TWIST-SPINNING SPHERES IN SPHERES

BY

W.-C. HSIANG AND B. J. SANDERSON[1]

I. Introduction

Let $\theta^{m,n}$, $n > 4$, denote the group of h-cobordism classes of pairs of spheres (S^m, Σ^n), where S^m denotes an m-sphere with its usual structure and Σ^n denotes an embedded n-sphere which may have an exotic structure, [2], [9].

Our aim is to introduce an operation, which will be called *twist-spinning*;

$$\phi : \theta^{m,n} \times \pi_l(SO(n) \times SO(m-n)) \rightarrow \theta^{m+l,n+l}.$$

When $m = n + 2$, the operation is twist-spinning as defined by Artin–Zeeman [1], [12], except that we have introduced tangential twisting by elements of $\pi_l(SO(n))$. The operation restricted to the embedded sphere Σ^n of the pair (S^m, Σ^n) is equivalent to a pairing of Milnor–Munkres [5], [6] (also Novikov [7]), except that the group $\pi_l(SO(n-1))$ has been replaced here by $\pi_l(SO(n))$. Another operation may be defined by replacing $\theta^{m,n}$ by $I^{m,n}$ the group of regular homotopy classes of immersions of S^n in S^m.

In §2, the operation is described and defined. In §3 it is related to a relative version of the Milnor–Munkres–Novikov pairing;

$$\pi_0(\text{Diff}_C(R^{m-1}, R^{n-1})) \otimes \pi_l(SO(n-1) \times SO(m-n)) \rightarrow \pi_0(\text{Diff}_C(R^{m+l-1}, R^{n+l-1})).$$

The resulting operation on normal bundles is investigated in §4 and found to be related to the Whitehead product pairing,

$$\pi_n BSO(m-n) \otimes \pi_{l+1} BSO(m-n) \rightarrow \pi_{n+l} BSO(m-n).$$

We are grateful to E. C. Zeeman for sending us a preprint of [12].

II. Twist-spinning

Let (S^m, Σ^n) we a pair of spheres representing an element of $\theta^{m,n}$. Let D^m_+, D^m_- denote the upper and lower hemispheres of S^m respectively. Then $S^m = D^m_+ \cup D^m_-$. Now the pair (S^m, Σ^n) is diffeomorphic to

$$(D^m_+ \cup D^m_-, D^n_+ \cup \Sigma^n - \text{Int}\, D^n_+)$$

where D^n_+ is a disc embedded in Σ^n and the inclusion $D^n_+ \subset D^m_+$ is assumed to be standard; further we may suppose the inclusion $\Sigma^n - \text{Int}\, D^n_+ \subset D^m_-$ coincides with the standard inclusion $D^n_- \subset D^m_-$ near the boundary of $\Sigma^n - \text{Int}\, D^n_+$.

Received April 27, 1964.

[1] During the preparation of this paper, the authors were partially supported by a National Science Foundation Grant.

Now consider $D^m_-(x_0)$ as the disc (S^{m-1}, t, x_0) in the join sphere

$$S^{m+l} = S^{m-1} \circ S^l = S^{m-1} \times D^{l+1} \cup D^m \times S^l$$

where x_0 is a point of S^l (Fig. 1).

The standard sphere $S^{n-1} = \partial(\Sigma^n - \operatorname{Int} D^n_+)$ bounds the standard disc D^n_- in D^m_-. Let D^{m-n} be the disc normal to D^n_- in D^m_-. As the point x moves along S^l, the disc $D^m_-(x) = D^n_- \times D^{m-n}$ is twisted by a representative of an element $[\gamma] = (\alpha, \beta)$ of

$$\pi_l(SO(n) \times SO(m-n)) = \pi_l(SO(n)) \oplus \pi_l(SO(m-n)).$$

The trace of $\Sigma^n - \operatorname{Int} D^n_+$ is an embedded homotopy sphere Σ^{n+l} in S^{m+l}. It is not difficult to see that the resulting operation, denoted by ϕ, is well defined.

We now give a more precise description of twist-spinning. This time we begin with an immersion $f : S^n \to S^m$, $m > n + 1$.

Let $S^n = D^n_+ \cup_g D^n_-$ and $S^m = D^m_+ \cup_h D^m_-$ and now suppose

$$f : D^n_+ \cup D^n_- \to D^m_+ \cup D^m_-$$

is standard on D^n_+ and on a neighborhood of the boundary of D^n_-. Let

$$S^{n+l} = S^{n-1} \times D^{l+1} \cup_1 D^n \times S^l$$
$$S^{m+l} = S^{m-1} \times D^{l+1} \cup_1 D^m \times S^l$$

be the standard decompositions of S^{n+l} and S^{m+l} respectively. For any element $[\gamma] = (\alpha, \beta)$ in $\pi_l(SO(n) \times SO(m-n))$, an immersion

$$\phi(f, \gamma) : S^{n+l} \to S^{m+l}$$

is defined by

$$\begin{array}{ccc} S^{n+l} & \xrightarrow{\quad\phi(f,\gamma)\quad} & S^{m+l} \\ = S^{n-1} \times D^{l+1} \cup_1 D^n \times S^l & & = S^{m-1} \times D^{l+1} \cup_1 D^m \times S^l \\ \Big\downarrow 1 \cup (g' \times 1)^{-1} \circ F^{-1} & & 1 \cup (h' \times 1)^{-1} \circ F^{-1} \Big\downarrow \\ \partial D^n_- \times D^{l+1} \cup_{F \circ (g \times 1)} D^n_- \times S^l & \xrightarrow[\text{incl} \cup f \times 1]{} & \partial D^m_- \times D^{l+1} \cup_{F \circ (h+1)} D^m_- \times S^l \end{array}$$

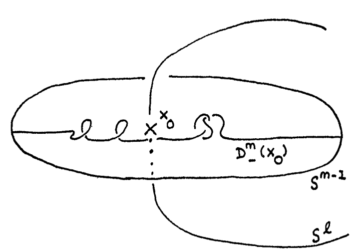

FIGURE 1

where h', g' are extensions of h, g respectively. The vertical maps are diffeomorphisms and

$$F : D^m_- \times S^l \to D^m_- \times S^l$$

is given by $F(x, y) = (\gamma(y)x, y)$. It is clear that the twist-spins of f, and an immersion f' regularly homotopic to f will be regularly homotopic since assuming f' is standard on D^n_+ and on a neighborhood of ∂D^n_- we may suppose the regular homotopy takes place in the interior of D^m_- since $m > n + 1$. It is also readily seen that the regular homotopy class of the twist-spun immersion is independent of the choice of representative for $[\gamma] = (\alpha, \beta)$. The corresponding description of twist-spinning h-cobordism classes of pairs (S^m, Σ^n) is simpler, since we do not need to worry about maps but only sphere pairs. As before suppose that the pair (S^m, Σ^n) is decomposed into

$$(D^m_+ \cup_h D^m_-, D^n_+ \cup_h (\Sigma^n - \operatorname{Int} D^n_+))$$

where h is a diffeomorphism of $(\partial D^m_-, \partial(\Sigma^n - \operatorname{Int} D^n_+))$ onto $(\partial D^m_+, \partial D^n_+)$. Then the twist-spin of the pair is

$$(\partial D^m_- \times D^{l+1} \cup_{F\circ(h\times 1)} D^m_- \times S^l,$$
$$\partial(\Sigma^n - \operatorname{Int} D^n_+) \times D^{l+1} \cup_{F\circ(h\times 1)} (\Sigma^n - \operatorname{Int} D^n_+) \times S^l)$$

which is the result of performing relative surgery on the pair

$$(S^m \times S^l, \Sigma^n \times S^l).$$

Now $\partial D^n_- \times D^{l+1} \cup_{F\circ(h\times 1)} D^m_- \times S^l$ is diffeomorphic to S^{m+l} since h extends to a diffeomorphism of D^m_- onto itself and F extends to $D^m_- \times S^l$. The submanifold is homeomorphic to a sphere since $\Sigma^n - \operatorname{Int} D^n_+$ is homeomorphic to a disc [9] and so $h \mid \partial D^n_+$ extends to a homeomorphism of $\Sigma^n - \operatorname{Int} D^n_+$ onto itself and F extends over $(\Sigma^n - \operatorname{Int} D^n_+)S^l$ similarly.

III. Bilinearity of twist-spinning and the operation of Milnor–Munkres

Let ϕ be the twist-spinning operation and let ϕ_{n-1} be defined by restricting the range of ϕ to $\theta^{m,n}\ \pi^{n-1}_l$, where π^{n-1}_l denotes the image of

$$\pi_l(SO(n-1) \times SO(m-n))$$

in

$$\pi_l(SO(n) \times SO(m-n)).$$

THEOREM 1. *The operation ϕ is linear on the second factor and ϕ_{n-1} is bilinear.*

Proof. Let (S^m, Σ^n) be a representative of the element σ in $\theta^{m,n}$ and let

$$\gamma_i : S^l \to SO(n) \times SO(m-n)$$

be a representative of the element $[\gamma_i] = (\alpha_i, \beta_i)$ $i = 1, 2$ in

$$\pi_l(SO(n) \times SO(m-n)).$$

We may assume that γ_1 is equal to the identity on the hemisphere D_-^l of S^l while γ_2 is equal to the identity on D_+^l. The map

$$\gamma : S^l \to SO(n) \times SO(m-n)$$

defined by

$$\gamma = \gamma_1 \quad \text{on} \quad D_+^l$$

$$\gamma = \gamma_2 \quad \text{on} \quad D_-^l$$

represents the element $[\gamma] = [\gamma_1] + [\gamma_2]$. It follows from the construction that $\phi((S^m, \Sigma^n), \gamma \mid D_+^l)$ and $\phi((S^m, \Sigma^n), \gamma \mid D_-^l)$ are two relative disc pairs (D_+^{m+l}, D_+^{n+l}), (D_-^{m+l}, D_-^{n+l}) with common boundary $\phi((S^m, \Sigma^n), \gamma \mid \partial D_+^l)$. Thus $\phi(\sigma, \gamma)$ is represented by (S^{m+l}, Σ^{n+l}) defined by attaching (D_+^{m+l}, D_+^{n+l}) and (D_-^{m+l}, D_-^{n+l}) along their common boundary. Consider the relative disc pairs

$$(\bar{D}_-^{m+l}, \bar{D}_-^{n+l}) = \phi((S^m, \Sigma^n), \gamma_1 \mid D_-^l)$$

$$(\bar{D}_+^{m+l}, \bar{D}_+^{n+l}) = \phi((S^m, \Sigma^n), \gamma_2 \mid D_+^l).$$

Now $(\bar{D}_-^{m+l}, \bar{D}_-^{n+l})$ and $(\bar{D}_+^{m+l}, \bar{D}_+^{n+l})$ also have boundary

$$\phi((S^m, \Sigma^n), \gamma \mid \partial D_+^l).$$

Joining $(\bar{D}_-^{m+l}, \bar{D}_-^{n+l})$ to (D_+^{m+l}, D_+^{n+l}) and $(\bar{D}_+^{m+l}, \bar{D}_+^{n+l})$ to (D_-^{m+l}, D_-^{n+l}) along the common boundary we have the twist-spun pairs

$$(S^{m+l}, \Sigma_1^{n+l}) = \phi((S^m, \Sigma^n), \gamma_1) \quad \text{and} \quad (S^{m+l}, \Sigma_2^{n+l}) = \phi((S^m, \Sigma^n), \gamma_2).$$

Since $(\bar{D}_+^{m+l}, \bar{D}_+^{n+l})$ is obtainable from $(\bar{D}_-^{m+l}, \bar{D}_-^{n+l})$ by reflection, there exists an h-cobordism between

$$(D_-^{m+l}, D_-^{n+l}) \cup (\bar{D}_+^{m+l}, \bar{D}_-^{n+l}) \# (\bar{D}_-^{m+l}, \bar{D}_-^{n+l}) \cup (D_+^{m+l}, D_-^{n+l}),$$

which is $(S^{m+l}, \Sigma_1) \# (S^{m+l}, \Sigma_2)$, and

$$(S^{m+l}, \Sigma^{n+l}) = (D_-^{m+l}, D_-^{n+l}) \cup (D_+^{m+l}, D_-^{n+l})$$

which cancels out the interior connected sum $(\bar{D}_+^{m+l}, \bar{D}_+^{n+l}) \# (\bar{D}_-^{m+l}, \bar{D}_-^{n+l})$. This completes the proof of linearity of ϕ on the second factor.

Now let $[\gamma] = (\alpha, \beta)$ be in the image of $\pi_l(SO(n-1) \times SO(m-n))$ under the standard inclusion, then

$$\gamma : S^l \to SO(n-1) \times SO(m-n) \subset SO(n) \times SO(m-n).$$

Let $\sigma_1, \sigma_2 \in \theta^{m,n}$ be represented by (S^m, Σ_i^n) $i = 1, 2$. Without loss of generality we may assume that $(S^m, \Sigma_1^n)((S^m, \Sigma_2^n))$ is standard on the upper hemisphere D_+^m and the left (right) half $D_{-l}^m(D_{-r}^m)$ of the lower hemisphere D_-^m (Fig. 2).

D_{-l}^m and D_{-r}^m meet in the disc $D^{m-1} = D_{-l}^m \cap D_{-r}^m$. Since $[\gamma]$ is in the image of $\pi_l(SO(n-1) \times SO(m-n))$ under the standard inclusion, the trace

$$\phi(\Sigma_1^n \cap D_{-r}^m, \gamma), \qquad (\phi(\Sigma_1^n \cap D_{-l}^m, \gamma))$$

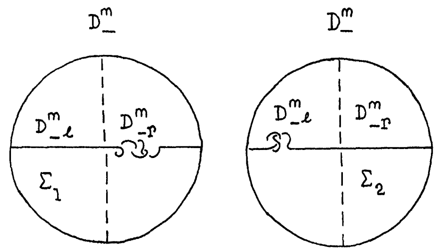

FIGURE 2

in $\phi(D^m_{-r}, \gamma)$, $(\phi(D^m_{-l}, \gamma))$ is a relative disc pair

$$(D^{m+l}_+, D^{n+l}_+), \qquad ((D^{m+l}_-, D^{n+l}_-)).$$

The disc pairs (D^{m+l}_+, D^{n+l}_+) and (D^{m+l}_-, D^{n+l}_-) have common boundary $(\phi(D^{m-1}, \gamma), \phi(D^{m-1} \cap \Sigma^n_i, \gamma))$ which is diffeomorphic to the standard pair (S^{m+l-1}, S^{n+l-1}). The element $\phi(\sigma_1 + \sigma_2, \gamma)$ is represented by the twist-spun pair (S^{m+l}, Σ^{n+l}) obtained by joining (D^{m+l}_+, D^{n+l}_+) and (D^{m+l}_-, D^{n+l}_-) along their common boundary. Now using arguments analogous to those used in the proof of the first half of the theorem we conclude that ϕ_{n-1} is bilinear.

Remark. One can prove similarly that ϕ is linear and ϕ_{n-1} is bilinear when $\theta^{m,n}$ is replaced by $I^{m,n}$.

The following corollary is an immediate consequence of Theorem 1.

COROLLARY 2. *The spin of a sphere pair (S^m, Σ^n), without twist, is h-cobordant to the standard pair and the result of twist-spinning the standard pair (S^m, S^n) by an element $[\gamma] \in \pi^{n-1}_l$ is h-cobordant to the standard pair.*

We now show that in general ϕ is non-trivial. Let $\pi_0(\mathrm{Diff}_C(R^{m-1}, R^{n-1}))$ denote the group of path components of orientation-preserving diffeomorphisms, of the standard pair (R^{m-1}, R^{n-1}) onto itself with compact support, in the C^∞ topology. An operation

$$\psi : \pi_0 \mathrm{Diff}_C(R^{m-1}, R^{n-1}) \times \pi_l(SO(n-1) \times SO(m-n)) \rightarrow \pi_0 \mathrm{Diff}_C(R^{m+l-1}, R^{n+l-1})$$

is defined as follows. Let $[h] \in \pi_0 \mathrm{Diff}_C(R^{m-1}, R^{n-1})$ and let

$$[\gamma] = (\alpha, \beta) \in \pi_l(SO(n-1) \times SO(m-n))$$

be represented by the map

$$f : R^l \rightarrow SO(n-1) \times SO(m-n)$$

with compact support. Let

$$F : R^{m-1} \times R^l \rightarrow R^{m-1} \times R^l$$

be given by $F(x, y) = (f(y) \cdot x, y)$. The operation ψ is defined by

$$\psi([h], [\gamma]) = (h \times 1)^{-1} \circ F \circ (h \times 1) \circ F^{-1}.$$

Now let $\pi_0 \operatorname{Diff}_C (R_+^m, R_+^n)$ denote the abelian group of path components of orientation-preserving diffeomorphisms with compact support of the closed half space pair (R_+^m, R_+^n) onto itself. Let

$$i_* : \pi_0 \operatorname{Diff}_C (R_+^m, R_+^n) \to \pi_0 \operatorname{Diff}_C (R^{m-1}, R^{n-1})$$

be induced by restriction. Let $\hat{\theta}^{m,n}$ denote the group of h-cobordism classes of pairs (Σ^m, Σ^n).

THEOREM 3. *Suppose $m - n > 2$ and $n > 4$. Then*

$$\frac{\pi_0 \operatorname{Diff}_C (R^{m-1}, R^{n-1})}{i_* \pi_0 \operatorname{Diff}_C (R_+^m, R_+^n)}$$

is isomorphic to $\hat{\theta}^{m,n}$, and the operation ψ induces a pairing

$$\hat{\psi} : \hat{\theta}^{m,n} \otimes \pi_l(SO(n-1) \times SO(m-n)) \to \hat{\theta}^{m+l,n+l}.$$

Proof. It follows from Corollary 3.2 of [10] that $\hat{\theta}^{m,n}$ is just the group of diffeomorphism classes of sphere pairs, if $m - n > 2$ $n > 4$, and any sphere pair is representable as a union $(D^m, D^n) \cup (D^m, D^n)$ where (D^m, D^n) denotes the standard disc pair. To see this consider an arbitrary sphere pair (Σ^m, Σ^n) as the union of two standard disc pairs and a third manifold pair. Smale's Corollary (3.2 of [10]) allows us to eliminate the manifold pair. The following theorem gives us the relation between $\hat{\psi}$ and ϕ.

THEOREM 4. *Suppose $m - n > 2$ and $n > 4$. Then*

$$\phi = \hat{\psi} \mid \theta^{m,n} \otimes \pi_l\, SO(n-1) \times SO(m-n).$$

Proof. Consider the relative surgery description of twist-spinning given in §1. Since F is defined on $D_-^m \times S^l$ and $h \times 1$ is defined on $\partial D_-^m \times D^{l+1}$ the twist-spin is diffeomorphic, and hence h-cobordant, to the pair

$$(\partial D_-^m \times D^{l+1} \cup_{(h+1)^{-1} \circ F \circ (h+1) \circ F^{-1}} D_-^m \times S^l,$$
$$\partial(\Sigma^n - \operatorname{Int} D_+^n) \times D^{l+1} \cup_{(h+1)^{-1} \circ F \circ (h+1) \circ F^{-1}} (\Sigma^n - \operatorname{Int} D_+^n) \times S^l)).$$

But now $(h + 1)^{-1} \circ F \circ (h + 1) \circ F^{-1} \mid \partial D_-^m \times S^l = 1$ and the result follows from the isomorphism of Theorem 3.

When $m = n$, $\hat{\psi}$ reduces to the operation used for investigating Diff S^{m-1}, of Milnor [5], Munkres [6] and Novikov [7]. From Theorem 4 and examples given in their work we deduce the non-triviality of ϕ. The following corollary is an immediate consequence of Theorem 4.

COROLLARY 5. *Suppose the Milnor–Munkres–Novikov operations applied to Σ^n gives Σ^{n+l} and suppose Σ^n is embeddable in S^{n+t}. Then Σ^{n+l} is embeddable in Σ^{n+l+t}.*

IV. Twist-spun normal bundles

In this section we shall show how the normal bundle of a twist-spun sphere is determined by the normal bundle of the original sphere and the normal twist. Let $BSO(m-n)$ be the classifying space of $SO(m-n)$, and let

$$\pi_n(BSO(m-n)) \otimes \pi_{l+1}(BSO(m-n)) \xrightarrow{W} \pi_{l+n}(BSO(m-n))$$

be the Whitehead product pairing.

THEOREM 6. *The following diagram is commutative:*

$$\begin{array}{ccc} \theta^{m,n} \times \pi_l(SO(n) \times SO(m-n)) & \longrightarrow & \theta^{m+l,n+l} \\ (\text{or } I^{m,n}) & & (\text{or } I^{m+l,n+l}) \\ \Big\downarrow \eta \times p & & \Big\downarrow \eta \\ \pi_n(BSO(m-n)) \otimes \pi_{l+1}(BSO(m-n)) & \xrightarrow{-W} & \pi_{n+l+1}(BSO(m-n)). \end{array}$$

Here η assigns to each embedding (or immersion) its normal bundle and p is projection followed by the transgression isomorphism

$$\sigma : \pi_l(SO(m-n)) \to \pi_{l+1}(BSO(m-n)).$$

It is well known that the Whitehead product is related to the Samelson product by the following diagram,

$$\begin{array}{ccc} \pi_i(SO(t)) \otimes \pi_j(SO(t)) & \xrightarrow{S} & \pi_{i+j}(SO(t)) \\ \Big\downarrow \sigma \otimes \sigma & & \Big\downarrow \sigma \\ \pi_{i+1}(BSO(t)) \otimes \pi_{j+1}(BSO(t)) & \xrightarrow{(-)^i W} & \pi_{i+j+1}(BSO(t)), \end{array}$$

where S denotes Samelson product [8].

A non-triviality of the Samelson product of the characteristic class of the normal bundle of the embedded (immersed) sphere with an element $\gamma \in \pi_l(SO(m-n))$ will lead to a twist-spun sphere with non-trivial normal bundle. As the condition on the characteristic class of the normal bundle of an embedded sphere is very restrictive [3], we are unable to produce any example, but there are several examples in the case of immersions. For instance it follows from [4], that there always exists an immersion $S^{4k} \subset S^{8k-1}$ such that the result of twist-spinning by some element in $\pi_{4k-1}(SO(4k-1))$ has non-trivial normal bundle.

Proof of Theorem 6. Let the embedding $\Sigma^n \subset S^m$, which is the standard inclusion $D_+^n \subset D_+^m$ on the upper hemisphere, be a representative of the element $\sigma \in \theta^{m,n}$. Let ν be the normal bundle of this embedding. The classifying map $f : \Sigma^n \to BSO(m-n)$ of ν can be described as follows. Let F_0 be the standard normal frame over D_+^n and let F_1 be a normal frame on $E^n - \operatorname{Int} D_+^n$.

The difference between F_0 and F_1 determines a map

$$\hat{g} : S^{n-1} = \partial(\Sigma^n - \operatorname{Int} D_+^n) \to SO(m - n).$$

Consider $SO(m - n)$ as the fibre of the universal bundle;

$$SO(m - n) \to ESO(m - n) \xrightarrow{p} BSO(m - n).$$

Since $ESO(m - n)$ is contractible, $\hat{g}$ extends to a map

$$g : (D_-^n = (\Sigma^n - \operatorname{Int} D_+^n), S^{n-1} = \partial(\Sigma^n - \operatorname{Int} D_+^n)) \to (ESO(m - n), SO(m - n)).$$

Then $f = p \circ g$. Now let $\Sigma^{n+l} \subset S^{m+l}$ be the result of spinning Σ^n in S^m around S^l. Then we have induced frames F_0', F_1' on the two halves $\partial D_-^n \times D^{l+1}$ and $D_-^n \times S^l$ of Σ^{n+l} in S^{m+l} and the difference is given by the map

$$\hat{g} \circ p_1 : \partial D_-^n \times S^l \to \partial D_-^n \to SO(m - n).$$

Now the frame F_0'' induced from F_0' by twist-spinning differs from F_0' by the map

$$\hat{h} \circ p_2 : \partial D_-^n \times S^l \to S^l \to SO(m - n),$$

where $\hat{h}$ is a representative for the element β of

$$\gamma = (\alpha, \beta) \in \pi_l(SO(n) \times SO(m - n)).$$

Thus extending $\hat{g} \circ p_1$, $\hat{h} \circ p_2$ to maps

$$G : (D_-^n \times S^l, \partial D_-^n \times S^l) \to (ESO(m - n), BSO(m - n))$$

and

$$H : (\partial D_-^n \times D^{l+1}, \partial D_-^n \times S^l) \to (ESO(m - n), SO(m - n))$$

we are able to define a map,

$$k : \Sigma^{n+l} \to BSO(m - n)$$

by

$$k = pG \quad \text{on} \quad D_-^n \times S^l$$

$$k = pH \quad \text{on} \quad \partial D_-^n \times D^{l+1}.$$

It follows from [11, p. 102] that k is the classifying map for the normal bundle of the embedding $\Sigma^{n+l} \subset S^{n+l}$. On the other hand, it follows from the definition of the Whitehead product that k is a representative of $[\nu, -\sigma(\beta)] = -[\nu, p(\gamma)]$. This completes the proof of our assertion for embeddings. The proof for immersions goes through in the same way.

References

1. E. Artin, *Zur Isotopie zweidemensionalen Flächen im R_4*, Abh. Math. Sem. Univ. Hamburg, vol. 4 (1926), pp. 174–177.

2. A. Haefliger, *Knotted $(4k - 1)$-spheres in $6k$-space*, Ann. of Math. (3), vol. 75 (1962), pp. 452–466.

3. W. C. HSIANG, J. LEVINE AND R. H. SZCZARBA, *On the normal bundle of a homotopy sphere embedded in euclidean space*, Topology, vol. 3 (1965), pp. 173–181.
4. I. M. JAMES AND E. THOMAS, *Which Lie groups are homotopy abelian?*, Proc. Nat. Acad. Sci., vol. 45 (1959), pp. 737–740.
5. J. MILNOR, *Diffeomorphisms of a sphere*, unpublished.
6. J. R. MUNKRES, *Killing exotic spheres*, unpublished.
7. S. P. NOVIKOV, *Homotopy properties of the group of diffeomorphisms of a sphere*, Dokl. Akad. Nauk SSSR, vol. 148 (1963), pp. 32–35.
8. H. SAMELSON, *A connection between the Whitehead and the Pontryagin product*, Amer. J. Math., vol. 75 (1953), pp. 744–752.
9. S. SMALE, *Generalized Poincaré's conjecture in dimensions greater than four*, Ann. of Math. (2), vol. 74 (1961), pp. 391–406.
10. ——— , *On the structure of manifolds*, Amer. J. Math. (3), vol. 84 (1962), pp. 387–399.
11. N. E. STEENROD, *The topology of fibre bundles*, Princeton, Princeton University Press, 1951. to appear.
12. E. C. ZEEMAN, *Twisting spun knots*, Trans. Amer. Math. Soc., to appear.

YALE UNIVERSITY
NEW HAVEN, CONNECTICUT