A LOCALLY COMPACT CONNECTED GROUP ACTING ON THE PLANE HAS A CLOSED ORBIT[1]

BY

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The theorem of the title has its origin in a question concerning topological semigroups: Suppose S is a topological semigroup with identity 1 on a manifold. It is known that the set $H(1)$ of all elements having an inverse with respect to 1 is a Lie group [5]. Let G be the component of the identity of $H(1)$ and let L be the boundary of G. The question arises whether L (if non-empty) necessarily contains an idempotent. This was shown to be so in [5] if S is a plane. We had recently shown that this is so if S is Euclidean three-space and L is topologically a plane. For each case, use was made of the Lemma 2.5 in [5] that if G has a closed left orbit and a closed right orbit in L then L contains an idempotent. The question was thus raised whether a connected Lie group can act on the plane without a closed orbit. Using a technique developed to prove the result in the second of the two cases above, together with the result of Professor Hofmann in [2] of this journal, we prove that it cannot. Finally, we include an argument sent to us by Professor Hofmann which extends this theorem to locally compact connected groups.

A result which we use repeatedly is Theorem 2 of [3]. This asserts that if a one-parameter group P acts as a transformation group on the plane and an orbit Px is unbounded in both directions (that is, Px is topologically a line and neither component of $Px \setminus \{x\}$ has compact closure) then $Pz = z$ for all $z \in (Px)^{-} \setminus Px$. In fact, this result is needed under the assumption that P acts, not on the entire plane E, but only on a closed subset of E. An examination of the proof in [3] reveals that this is actually what is proved. This theorem allows us to obtain a closed orbit when an orbit exists which is unbounded in both directions. In case every orbit Px has one of its ends bounded we apparently need the following lemma. (By an "end" of Px is meant one of the components of $Px \setminus \{x\}$.)

Lemma 1. *Let S be a subset of the plane and suppose that the multiplicative group P of positive real numbers acts as a transformation group of S. Let R denote one of the components of $P \setminus \{1\}$. Suppose x, y, z are points of S such that*

$$y \in (Rx)^{-} \setminus Px \quad \text{and} \quad z \in (Ry)^{-} \setminus Py.$$

Then $Pz = z$.

Proof. Order P so that $p > 1$ if and only if $p \in R$. Assume $Pz \neq z$. Then there exists an interval $A = [a, b]$, $a < 1 < b$, such that the map $p \to pw$ is a

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homeomorphism on A for each w sufficiently close to z. Since $z \in (Ry)^{-} \setminus Py$, there is an unbounded sequence $\{p_n\}_n$, $p_n \in R$, such that $p_n y \to z$. Let D be a small disc about z. We may suppose that each of the arcs Az and $Ap_n y$, $n = 1, 2, \cdots$, cuts D. There exists a sub-arc of $Az \cap D$ which contains z and separates D into exactly two components. One of these will contain an infinite number of the points $p_n y$. Let E denote such a component and assume, without loss of generality that each $p_n y \in E$.

Choose a point q in the boundary of E but not in Az. Let B be an arc from z to q lying, except for its end points, in the interior of E. Say that an arc goes across E if it has its end points on the boundary of E but not in Az, if these end points are on opposite sides of B and if, except for its end points, it is contained in the interior of E.

Consider the collection of components of the intersection of Ry with the interior of E. The closure of each member of this collection is a sub-arc of Ry. Let $\mathfrak{C}$ denote the collection of these arcs which go across E. It is clear that $\mathfrak{C}$ can be linearly ordered so that later terms of $\mathfrak{C}$ are nearer Az than earlier terms. Even more: each member of $\mathfrak{C}$ has an immediate successor and the members of $\mathfrak{C}$ may be arranged in a sequence $C_1, C_2, \cdots, C_n, \cdots$ so that in general C_{n+1} is the successor of C_n. Now choose a (possibily new) sequence $r_n y \in Ry$ so that $r_n y$ converges to z and $r_n y \in C_n$ for each n. Since each $r_n y$ belongs to $(Rx)^{-} \setminus Px$, there exists a sequence $\{q_n\}_n$, $q_n \in R$, such that $q_n x$ is between $Ar_{n-1} y$ and $Ar_{n+1} y$ and so that $q_n x$ converges to z. The collection of arcs Az, $\{Ar_n y\}_n$ and $\{Aq_n x\}_n$ taken together forms an equicontinuous collection of arcs (see the proof of Theorem 1 of [2], for example). Therefore, by [1], we may assume that each of these arcs is a straight line segment.

The discussion is now facilitated somewhat by thinking of Az as lying along the X-axis with z at the origin and az "to the left" of the origin. Let $a_1, b_1 \in P$ be such that $a < a_1 < 1 < b_1 < b$. Let L_1 be the line perpendicular to Az at $a_1 z$ and let L_2 be the line perpendicular to Az at $b_1 z$. There is no loss in generality in assuming that each $ar_n y$ and each $aq_n x$ lies to the left of L_1 and each $br_n y$ and each $bq_n x$ lies to the right of L_2.

An arc will be said to cross $L_1 L_2$ if it has one end point on L_1, the other end point on L_2 and except for these points is contained between L_1 and L_2. An arc will be said to cross $L_1 L_2$ in the *right* direction if it is a sub-arc of an arc having the form Av for some $v \in E$, if it crosses $L_1 L_2$ and if in the order it inherits from A, the smallest point is on L_1 and the largest point is on L_2. If for such an arc the largest point is on L_1 and the smallest point is on L_2, it will be said to cross $L_1 L_2$ in the *wrong* direction.

Let $c_n y$ denote the intersection of L_1 and $Ar_n y$ and let $d_n y$ denote the intersection of L_2 and $Ar_n y$. Let Q_n denote the quadrilateral whose vertices are $c_n y$, $d_n y$, $c_{n+1} y$ and $d_{n+1} y$. There are now two cases according to whether $c_n > c_{n+1}$ or $c_{n+1} > c_n$. We consider the first case. The argument is similar in the second case and is omitted. Since C_{n+1} is the successor of C_n, no sub-arc of $[d_n, d_{n+1}]y$ crosses $L_1 L_2$ between $Ar_n y$ and $Ar_{n+1} y$. Furthermore there exist arbitrarily large integers n such that $q_n x$ is between $Ar_n y$ and $Ar_{n+1} y$. Choose such an integer. By [4, p. 173] there is an arc T joining $d_n y$ to $d_{n+1} y$ which, except for its end points, is contained in Q_n and which has only end points in common with $[d_n, d_{n+1}]y$. Similarly there is an arc S joining $c_n y$ to $c_{n+1} y$ which, except for its end points, is contained in Q_n, which misses not only $[d_n, d_{n+1}]y$ but T as well. The arcs S and T can be chosen to be polygonal and to intersect $Aq_n x$ in one point each. Let C be the simple closed curve formed by joining T and $[d_n, d_{n+1}]y$. Then $Aq_n x$ has points on opposite sides of C. Since $Aq_n x$ intersects C in only one point, its end points are on opposite sides of C. Certainly the intersection of $Aq_n x$ and S is on the opposite side of C from $bq_n x$. Since S has no points in common with C, $c_n y$ and $bq_n x$ lie on opposite sides of C. Again, since $[ar_n, c_n]y$ has no points in common with C, $ar_n y$ and $bq_n x$ lie on opposite sides of C. Since $ar_n y \in (Rx)^{-} \setminus Px$, there exist numbers $r, s \in R$ such that $s > r > bq_n$ and such that rx is on L_2 while sx is on L_1. Thus there is a sub-arc of $[r, s]x$ which crosses $L_1 L_2$ in the wrong direction.

We sketch a proof that all points in S sufficiently near z must lie on arcs which cross $L_1 L_2$ in the right direction. Since the preceding result contradicts this fact, we conclude that $Pz = z$ and the proof of the lemma will be complete.

Let D_1, D_2 be two discs whose radii are slightly larger than one-half the distance from L_1 to L_2 and whose centers are at $a_1 z$ and $b_1 z$ respectively. Let cz be a point in the intersection of the interiors of D_1 and D_2. It is sufficient for our purposes to assume $az \in D_1$ and $bz \in D_2$ so that $[a, c]z \subset D_1$ and $[c, b]z \subset D_2$. Corresponding to each $t \in [a, c]$ there is an interval V containing t and a neighborhood W of z such that $VW \subset D_1$. By compactness, there exist a finite number of intervals $V_1, \cdots, V_n$ which cover $[a, c]$ and an open set W_1, containing z such that $V_i W_1 \subset D_1$ for each $i = 1, \cdots, n$. We may evidently assume $c \in V_n$ and that in fact $V_n W_1 \subset D_1 \cap D_2$. Similarly there exist open intervals $V_1, \cdots, V_n$ covering $[c, b]$ and an open set W_2 containing z such that $V_i W_2 \subset D_2$ for $i = 1, \cdots, m$. If $w \in W_1 \cap W_2$ then, since $Aw = [a, c]w \cup [c, b]w$, Aw contains no sub-arc which crosses $L_1 L_2$ in the wrong direction. The proof of the lemma is complete.

LEMMA 2. *Let G be a connected Lie group acting as a transformation group on a space M. Let $x \in M$ and let P be a one-parameter subgroup of G. Then the following are equivalent.*

(1) $Gx \neq \{x\}$ *and* $Gx = Px$;

(2) $\dim Gx = 1$ *and P has no conjugate in the isotropy subgroup G_x of G.*

Proof. Suppose $Gx \neq \{x\}$ and $Gx = Px$. Then $\dim Gx = 1$. If $gPg^{-1} \subset G_x$ for some $g \in G$ then $P \subset g^{-1}G_x g$ so $Pg^{-1}x = g^{-1}x$. However $g^{-1}x = px$ for some $p \in P$ so $Ppx = px$. On the other hand $Pp = P$ while $Px = Gx \neq px$. Thus P has no conjugate in Gx.

Suppose $\dim Gx = 1$ and that P has no conjugate in G_x. Let G act on the left coset space G/G_x by left multiplication. Let $m = G_x$. Now $Gx = Px$ provided $Gm = Pm$ under this action. Since G/G_x is a one-dimensional manifold, it is a line or a circle. Therefore, if $Gm \neq Pm$ then there exists $v \in (Pm)^{-}\setminus Pm$ and $Pv = v$ since otherwise $Pv \cap Pm \neq \emptyset$. There is $g \in G$ such that $v = gm$. Since the isotropy group of v is gG_xg^{-1}, $P \subset gG_x g^{-1}$, so $g^{-1}Pg \subset G_x$ which is a contradiction. Obviously, if $\dim Gx = 1$ then $Gx \neq \{x\}$.

Hereafter, if G is a transformation group on a space M and $x \in M$ then G_x will denote the component of the identity of the isotropy group of x. We have already observed that Theorem 2 of [3] is true for arbitrary closed subsets of the plane. The same observation holds for Theorem 1 and we use it in this form without further mention. That theorem asserts that if P operates as a transformation group on the plane then every orbit of P is either a point, a simple closed curve or topologically a line.

LEMMA 3. *Let G be a connected Lie group acting as a transformation group on a closed subset S of the plane. If for some $x \in S$, Gx is a line and G_x is a normal subgroup then G has a closed orbit.*

Proof. There exist one-parameter subgroups $P_1, \cdots, P_n$ such that $P_1 P_2 \cdots P_n$ generates G and no P_i is contained in G_x, $i = 1, 2, \cdots, n$. Since G_x is normal, no P_i has a conjugate in G_x so $Gx = P_i x$ for each $i = 1, 2, \cdots, n$ by the previous lemma. Furthermore, since G_x is normal, $G_x y = y$ for all $y \in (Gx)^{-}$. Therefore, if $y \in (Gx)^{-}$ then either $\dim Gy = 0$ and $Gy = y$ or $\dim Gy = 1$ and $G_y = G_x$. It follows that $Gy = P_i y$ for $i = 1, 2, \cdots, n$ and $y \in (Gx)^{-}$. If Gx is unbounded in both directions and $y \in (Gx)^{-}\setminus Gx$ then $P_i y = y$ for each i by Theorem 2 of [3]. Since $P_1 P_2 \cdots P_3$ generates G, $Gy = y$ so G has a closed orbit.

Suppose Gx is not unbounded in both directions. Let C be a component of $Gx\setminus\{x\}$ such that C^{-} is compact. For each $i = 1, 2, \cdots n$, let R_i be the component of $P_i\setminus\{1\}$ such that $R_i x = C$. Let $y \in C^{-}\setminus Gx$. If Gy is closed, there is nothing further to prove. Otherwise, Gy is a line contained in C^{-}. Furthermore, $R_i y = R_j y$ for $i, j = 1, 2, \cdots, n$. For since Gx is a line, $G\setminus G_x$ is the union of two components A and B. Since G_x is normal, if s belongs to one of these components then s^{-1} belongs to the other and each component is a subsemigroup of G. It follows that for each i, j, R_i and R_j belong to the same component of $G\setminus G_x$. For if R_i and R_j belong to different components then there exist elements $s \in A$ and $y \in B$ such that $sx = tx$. Hence $t^{-1}s \in G_x$. However, both t^{-1} and s belong to A which is impossible since A is closed under multiplication. Now if R_i and R_j belong to a common component of $G\setminus G_x$ then $R_i y = R_j y$. To see this, recall that $P_i y = P_j y$ so $R_i y = S_j y$ where S_j is the component of $P_j\setminus\{1\}$ which belongs to the component of $G\setminus G_x$ which contains R_i. Since this is R_j, it follows that $R_i y = R_j y$ for all i, j.

Since $R_1 y$ is contained in a compact set, there exists an element

$$z \in (R_1 y)^{-} \backslash P_1 y.$$

Since $R_1 y = R_i y$, $z \in (R_i y)^{-} \backslash P_i y$ for each i. By Lemma 1, $P_i z = z$ for each i. Hence $Gz = z$ and it follows that G has a closed orbit.

THEOREM 1. *Let G be a connected Lie group acting as a transformation group on a closed connected subset S of the plane. Then there exists $w \in S$ such that Gw is closed.*

Proof. If there exists $x \in S$ with $Gx = S$, there is nothing to prove. Suppose there exists $v \in S$ with $\dim Gv = 2$ but $Gv \neq S$. Then Gv has a boundary point x and for every such point, $\dim Gx < 2$. For if $\dim Gx = 2$ then, since Gx is homogeneous Gx is open and hence $Gx \cap Gv \neq \emptyset$ which is impossible. Now if $\dim Gy = 0$ for any $y \in S$, $Gy = y$ so G has a closed orbit. The proof of the theorem has thus been reduced to the following situation: Either G already has a closed orbit, or there exists $x \in S$ such that Gy is a line for all $y \in (Gx)^{-}$. Furthermore, by the previous lemma, we may assume that G_x is not normal.

Let $\mathfrak{H}$ be the sub-algebra of the Lie algebra of G corresponding to G_x. Let N be the normal subgroup of G which is contained in G_x and which corresponds to the ideal $\mathfrak{J}$ of Theorem I of [2]. Of course, N may not be closed, but N^{-} is still normal and contained in G_x. Thus there is no loss in generality in assuming that G/N is either abelian, locally isomorphic to $sl(2)$ or isomorphic to the non-commutative group on the plane.

Let $H = G/N$. Since $Ny = y$ for all $y \in (Gx)^{-}$, H operates on $(Gx)^{-}$ according to the rule

$$(gN)y = gy.$$

Furthermore, every orbit of H on $(Gx)^{-}$ is an orbit of G. Hence if we prove that H has a closed orbit it follows that G has a closed orbit.

We consider the possibilities for H separately. If H is abelian then H has a closed orbit by Lemma 3 since every subgroup of H is normal. Suppose H is locally isomorphic to $sl(2)$. Thus H is isomorphic to the quotient of the covering group K of $sl(2)$ modulo a discrete central subgroup. Now H_x is a planar subgroup. Let P be a one-parameter group of H which is the image under the natural map of a one-parameter subgroup of K which intersects the center non-trivially. Such P can have no conjugate in any planar subgroup of H. Hence, $Hx = Px$. In fact, $Hy = Py$ for all $y \in (Hx)^{-}$ since under our present assumptions, H_y is a planar group for each such y. If Hx is unbounded in both directions then Hx is closed since if $y \in (Hx)^{-} \setminus Hx$, $Py = y$ by [3] (as extended to arbitrary closed sets in the plane). This is a contradiction. Therefore suppose that Hx is not unbounded in both directions and let R denote a component of $P \setminus \{1\}$ such that $(Rx)^{-}$ is compact. Choose $y \in (Rx)^{-} \setminus Px$. We may suppose that Py is not closed. Thus Ry is not closed so there exists $z \in (Ry)^{-} \setminus Py$. But then by Lemma 1, $Pz = z$ which is a contradiction. We have proved that if G/N is locally isomorphic to $sl(2)$ then G has a closed orbit.

Finally, suppose H is the non-commutative group on the plane. If H_x is the normal one-parameter subgroup Q of H there is nothing further to prove in virtue of Lemma 3. Otherwise, $Hx = Qx$ since then Q is the only one-parameter subgroup of H having no conjugate in H_x. If Hx is unbounded in both directions we may assume that Hx is closed since otherwise there exists $z \in (Hx)^{-} \setminus Hx$ and for such z, $Qz = z$ as we know. Hence by Lemma 3 again, H has a closed orbit. Thus suppose one end Rx of Qx has compact closure and let $y \in (Rx)^{-} \setminus Qx$. Unless Hy is closed, there is $z \in (Ry)^{-} \setminus Qy$. But then by Lemma 1, $Qz = z$ so by Lemma 3 again, H has a closed orbit. All cases have now been considered and the proof of the theorem is complete.

In virtue of Lemma 2.5 of [5] mentioned in the beginning we have the following:

COROLLARY. *Let S be a topological semigroup with identity* 1. *Assume that the set G of all elements in S which have an inverse with respect to* 1 *is a connected Lie group. Let L be a connected non-empty ideal in S which is homeomorphic to a closed subset of the plane. Then L contains an idempotent.*

Professor Hofmann has observed to us that Theorem 1 actually holds for any locally compact group G such that G/G_0 is compact, where G_0 is the component of the identity in G. His observations run as follows: Under this assumption G is the projective limit of Lie groups (cf. [6]). Let M be a compact normal subgroup such that G/M is a Lie group. Let N be the subgroup of all $g \in G$ such that $gx = x$ for all $x \in S$. Then N is closed and normal. So is $M \cap N$. Now $M/M \cap N$ is a compact group acting effectively on the plane; hence the component of the identity of $M/M \cap N$ is a Lie group [6, p. 259] and hence $M/M \cap N$ is finite-dimensional. If $M/M \cap N$ is not itself a Lie group it contains a totally disconnected non-discrete subgroup [6, p. 237] which contradicts the fact that no such group can act effectively on the plane [6, p. 249]. Thus $M/M \cap N$ is a Lie group. But G/M is a Lie group with a finite number of components, so $G/M \cap N$ is a Lie group with a finite number of components. Therefore G/N is a Lie group. But G/N is the group which actually acts in the plane. Thus the following theorem is proved:

THEOREM 2. (Hofmann). *Let G be a locally compact group acting as a group of homeomorphisms on the plane and let G_0 be its identity component. If G/G_0 is compact then G has a closed orbit.*

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