adele, then the function $f(k) = k\theta$ is uniformly distributed.*

Proof. Rather than estimate the sums (6), we proceed directly. For T-ideles i and j, and $a \in \mathbf{A}^T$, put

$$F_j(i, a) = \# (\theta M^T(j) + K^T) \cap P(i, a).$$

It is easy to verify that

$$\int_{D^T} F_j(i, a)\, d\mu^T(a) = \# M^T(j) \mu^T P(i, a). \tag{7}$$

For any T-idele ε, there exists by Lemma 3.7, an idele m such that for any $b \in \mathbf{A}^T$ the inequality

$$k\theta + q - b \leq \varepsilon$$

has a solution $k, q \in K^T$, with $k \leq m$.

From this we obtain

$$F_{j+m}(i \oplus \varepsilon, a) \geq F_j(i, b), \tag{8}$$

for any T-adeles a and b. In fact suppose

$$r, s \in K^T, \quad r \leq j \quad \text{and} \quad r\theta + s \in P(i, a),$$

so that $r\theta + s - a \leq i$. Take $k, q \in K^T$ such that $k \leq m$ and $k\theta + q + a - b \leq \varepsilon$; then

$$(k + r)\theta + (q + s) - b \leq i \oplus \varepsilon \quad \text{and} \quad k + r \leq j \oplus m.$$

Integrating both sides of (8) over D^T with respect to a, we obtain, by use of (7)

$$\# M^T(j + m)\mu^T P(i \oplus \varepsilon, a) \geq F_j(i, b) |d|^{1/2},$$

or

$$\frac{F_j(i, b)}{\# M^T(j)} \leq \frac{\# M^T(j \oplus m)}{\# M^T(j)} \cdot \frac{\mu^T P(i \oplus \varepsilon, a)}{|d|^{1/2}}.$$

Letting $j \to \infty$, and then $\varepsilon \to 0$, we obtain

$$\begin{aligned} \limsup_{j\to\infty} (F_j(i, b)/ \# M^T(j)) &\leq \mu^T P(i, a)/|d|^{1/2} \\ &= \mu^T P(i, b)/|d|^{1/2}. \end{aligned}$$

Similarly, by integrating the right side of (8) with respect to b; then letting $j \to \infty$, and then $\varepsilon \to 0$, we obtain the opposite inequality with "lim sup" replaced by "lim inf". Hence

$$\lim_{j\to\infty} F_j(i, b)/\#M^T(j) = \mu^T P(i, b)/|d|^{1/2};$$

but this is the definition of uniform distribution.

COROLLARY 4.3. *For any irrational T-idele θ and any nonzero $b \in \mathbf{D}^T$,*

$$\lim_{j\to\infty}\left(\sum\nolimits_a \chi_1(ab\theta)/|j|^T\right) = 0,$$

where the summation is over those $a \in K^T$ satisfying $a \leq j$.

THEOREM 4.4. *Let $f(q)$ be a function from K^T to $\mathbf{A}^T$. If for each non-zero $h \in K^T$, the function $g_h(q) = f(q + h) - f(q)$ is uniformly distributed, then f is uniformly distributed.*

We first prove (following [2, p. 71, Lemma 3])

LEMMA 4.5. *Let u be a complex-valued function with domain K^T. Then for T-ideles Q and R we have*

$$\begin{aligned} m^T(R)^2 \Big|\sum\nolimits_{q\leq Q} u(q)\Big|^2 \leq m^T(Q \oplus R)\Big[m^T(R)\sum\nolimits_{q\leq Q} |u(q)|^2 \\ + 2\sum\nolimits_h A(R, h)\cdot\Big|\sum\nolimits_q u(q)u(q+h)\Big|\Big], \end{aligned} \tag{9}$$

where the summations are over h, $q \in K^T$; in the next to last sum, $0 \neq h \leq R \oplus R$; and in the last sum $q \leq Q$ and $q + h \leq Q$; $A(R, h)$ is the (finite) number of solutions to $r - s = h$ with $r, s \leq R$, $r, s \in K^T$; $m^T(R) = \#M^T(R)$.

Proof. We may assume that $u(q) = 0$ if $q \notin P(Q, 0)$. Throughout this proof all summation indices will run through specified subsets of K^T. Then

$$m^T(R)\sum\nolimits_{q\leq Q} u(q) = \sum\nolimits_{p\leq Q\oplus R}\sum\nolimits_{r\leq R} u(p - r).$$

By the Schwarz inequality,

$$m^T(R)^2\Big|\sum\nolimits_{q\leq Q} u(q)\Big|^2 \leq m^T(Q \oplus R)\sum u(p - r)\bar{u}(p - s), \tag{10}$$

where the last sum is over p, r, s satisfying $p \leq Q \oplus R$, $r \leq R$, $s \leq R$. Now any non-zero term of the form $|u(q)|^2$ appears in the last sum of (10) precisely $M^T(R)$ times, namely when $q = p - r = p - s$. For given q and h, terms of the form $u(q)\bar{u}(q + h)$ or $\bar{u}(q)u(q + h)$ can only be non-zero when both $q \leq Q$ and $q + h \leq Q$; then each such term occurs once for each solution of $q = p - r$, $q + h = p - s$; hence once for each solution of $r - s = h$ with $r \leq R$, $s \leq R$. Thus (10) is

$$\begin{aligned} m^T(R)^2\Big|\sum\nolimits_{q\leq Q} u(q)\Big|^2 \leq m^T(Q \oplus R)\Big[\sum\nolimits_{q\leq Q} |u(q)|^2 m^T(R) \\ + \sum\nolimits_h A(R, h)\sum\nolimits_q (u(q)\bar{u}(q + h) + \bar{u}(q)u(q + h))\Big], \end{aligned}$$

where the next to last sum is over non-zero $h \leq R \oplus R$ and the last sum is over q satisfying $q \leq Q$, $q + h \leq Q$. Taking absolute values in the last sum on the right completes the proof.

Proof of Theorem 4.4. Let b be a non-zero element of $\mathbf{D}^T$ and put $u(q) = \chi_1(bf(q))$. For each $h \in K^T$, $(f(q + h) - f(q))$ is uniformly distributed. Hence by Theorem 4.1

$$\sum_{q \leq Q} \chi_1(b(f(q + h) - f(q))) = \sum_{q \leq Q} u(q)\bar{u}(q + h) = O(|Q|^T)$$

as $Q \to \infty$.

Since $|Q \oplus h|^T/|Q|^T \to 1$ as $Q \to \infty$, we have

$$\sum_q u(q + h)\bar{u}(q) = O(|Q|^T),$$

where the sum is over q satisfying $q \leq Q$, $q + h \leq Q$. We now fix a T-adele R and apply Lemma 4.5. Since $|u(q)| = 1$ and $A(R, h) \leq M^T(R)$, we obtain

$$m^T(R)^2 \left| \sum_{q \leq Q} u(q) \right|^2 \leq m^T(Q \oplus R)m^T(R)[m^T(Q) + m^T(R \oplus R) \cdot o(|Q|^T)].$$

Dividing by $m^T(R)^2 m^T(Q)^2$, we obtain, as $Q \to \infty$,

$$\left| \sum_{q \leq Q} u_q \right|^2 / (|Q|^T)^2 \leq c/m^T(R) + o(1)$$

where c is a constant independent of R and Q. Since R is arbitrary,

$$\left| \sum_{q \leq Q} \chi_1(bf(q)) \right| = \left| \sum_{q \leq Q} u(q) \right| = o(|Q|^T)$$

as $Q \to \infty$. By Theorem 4.1, the function $f(q)$ is uniformly distributed.

THEOREM 4.6. *Suppose $f(x) = \sum_{k=0}^{s} a_k x^k$ is a polynomial with coefficients $a_k \in \mathbf{A}^T$. If at least one of the coefficients a_j, with $j > 0$, is irrational, then $f(q)$ is uniformly distributed.*

Proof. Induction on s, the degree of f. For $s = 1$, the theorem follows from Theorem 4.2. Suppose now that $s > 1$, and the theorem has been proved for $s - 1$. If α_s is irrational, then for each $h \in K^T$, $f(q + h) - f(q)$ is a polynomial of degree $s' < 1$ with leading coefficient irrational; hence by the induction hypothesis $f(q + h) - f(q)$ is u(iformly distributed for each $h \in K^T$. By Theorem 4.6, so is $f(q)$. If α_s is rational, then there exists $t > 0$ such that α_t is irrational and $\alpha_{t+1}, \alpha_{t+2}, \cdots, \alpha_s$ are rational. Let $n \in K^T$ be such that $n\alpha_{t+1}, \cdots, n\alpha_s$ are in K^T. It clearly suffices to show that the functions $f(nq + m)$ are uniformly distributed for each $m \in K^T$. But $f(nq + m)$ when written as a polynomial in q, differs only by an element of K^T from a polynomial of degree t whose leading coefficient is irrational. Hence by the inductive hypothesis $f(nq + m)$ is uniformly distributed.

5. Metrical theory

If a and b are real numbers, we say that $a < b \pmod 1$ if there exists $m \in \mathbf{Z}$ such that $|a - m| < b$. Let $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots$ be a sequence of positive real numbers. A classical theorem of Khintchine [2] asserts that the inequalities $n\theta - \alpha < \varepsilon_n \pmod 1$ for $n = 1, 2, 3, \cdots$, have infinitely many solutions for almost all pairs $(\theta, \alpha) \in \mathbf{R}^2$ if and only if $\sum_{n=1}^{\infty} \varepsilon_n$ diverges. Another theorem of Khintchine asserts that if the ε_n are monotically decreasing, then the inequalities $n\theta < \varepsilon_n \pmod 1$ have infinitely many solutions for almost all θ if and only if $\sum_{n=1}^{\infty} \varepsilon_n$ diverges. The purpose of this section is to prove the analogous theorems for T-adeles, (at least when $T \neq \Omega$); see Theorems 5.6, 5.7, 5.12.

If a and b are T-adeles, we say that $a < b \pmod{K^T}$ if there exists $m \in K^T$ such that $a - m < b$. Throughout this section, $c_1, c_2, c_3, \cdots$ will denote real positive constants, which in every case can be evaluated, but the exact values are irrelevant for our purposes. We note here that all sets of T-adeles referred to in this section are obviously measurable.

LEMMA 5.1. *Let i be a T-idele; let a be a T-adele satisfying $| a |^T < 2^{-N}$; and let $k \neq 0$ be an element of K^T. Put*

$$S = \{\theta \in \mathbf{A}^T : \theta \leq i \quad \textit{and} \quad k\theta \leq a \pmod{K^T}\}.$$

Then there exist positive real numbers d_1 and d_2 not depending on k or a such that

$$d_1 | a |^T \leq \mu^T(S) \leq d_2 | a |^T. \tag{1}$$

Proof. For each $q \in K^T$, put

$$S_q = \{\theta \in \mathbf{A}^T : \theta \leq i \quad \text{and} \quad k\theta - q \leq a\}.$$

We first show that the S_q are pairwise disjoint. In fact, suppose that $\theta \in S_q \cap S_{q'}$, with $q \neq q'$. Then $k\theta - q \leq a$ and $k\theta - q' \leq a$. Hence $| q - q' |_v \leq H_v | a |_v$ for all $v \in T$. Then $\prod_{v \in T} | q - q' |_v \leq 2^N | a |^T < 1$, hence $q - q' = 0$. Thus $\mu(S) = \sum_q \mu(S_q)$. If $\theta \in S_q$, then for each $v \in T$, we have $| \theta - q/k |_v \leq | a/k |_v$ and $| \theta |_v \leq | i |_v$. Hence

$$| q/k |_v \leq H_v \max(| i |_v, | a/k |_v) \tag{2}$$

Then S_q is empty if q does not satisfy (2) for all $v \in T$. Suppose q satisfies (2). For each $v \in T$, put

$$S_q^v = \{x \in K_v : | x |_v \leq | i |_v, | kx - q |_v \leq | a |_v\}.$$

Then $S_q = \prod_v S_q^v$ and $\mu^T(S_q) = \prod_v \mu_v(S_q^v)$. If v is p-adic, an easy computation shows that

$$\mu_v(S_q^v) = \min(| i |_v, | a/k |_v);$$

while if v is infinite, then

$$\mu_v(S_q^v) \leq 8 \min(| i |_v, | a/k |_v),$$

and if q satisfies

$$| q/k |_v \leq \max(| i |_v, | a/k |_v), \tag{3}$$

then

$$\mu_v(S_q^v) \geq \tfrac{1}{2} \min(| i |_v, | a/k |_v).$$

Combining these results, we see that if q satisfies (2) for all $v \in T$, then

$$\mu^T(S_q) \leq 8^N \prod_v \min(|i|_v, |a/k|_v);$$

if in addition q satisfies (3) for all $v \in \Omega_\infty$, then

$$\mu^T(S_q) \geq 2^{-N} \prod_v \min(|i|_v, |a/k|_v).$$

By Lemma 2.7, there exists real constants $c_1, c_2 > 0$ such that the number of $q \in K^T$ satisfying (2) for all $v \in T$ is less than

$$c_1 \prod_v \max(|ik|_v, |a|_v),$$

while the number of $q \in K^T$ satisfying (3) for all $v \in T$ is greater than

$$c_2 \prod_v \max(|ik|_v, |a|_v).$$

Thus

$$\mu^T(S) \leq 8^N c_1 \prod_v \min(|i|_v, |a/k|_v) \max(|ik|_v, |a|_v) = 8^N c_1 |i|^T |a|^T.$$

Similarly $\mu^T(S) \geq 2^{-N} c_2 |i|^T |a|^T$.

LEMMA 5.2. *Let a be a T-adele with $|a|^T < 2^{-N}$, and let $k \neq 0$ be in K^T. Put $S = \{\theta \in D^T : k\theta \leq a (\operatorname{mod} K^T)\}$. Then there exist positive constants c_1 and c_2, not depending on a or k such that $c_1 |a|^T \leq \mu^T(S) \leq c_2 |a|^T$.*

Proof. Clearly $\mu^T(S)$ is independent of the choice of fundamental domain D^T. Since there exist sets of the form $P(i, 0)$ which include a fundamental domain and others which are included in a fundamental domain, this lemma is an immediate consequence of Lemma 5.1.

For $a, x \in \mathbf{A}^T$, put $\delta_a(x) = 1$ if $x \leq a \ (\operatorname{mod} K^T)$, otherwise $\delta_a(x) = 0$. Then if $k \in K^T$, $k \neq 0$, and if $|a|^T < 2^{-N}$, we have by Lemma 5.2

$$c_1 |a|^T \leq \int_{D^T} \delta_a(kx)\, d\mu^T(x) \leq c_2 |a|^T.$$

Since any translate $D^T - \alpha$, $\alpha \in \mathbf{A}^T$, of a fundamental domain is a fundamental domain, we obtain:

LEMMA 5.3. *If $a, \alpha \in \mathbf{A}^T$, $k \in K^T$, $k \neq 0$, and $|a|^T < 2^{-N}$, then*

$$c_1 |a|^T \leq \int_{D^T} \delta_a(kx - \alpha)\, d\mu^T(x) \leq c_2 |a|^T.$$

LEMMA 5.4. *There exist real positive constants c_3, c_4 such that if $a, b \in \mathbf{A}^T$, $p, q \in K^T$ and $p \neq q$, then*

$$c_3 |ab|^T \leq \iint \delta_a(p\theta + \alpha)\delta_b(q\theta + \alpha)\, d\mu^T(\theta)\, d\mu^T(\alpha) \leq c_4 |ab|^T,$$

while if $p = q$, then

$$c_3 |a|^T \leq \iint \delta_a(p\theta + \alpha)\, d\mu^T(\theta)\, d\mu^T(\alpha) \leq c_4 |a|^T,$$

where both integrals are over $(\theta, \alpha) \in D^T \times D^T$.

Proof. The inequalities for the second integral follow from Lemma 5.3 by integrating first with respect to θ. By substituting $s = q - p$, $\alpha' = \alpha + p\theta$, the first integral is transformed into

$$\iint \delta_a(\alpha')\, \delta_b(s\theta + \alpha')\, d\mu^T(\theta)\, d\mu^T(\alpha'),$$

where the integral is still over $D^T \times D^T$. Integrating first with respect to θ, then with respect to α', by applying Lemma 5.3 completes the proof.

LEMMA 5.5. *Suppose there exists a completion* $v_0 \notin T$ (*i.e.* $T \neq \Omega$), *and that* E *is a measurable subset of* $(\mathbf{A}^T)^m$ *with* $(\mu^T)^m(E) > 0$, *where*

$$(\mu^T)^m = \mu^T \times \mu^T \times \cdots \times \mu^T,$$

he product measure on $(\mathbf{A}^T)^m$. *Then almost all points of* $(\mathbf{A}^T)^m$ *are contained in* $K^T \cdot E + (K^T)^m$.

Proof. Sets of the form $(P(i, b))^m$, $i \in \mathbf{I}^T$, $b \in \mathbf{A}^T$, form a neighborhood basis for $(\mathbf{A}^T)^m$. By the density theorem for Haar measures on locally compact groups [3], there exists, for any real positive δ, a set of the form $(P(i, b))^m$ such that $|\, i\,|^T \leq 1$ and

$$(\mu^T)^m[(P(i, b))^m \cap E] > (1 - \delta)(\mu^T)^m(P(i, b))^m.$$

Since $T \neq \Omega$, there exists $q \in K^T$ with $|\, q\,|^T = \sigma > 1$. Let t be the least non-negative integer such that $\sigma^t\, |\, i\,|^T > c$, where c is the constant of Lemma 2.6. Call $F_t = (P(iq^t, bq^t))^m$. Then

$$(\mu^T)^m[F_t \cap q^t E] > (1 - \delta)(\mu^T)^m F_t\,.$$

But by Lemma 2.6, F_t contains a fundamental domain G of $(\mathbf{A}^T)^m/(K^T)^m$. Then

$$(\mu^T)^m(q^t E \cap G) > (\mu^T)^m G - \delta(\mu^T)^m F_t\,.$$

Now, $(\mu^T)^m F_t$ is bounded, (since $\sigma^t\, |\, i\,|^T$ is bounded) say by M. Translating $q^t E$ by $(K^T)^m$ we see that

$$(\mu^T)^m((q^T E + (K^T)^m) \cap (D^T)^m) > (\mu^T)^m((D^T)^m) - \delta M.$$

Since δ is arbitrary, the proof is complete. We do not know if Lemma 5.5 remains true when the hypothesis $T \neq \Omega$ is omitted.

THEOREM 5.6. *Suppose, for each* $q \in K^T$, *we are given a* T-*adele* ε_q *such that* $\sum_{q \in K^T} |\, \varepsilon_q\,|^T < \infty$. *Then, for each* $\alpha \in \mathbf{A}^T$, *and for almost all* $\theta \in \mathbf{A}^T$, *there exist only finitely many solutions* $q \in K^T$ *to*

$$q\theta - \alpha \leq \varepsilon_q \pmod{K^T}. \tag{4}$$

Proof. Without loss of generality, assume that $|\, \varepsilon_q\,|^T < 2^{-N}$ for all $q \in K^T$.

By Lemma 5.2, for fixed q and α, the set of $\theta \in D^T$ satisfying (4) has measure at most $c_2 \mid \varepsilon_q \mid^T$. Assume now that α is fixed. Then if L is a finite subset of K^T, the set of $\theta \in D^T$ for which (4) has a solution $q \notin L$ has measure at most $c_2 \sum_{q \notin L} \mid \varepsilon_q \mid^T$. Thus, the set of $\theta \in \mathbf{D}^T$ for which (4) has infinitely many solutions has measure at most $\inf_L c_2 \sum_{q \notin L} \mid \varepsilon_q \mid^T = 0$.

THEOREM 5.7. *Suppose that $T \neq \Omega$, and that for $q \in K^T$, we are given a T-adele ε_q. Then there exists, for almost all $(\theta, \alpha) \in \mathbf{A}^T \times \mathbf{A}^T$, infinitely many solutions to $q\theta - \alpha \leq \varepsilon_q \pmod{K^T}$ if and only if $\sum_q \mid \varepsilon_q \mid^T = \infty$.*

Proof. If $\sum_q \mid \varepsilon_q \mid^T < \infty$, the theorem follows immediately from Theorem 5.6. Hence assume that $\sum_q \mid \varepsilon_q \mid^T = \infty$. For any T-idele h, let $\Delta_h(\theta, \alpha)$ be the number of $q \in K^T$, $q \neq 0$, satisfying $q\theta - \alpha \leq \varepsilon_q$ and $q \leq h$. Put

$$M_1(h) = \iint \Delta_h(\theta, \alpha)\, d\mu^T(\theta)\, d\mu^T(\alpha)$$

$$M_2(h)^2 = \iint \Delta_h(\theta, \alpha)^2\, d\mu^T(\theta)\, d\mu^T(\alpha),$$

where both integrals are over $D^T \times D^T$. For $x \in \mathbf{A}^T$, put $\delta_q(x) = 1$ if $x \leq \varepsilon_q$ otherwise $\delta_q(x) = 0$. Then $\Delta_h(\theta, \alpha) = \sum_{q \leq h} \delta_q(q\theta - \alpha)$. Hence

$$M_1(h) = \sum_{q \leq h} \iint \delta_q(q\theta - \alpha)\, d\mu^T(\theta)\, d\mu^T(\alpha).$$

By Lemma 5.4, $M_1(h) \geq c_3 \sum_{q \leq h} \mid \varepsilon_q \mid^T$. Similarly,

$$M_2(h)^2 = \sum_{p \leq h} \sum_{q \leq h} \iint \delta_q(q\theta - \alpha)\, \delta_p(p\theta - \alpha)\, d\mu^T(\theta)\, d\mu^T(\alpha),$$

hence, by Lemma 5.4,

$$M_2(h)^2 \leq c_4 (\textstyle\sum_{q \leq h} \mid \varepsilon_q \mid^T)^2 + c_4 \sum_{q \leq h} \mid \varepsilon_q \mid^T.$$

Since $\sum_q \mid \varepsilon_q \mid^T = \infty$, there exists a real, positive constant c_5, such that $M_1(h) \geq c_5 M_2(h)$ for all large h (say, for all $h \geq 1$). By a lemma of Paley-Zygmund [2, p. 122], the set of θ for which $\Delta_h(\theta, \alpha) \geq (c_5/2) M_1(h)$ has measure at least $(c_5^2/4)(\mu^T(D^T))^2$. Since $\Delta_h(\theta, \alpha)$ increases as h increases (in the sense of $\leq$) there is a set $S \subset D^T \times D^T$ of positive measure for which $\Delta_h(\theta, \alpha) \to \infty$ as $h \to \infty$, whenever $(\theta, \alpha) \in S$. Now let τ_q be a T-idele for each $q \in K^T$, such that $\sum_q \mid \varepsilon_q \tau_q \mid^T = \infty$ and such that for every T-idele i, there exists a T-idele j such that if $q \notin P(j, 0)$, then $\tau_q \leq i$ (roughly speaking $\tau_q \to 0$ as $q \to \infty$). It is easy to construct such τ_q. By what we have already proved, there exists a set $S \subset D^T \times D^T$ of positive measure such that the inequalities $q\theta - \alpha \leq \tau_q \varepsilon_q \pmod{K^T}$ have infinitely many solutions $q \in K^T$ whenever $(\theta, \alpha) \in S$. By Lemma 5.5, almost all points of $(\mathbf{A}^T)^2$ can be written in the form $(x\theta + y, x\alpha + z)$ where $x, y, z \in K^T$ and $(\theta, \alpha) \in S$. For any such point there are infinitely many $q \in K^T$ such that

$$q\theta - \alpha \leq \tau_q \varepsilon_q \pmod{K^T};$$

hence

$$q(x\theta + y) - (x\alpha + z) \leq \varepsilon_q \tau_q x \pmod{K^T}.$$

Since $\tau_q x \leq 1$ with only finitely many exceptions (by the choice of τ_q), the inequality

$$q(x\theta + y) - (x\alpha + z) \leq \varepsilon_q \pmod{K^T}$$

has infinitely many solutions.

Let $q_1, q_2, q_3, \cdots$ be a sequence of non-zero elements ϵK^T. We shall call such a sequence *regular* if there exists a T-idele j_1 and finite subsets of K, $V_1, V_2, V_3, \cdots$ with cardinalities $v_1, v_2, v_3, \cdots$ satisfying the following:

R1. $q_n V_n \subset K^T$, for $n = 1, 2, 3, \cdots$.

R2. $V_n \subset P(j_1, 0)$, for $n = 1, 2, 3, \cdots$.

R3. The V_n are pairwise disjoint.

R4. $(1/q_n) \leq j_1$ for all n for which $v_n \neq 0$.

R5. There exists a real positive constant c_6 such that

$$\textstyle\sum_{n=1}^{m} v_n/|\, q_n \,|^T \geq c_6\, m,$$

for all sufficiently large positive integers m.

If $a, b \epsilon \mathbf{Z}$, then as usual (a, b) denotes the greatest common divisor of a and b.

LEMMA 5.8. *Suppose T is finite and k is a fixed positive integer; then the sequence $q_n = n^k$ is regular.*

Proof. Let P be the product of those primes p for which $|\, p \,|_v < 1$ for some $v \epsilon T$. Let $\xi_1, \xi_2, \cdots, \xi_N$ be an integral basis for the ring of algebraic integers of K. If $(n, P) = 1$, let V_n be the set of those $x \epsilon K$ which can be written in the form $x = (\sum_{i=1}^{N} x_i \xi_i)/n^k$ where $x_i \epsilon \mathbf{Z}$ $(x_i, P) = 1$ and $1 \leq x_i \leq n^k$. If $(n, P) \neq 1$, let V_n be the empty set. Then, clearly R1–4 are satisfied with $j_1 = 1$. We now prove R5. Put $\varphi_P(n) = \varphi(n)$ if $(n, P) = 1$, otherwise put $\varphi_P(n) = 0$ ($\varphi(n)$ is the Euler Totient). As $|\, n^k \,|^T \leq n^{kN}$, we have

$$\begin{aligned}(1/m) \textstyle\sum_{n=1}^{m} v_n/|\, q_n \,|^T &\geq (1/m) \textstyle\sum_{n=1}^{m} (\varphi_P(n)/n^k)^N \\ &\geq ((1/m) \textstyle\sum_{n=1}^{m} \varphi_P(n^k)/n^k)^N,\end{aligned}$$

by Holder's inequality. Since $\varphi_P(n^k) = n^{k-1}\varphi_P(n)$, it is enough to show the existence of a positive constant c_7 such that

$$s_m = (1/m) \textstyle\sum_{n=1}^{m} \varphi_P(n)/n \geq c_7 .$$

But $s_m \geq t_m = (1/m) \sum_{n=1}^{\prime m} \varphi(n)/n$ where $\sum'$ indicates summation over n congruent to 1 modulo P. Now $\varphi(n) = n \sum_{d|n} \mu(d)/d$ where μ is the Moebius function. As there are $m/(Pd) + O(1)$ multiples of d between 1 and m which are congruent to 1 modulo P, we have

$$\begin{aligned} t_m &= (1/m) \textstyle\sum_{d=1}^{m} (m/(Pd) + O(1))\mu(d)/d \\ &= (1/P) \textstyle\sum_{d=1}^{m} \mu(d)/d^2 + O((1/m) \textstyle\sum_{d=1}^{m} 1/d) \\ &\to 6/(P\pi^2) \quad \text{as} \quad m \to \infty \qquad\qquad [4, \text{p. } 250, \text{Th. } 287]. \end{aligned}$$

LEMMA 5.9. *Suppose $k \in \mathbf{Z}$, $k \geq 2$ and $|\, k\,|_v \geq 1$ for all $v \in T$. Then $q_n = k^n$ is a regular sequence.*

Proof. Let V_n be the set of those $x \in K$ which can be written in the form $x = (\sum_{i=1}^{N} x_i\, \xi_i)/k^n$, where each x_i is an integer relatively prime to k and $1 \leq x_i \leq k^n$. As in Lemma 5.8, R1–4 are easily verified with $j_1 = 1$. Now $v_n = (\varphi(k^n))^N$ and $|\, k^n\,|^T = k^{nN}$, since $|\, k\,|_v = 1$ at all p-adic $v \in T$. Hence,

$$\begin{aligned} (1/m) \textstyle\sum_{n=1}^{m} v_n/|\, q_n\,|^T &= (1/m) \textstyle\sum_{n=1}^{m} (\varphi(k^n)/k^n)^N \\ &= (1/m) \textstyle\sum_{n=1}^{m} (\varphi(k)/k)^N \\ &= (\varphi(k)/k)^N > 0, \end{aligned}$$

since $\varphi(k^n) = k^{n-1}\varphi(k)$; hence R5 is satisfied.

REMARK 5.10. *We list some other regular sequences, leaving the proof to the reader. We make the blanket restrictions that any element q_n in any of the following sequences shall satisfy $|\, q_n\,|_v = 1$ for all p-adic $v \in T$, and that the elements in the sequence shall be distinct:*

(a) *Any infinite sequence of distinct primes $p \in \mathbf{Z}$.*

(b) *Let r be a fixed positive integer. Any sequence, every element of which has at most r distinct prime factors (this generalizes Lemma 5.9).*

(c) *For a fixed positive integer k, the subsequence of $\{n^k\}$ consisting of those n^k where n lies in a fixed arithmetic progression (this generalizes Lemma 5.8).*

REMARK 5.11. *Suppose q_n is a sequence of elements of K^T and has a subsequence q_{n_i} which is regular. If there is a real constant $k > 0$ such that $n_i < ki$ and the n_i are increasing then the sequence q_n is regular. (One chooses V_n to be the empty set if n is not of the form n_i.)*

THEOREM 5.12. *Suppose $T \neq \Omega$, that $q_1, q_2, q_3, \cdots$ is a regular sequence of elements of K^T and that $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots$ is a sequence of T-ideles for which $|\, \varepsilon_1\,|^T \geq |\, \varepsilon_2\,|^T \geq |\, \varepsilon_3\,|^T \geq \cdots$, and for which there exists a T-idele j_2 such that $\varepsilon_i \leq j_2$ for all i. Then there exist, for almost all $\theta \in \mathbf{A}^T$, infinitely many solutions to $q_n\, \theta \leq \varepsilon_n \pmod{K^T}$ if and only if $\sum_n |\, \varepsilon_n\,|^T = \infty$.*

Proof. If $\sum_n |\, \varepsilon_n\,|^T < \infty$, then the theorem follows from Theorem 5.6. Assume then that $\sum |\, \varepsilon_n\,|^T = \infty$, and without loss of generality that $|\, \varepsilon_n\,|^T < 2^{-N}$ for all n. We may also assume that R5 holds for all $m \geq 1$. Since q_n is regular, we have a sequence of sets $V_1, V_2, V_3, \cdots$ with cardinalities $v_1, v_2, v_3, \cdots$ satisfying R1–5. Put $j = j_1 \oplus (j_2/j_1)$ (j_1 is defined in R1 and R4). Then if $\theta \in \mathbf{A}^T$ satisfies $\theta - p \leq \varepsilon_n/q_n$, where $p \in V_n$, we have $\theta \leq j$. For $\theta \in \mathbf{A}^T$, put $\beta_n(\theta) = 1$ if $\theta \leq \varepsilon_n/q_n$, otherwise $\beta_n(\theta) = 0$. Put $\gamma_n(\theta) =$

$\sum_x \beta_n(\theta - x)$ where the sum is over $x \in V_n$; $\gamma_n(\theta)$ is the number of $x \in V_n$ satisfying $\theta - x \leq \varepsilon_n/q_n$; if $x' \in V_n$ also satisfies this inequality, then $q_n x - q_n x' \leq \varepsilon_n \oplus \varepsilon_n$; hence,

$$| q_n x - q_n x' |^T < 2^N | \varepsilon_N |^T < 1 \quad \text{and} \quad x = x',$$

since $q_n x, q_n x' \in K^T$. Thus $\gamma_n(\theta)$ is 0 or 1. For positive $h \in \mathbf{Z}$, put $\Gamma_h(\theta) = \sum_{n=1}^h \gamma_n(\theta)$ and

$$M_1(h) = \int \Gamma_h(\theta)\, d\mu^T(\theta), \qquad M_2(h)^2 = \int \Gamma_h(\theta)^2\, d\mu^T(\theta),$$

where the integrals are over $\theta \in \mathbf{A}^T$, $\theta \leq j$. Note that $\Gamma_h(\theta) = 0$ outside of the range of integration.

LEMMA 5.13. *There exists real $c_8 > 0$ such that $M_1(h) \geq c_8 \sum_{n=1}^h | \varepsilon_n |^T$.*

Proof. We have

$$\int_{\theta \leq j} \gamma_n(\theta)\, d\mu^T(\theta) = \sum_{x \in V_n} \int_{\theta \leq j} \beta_n(\theta - x)\, d\mu^T(\theta)$$
$$= v_n C_K | \varepsilon_n/q_n |^T.$$

Hence,

$$\begin{aligned} M_1(h) &= C_K \textstyle\sum_{n=1}^h v_n | \varepsilon_n/q_n |^T \\ &= C_K[\textstyle\sum_{n=1}^{h-1} (\sum_{r=1}^n v_r/| r |^T)(| \varepsilon_n |^T - | \varepsilon_{n+1} |^T) \\ &\quad + (\textstyle\sum_{r=1}^h v_r/| r |^T)| \varepsilon_n |^T] \\ &\geq c_8[\textstyle\sum_{n=1}^{h-1} n(| \varepsilon_n |^T - | \varepsilon_{n+1} |^T) + h | \varepsilon_n |^T] \\ &= c_8 \textstyle\sum_{n=1}^n | \varepsilon_n |^T \end{aligned}$$

by R5.

LEMMA 5.14. *There exists $c_9 > 0$ such that if $m \neq n$, then*

$$I_{mn} = \int_{\theta \leq j} \gamma_m(\theta)\gamma_n(\theta)\, d\mu^T(\theta) \leq c_9 | \varepsilon_m |^T | \varepsilon_n |^T.$$

Proof. Clearly, $I_{mn} = \sum_{s \in V_m} \sum_{t \in V_n} \mu^T(W_{st})$, where

$$W_{st} = \{\theta \in \mathbf{A}^T : \theta - s \leq \varepsilon_m/q_m , \theta - t \leq \varepsilon_n/q_n\}.$$

Then,

$$\mu^T(W_{st}) \leq c_{10} \textstyle\prod_{v \in T} \min (| \varepsilon_m/q_m |_v , | \varepsilon_n/q_n |_v). \tag{5}$$

We now estimate how many W_{st} are non-empty. Note that $s \neq t$ since by R3, V_m and V_n are disjoint. Put $s' = q_m s$, $t' = q_n t$ so that $s', t' \in K^T$ and $s' \leq jq_m$, $t' \leq jq_n$ by R1 and R2. Put $a = q_m q_n(s - t) = q_n s' - q_m t' \neq 0$; then $a \in K^T$, moreover

$$| a |_v \leq \max (| q_m |_v , | q_n |_v)$$

if $v \notin T$; while if $v \in T$, then, since $| s - t |_v \leq H_v \max (| \varepsilon_m/q_m |_v , | \varepsilon_n/q_n |_v)$ we have

$$|a|_v \leq H_v \max(|q_n \varepsilon_m|_v, |q_m \varepsilon_n|_v).$$

Thus by Lemma 2.8, the total number of distinct a which may occur is less than

$$c_{11} \prod_{v \in T} \max(|q_n \varepsilon_m|_v, |q_m \varepsilon_n|_v) \prod_{v \notin T} \max(|q_m|_v, |q_n|_v). \tag{6}$$

For a given value of a, we now determine an upper bound for the number of pairs s', t' which satisfy $a = q_n s' - q_m t'$. Once s' is known, t' is determined, hence we need only estimate the number of s' for which there exists t' satisfying $q_n s' - q_m t' = a$. Suppose that $q_n s'' - q_m t'' = a$. Then $q_n(s' - s'') = q_m(t' - t'')$; hence

$$|s' - s''|_v \leq \min(1, |q_m/q_n|_v) \qquad \text{if} \quad v \notin T,$$

while

$$|s' - s''|_v \leq H_v |jq_m|_v \qquad \text{if} \quad v \in T.$$

Thus by Lemma 2.9, the total number of pairs $s' - s''$ which can occur is less than

$$\begin{aligned} &c_{12} |q_m|^T \prod_{v \notin T} (\min(|q_m|_v, |q_n|_v)/|q_n|_v) \\ &\qquad = c_{12} |q_m|^T |q_n|^T \prod_{v \notin T} \min(|q_m|_v, |q_n|_v), \end{aligned} \tag{7}$$

by Artin's product formula (see Section 1). The total number of possible values of s' is less than

$$\begin{aligned} &1 + c_{12} |q_m|^T |q_n|^T \prod_{v \notin T} \min(|q_m|_v, |q_n|_v) \\ &\qquad \leq c_{13} |q_m|^T |q_n|^T \prod_{v \notin T} \min(|q_m|_v, |q_n|_v) \end{aligned} \tag{8}$$

since the product in (7) is bounded away from 0. Then I_{mn} is less than the product of (5), (6), and (8). Hence

$$\begin{aligned} I_{mn} &\leq c_{14} \prod_{v \in T} \min(|\varepsilon_m/q_m|_v, |\varepsilon_n/q_n|_v) \\ &\qquad \cdot \prod_{v \in T} \max(|q_n \varepsilon_m|_v, |q_m \varepsilon_n|_v) \prod_{v \notin T} \max(|q_m|_v, |q_n|_v) \\ &\qquad \cdot |q_m q_n|^T \prod_{v \notin T} \min(|q_m|_v, |q_n|_v) \\ &= c_{14} \prod_{v \in T} [\min(|\varepsilon_m/q_m|_v, |\varepsilon_n/q_n|_v) \cdot \max(|q_n \varepsilon_m|_v, |q_m \varepsilon_n|_v)] \\ &\qquad \cdot |q_m q_n|^T \prod_{v \notin T} [\max(|q_m|_v, |q_n|_v) \min(|q_m|_v, |q_n|_v)] \\ &= c_{14} \prod_{v \in T} (|\varepsilon_m/q_m|_v \cdot |\varepsilon_n/q_n|_v \cdot |q_m|_v \cdot |q_n|_v) \\ &\qquad \cdot |q_m q_n|^T \prod_{v \notin T} |q_m q_n|_v \\ &= c_{14} |\varepsilon_m|^T |\varepsilon_n|^T, \end{aligned}$$

by Artin's product formula.

LEMMA 5.15. *We have* $M_2(h) \leq c_{15} M_1(h)$.

Proof.

$$M_2(h)^2 = \int_{\theta \le j} \Gamma_h(\theta)^2\, d\mu^T(\theta)$$
$$= \sum_{\substack{m,n=1\\ m\neq n}}^{h} \int_{\theta\le j} \gamma_m(\theta)\gamma_n(\theta)\, d\mu^T(\theta) + \sum_{m=1}^{h} \int_{\theta \le j} \gamma_m(\theta)\, d\mu^T(\theta),$$

since $\gamma_m(\theta) = 0$ or 1. Then by Lemma 5.14,

$$M_2(h)^2 \le c_{14} \textstyle\sum_{m,n=1}^{h} |\, \varepsilon_m |^T |\, \varepsilon_n |^T + M_1(h)$$

or

$$M_2(h)^2 \le c_{15}\, M_1(h)^2 + M_1(h) \le c_{16}^2\, M_1(h)^2.$$

LEMMA 5.16. *There exists a set of positive measure in* $\mathbf{A}^T$ *on which* $\Gamma_h(\theta) \to \infty$, *as* $h \to \infty$.

Proof. By Lemma 5.15, $M_1(h) \ge c_{16}\, M_2(h)$, hence by the Paley-Zygmund Lemma [2, p. 122], the set of θ where $\Gamma_h(\theta) \ge (c_{16}/2)M_1(h)$ has measure $\ge (c_{16}^2/4)\mu^T P(j, 0)$.

Proof of Theorem 5.12 (Concluded). Let τ_n, $n = 1, 2, 3, \cdots$ be a sequence of T-ideles such that $\tau_1 \ge \tau_2 \ge \tau_3 \ge \cdots$, that $\sum_{n=1}^{\infty} |\, \varepsilon_n\, \tau_n |^T = \infty$, and that for every T-idele i there exists an integer n_0 such that $\tau_n \le i$ if $n \ge n_0$. It is easy to construct such sequences.

By Lemma 5.16 applied to the $\varepsilon_n\, \tau_n$ instead of the ε_n, there exists a set of positive measure S such that the inequalities $q_n\, \theta \le \varepsilon_n\, \tau_n$ have infinitely many solutions for $\theta \in S$. By Lemma 5.5 almost all points of $\mathbf{A}^T$ can be written in the form $x\theta + y$, with $\theta \in S$, x, $y \in K^T$. For any such point there are infinitely many solutions to $q_n\, \theta \le \varepsilon_n\, \tau_n$, hence to $q_n(x\theta + y) \le \varepsilon_n\, \tau_n\, x (\bmod K^T)$.

For all large n, we have $\tau_n\, x \le 1$, hence there are infinitely many solutions to $q_n(x\theta + y) \le \varepsilon_n (\bmod K^T)$.

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