LIE ALGEBRAS WITH SUBALGEBRAS OF CO-DIMENSION ONE

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In the study of Lie groups of dimension n acting on a space and having a 1-dimensional orbit—and in particular in the study of the boundary of certain topological semigroups on manifolds with boundary—the following problem arises: If G is an n-dimensional connected Lie group and H a closed $(n - 1)$-dimensional subgroup, is there a one-parameter group E such that $G = HE$ and no conjugate of E is contained in H? If so, "how many" such one-parameter groups exist? We shall give a complete answer to this question. In order to do so we classify the n-dimensional real Lie algebras possessing a subalgebra of dimension $n - 1$ (Theorem I). Thus we establish the fact that the Lie algebra of G contains at least $n - 1$ linearly independent vectors such that no conjugate of a one-parameter group generated by one of these is ever contained in H; in many cases there are even n linearly independent vectors with this property.

In order to make the proof fairly self contained we first deal with simple Lie algebras; the results so obtained may also be produced by a close inspection of the classification of simple Lie algebras.

LEMMA 1. *Let $\mathfrak{G}$ be a compact simple Lie algebra over an ordered field. Suppose that* $\dim \mathfrak{G} = n$ *and that $\mathfrak{H}$ is a subalgebra of dimension $n - d$. Then* $2n \leq d(d + 1)$.

Proof. Since usually the term of a compact Lie algebra is applied to real Lie algebras we first remark, that under a compact Lie algebra over an ordered field we understand a Lie algebra whose Killing form is negative definite. Now we let $\mathfrak{V}$ be an orthogonal complement of $\mathfrak{H}$ in $\mathfrak{G}$ with respect to the Killing form. Then $\dim \mathfrak{V} = d$. Preserving the Killing form on $\mathfrak{G}$ under the adjoint action, $\mathfrak{H}$ is represented in the Lie algebra of the orthogonal group $O(\mathfrak{V})$ on $\mathfrak{V}$. Let $\mathfrak{K}$ be the kernel of this representation. Then

$$[\mathfrak{K}, \mathfrak{G}] = [\mathfrak{K}, \mathfrak{V}] + [\mathfrak{K}, \mathfrak{H}]$$

which is in $\mathfrak{K}$ because the first summand vanishes and $\mathfrak{K}$ is an ideal in $\mathfrak{H}$. This shows that $\mathfrak{K}$ is an ideal of $\mathfrak{G}$. Since $\mathfrak{G}$ is simple we have $\mathfrak{K} = 0$; so $n - d = \dim \mathfrak{H} \leq \dim O(\mathfrak{V}) = d(d - 1)/2$. Consequently $2n \leq d(d - 1)$.

It may be remarked that equality holds if $\mathfrak{G}$ is the Lie algebra of $SO(m)$ and $\mathfrak{H}$ is the Lie algebra of the subgroup $SO(m - 1)$.

LEMMA 2. *Let $\mathfrak{G}$ be a real simple n-dimensional Lie algebra and $\mathfrak{H}$ a subalgebra of dimension $n - 1$. Then* $\dim \mathfrak{G} = 3$ *and* $\mathfrak{G} \cong sl(2)$, *the Lie algebra of* $Sl(2)$.

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Proof. Let $\mathfrak{G} \otimes \mathbf{C}$ be the complexification of $\mathfrak{G}$. Then $\mathfrak{H} \otimes \mathbf{C}$ is a subalgebra of $\mathfrak{G} \otimes \mathbf{C}$ which is $(2n - 2)$-dimensional as a real Lie algebra. Let $\mathfrak{G}_c \subset \mathfrak{G} \otimes \mathbf{C}$ be a compact real form of $\mathfrak{G} \otimes \mathbf{C}$ and let $\mathfrak{K} = (\mathfrak{H} \otimes \mathbf{C}) \cap \mathfrak{G}_c$. Then $\mathfrak{K}$ is a real subalgebra of $\mathfrak{G}_c$ and $\dim_{\mathbf{R}} \mathfrak{K} \geq \dim_{\mathbf{R}} \mathfrak{G}_c - 2$. If $\mathfrak{K} = \mathfrak{G}_c$, then the complex linear combinations of $\mathfrak{K} = \mathfrak{G}_c$ span all of $\mathfrak{G} \otimes \mathbf{C}$ which is not even the case if we take the complex linear combinations of $\mathfrak{H} \otimes \mathbf{C}$. Hence

$$\dim_{\mathbf{R}} \mathfrak{G}_c - 2 \leq \dim_{\mathbf{R}} \mathfrak{K} \leq \dim_{\mathbf{R}} \mathfrak{G}_c - 1.$$

But then, after Lemma 1, $\dim_{\mathbf{R}} \mathfrak{G}_c = 3$ which implies $\dim_{\mathbf{C}} \mathfrak{G} \otimes \mathbf{C} = \dim_{\mathbf{R}} \mathfrak{G} = 3$. Then $\mathfrak{G}$ is isomorphic to the Lie algebra of $SO(3)$ or of $Sl(2)$ but only the latter contains 2-dimensional subalgebras.

The following lemmas deal with the nonsimple case.

LEMMA 3. *Let $\mathfrak{G}$ be an n-dimensional real Lie algebra, $\mathfrak{N}$ its nil radical and $\mathfrak{H}$ an $(n - 1)$-dimensional subalgebra of $\mathfrak{G}$. Then $\mathfrak{H} \cap \mathfrak{N}$ is an ideal in $\mathfrak{G}$.*

Proof. If $\mathfrak{H} \cap \mathfrak{N} = \mathfrak{N}$, then the claim is trivially true. If $\mathfrak{H} \cap \mathfrak{N} \neq \mathfrak{N}$, then $\dim \mathfrak{H} \cap \mathfrak{N} = \dim \mathfrak{N} - 1$. Consequently, $\mathfrak{H} \cap \mathfrak{N}$ is certainly a maximal subalgebra of the nilpotent algebra $\mathfrak{N}$; it is therefore an ideal in $\mathfrak{N}$. Trivially, $\mathfrak{H} \cap \mathfrak{N}$ is an ideal in $\mathfrak{H}$. Since $\mathfrak{H}$ is a maximal subalgebra of $\mathfrak{G}$, and since $\mathfrak{N} \not\subset \mathfrak{H}$ we have $\mathfrak{G} = \mathfrak{H} + \mathfrak{N}$. Now $\mathfrak{H} \cap \mathfrak{N}$ is an ideal in $\mathfrak{H} + \mathfrak{N} = \mathfrak{G}$.

LEMMA 4. *With the notation of Lemma 3 let $\mathfrak{H} \cap \mathfrak{N} \neq \mathfrak{N}$ and put*

$$\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{H} \cap \mathfrak{N}, \quad \bar{\mathfrak{N}} = \mathfrak{N}/\mathfrak{H} \cap \mathfrak{N} \quad \textit{and} \quad \bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{H} \cap \mathfrak{N}.$$

Then $\bar{\mathfrak{G}}$ is a split extension of the 1-dimensional ideal $\bar{\mathfrak{N}}$ with the subalgebra $\bar{\mathfrak{H}}$. Moreover, $\bar{\mathfrak{H}}$ is a direct sum $\bar{\mathfrak{A}} \oplus \bar{\mathfrak{B}} \oplus \bar{\mathfrak{S}}$, where $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are abelian ideals in $\bar{\mathfrak{H}}$, and $\bar{\mathfrak{S}}$ is a semisimple subalgebra, where $\bar{\mathfrak{B}} \oplus \bar{\mathfrak{S}}$ is in the centralizer of $\bar{\mathfrak{N}}$ and where $\dim \bar{\mathfrak{A}} \leq 1$. *Furthermore, if* $\dim \bar{\mathfrak{A}} = 1$, *then $\bar{\mathfrak{N}} + \bar{\mathfrak{A}}$ is the 2-dimensional solvable nonabelian Lie algebra. Altogether*

$$\bar{\mathfrak{G}} = \bar{\mathfrak{N}} \oplus \bar{\mathfrak{A}} \oplus \bar{\mathfrak{B}} \oplus \bar{\mathfrak{S}},$$

and $\bar{\mathfrak{B}} \oplus \bar{\mathfrak{S}}$ is an ideal in $\bar{\mathfrak{G}}$.

Proof. Since $\mathfrak{H} \cap \mathfrak{N} \neq \mathfrak{N}$ and $\mathfrak{H}$ is $(n - 1)$-dimensional in $\mathfrak{G}$, clearly $\bar{\mathfrak{N}}$ is a 1-dimensional ideal in $\bar{\mathfrak{G}}$. From $\dim \bar{\mathfrak{G}} = \dim \bar{\mathfrak{H}} + 1$ and $\bar{\mathfrak{N}} \cap \bar{\mathfrak{H}} = \bar{0}$ we conclude $\bar{\mathfrak{G}} = \bar{\mathfrak{N}} \oplus \bar{\mathfrak{H}}$. The radical of $\bar{\mathfrak{H}}$ is isomorphic to the radical of $\bar{\mathfrak{G}}$ modulo $\bar{\mathfrak{N}}$ which is isomorphic to the radical of $\mathfrak{G}$ modulo $\mathfrak{N}$. This factor algebra is abelian because the nil radical contains the derived algebra of the radical. Therefore the radical of $\bar{\mathfrak{H}}$ is abelian. The restriction of the adjoint representation of $\bar{\mathfrak{H}}$ to $\bar{\mathfrak{N}}$ defines a homomorphism of $\bar{\mathfrak{H}}$ into the algebra of derivations of $\bar{\mathfrak{N}}$ which here is simply the real field. The kernel $\bar{\mathfrak{K}}$ of this homomorphism is the intersection of the centralizer of $\bar{\mathfrak{N}}$ with $\bar{\mathfrak{H}}$, and its dimension is $\geq \dim \bar{\mathfrak{H}} - 1$ since the range has dimension 1. According to the theorem of Levi $\bar{\mathfrak{K}}$ splits into a direct sum of the radical $\bar{\mathfrak{B}}$ (which, as we know, is abelian) and a semisimple subalgebra $\bar{\mathfrak{S}}$. Likewise $\bar{\mathfrak{H}}$ is a direct vector space sum of its radical $\bar{\mathfrak{C}}$ (which is abelian) and a semisimple subalgebra which according to the theorem of Malcev and Harish Chandra can be picked to be $\bar{\mathfrak{S}}$. Since the semisimple algebra is completely reducible when acting under the adjoint representation on $\bar{\mathfrak{C}}$, we deduce that $\bar{\mathfrak{C}}$ splits into a direct sum of and an at most 1-dimensional subspace $\bar{\mathfrak{A}}$ invariant under $\bar{\mathfrak{S}}$ under the adjoint representation. This means that $\bar{\mathfrak{A}}$ is an ideal in $\bar{\mathfrak{H}}$. Since $\bar{\mathfrak{K}} = \bar{\mathfrak{B}} \oplus \bar{\mathfrak{S}}$ is an ideal in $\bar{\mathfrak{H}}$ and $\bar{\mathfrak{B}}$ is the radical of $\bar{\mathfrak{K}}$, clearly $\bar{\mathfrak{B}}$ is an ideal in $\bar{\mathfrak{H}}$. If $\bar{\mathfrak{A}}$ is 1-dimensional then the algebra $\bar{\mathfrak{N}} \oplus \bar{\mathfrak{A}}$ is not abelian, but it is solvable and has dimension 2. Since $\bar{\mathfrak{B}} \oplus \bar{\mathfrak{S}}$ is in the centralizer of $\bar{\mathfrak{N}} + \bar{\mathfrak{A}}$ the sum $(\bar{\mathfrak{N}} \oplus \bar{\mathfrak{A}}) \oplus (\bar{\mathfrak{B}} \oplus \mathfrak{S})$ is a direct sum of ideals.

LEMMA 5. *With the notation of Lemmas* 3 *and* 4 *let* $\mathfrak{N} \subset \mathfrak{H}$. *Let* $\mathfrak{N}$ *be the radical of* $\mathfrak{G}$. *Then* $\mathfrak{H} \cap \mathfrak{N}$ *is an ideal in* $\mathfrak{G}$.

Proof. If $\mathfrak{H} \cap \mathfrak{N} = \mathfrak{N}$ then the claim is trivially true. If $\mathfrak{H} \cap \mathfrak{N} \neq \mathfrak{N}$, then $\mathfrak{N}' \subset \mathfrak{N} \subset \mathfrak{H} \cap \mathfrak{N}$, where $\mathfrak{N}'$ is the derived algebra of $\mathfrak{N}$. Hence $\mathfrak{H} \cap \mathfrak{N}$ is an ideal in $\mathfrak{N}$. Clearly $\mathfrak{H} \cap \mathfrak{N}$ is an ideal in $\mathfrak{H}$. Thus $\mathfrak{H} \cap \mathfrak{N}$ is an ideal in $\mathfrak{N} + \mathfrak{H}$ which is equal to $\mathfrak{G}$, since $\mathfrak{N} \not\subset \mathfrak{H}$.

LEMMA 6. *With the previous notation, assume that* $\mathfrak{N} \not\subset \mathfrak{H}$. *Let*

$$\mathfrak{G}^* = \mathfrak{G}/\mathfrak{H} \cap \mathfrak{N}, \qquad \mathfrak{N}^* = \mathfrak{N}/\mathfrak{H} \cap \mathfrak{N}, \qquad \mathfrak{H}^* = \mathfrak{H}/\mathfrak{H} \cap \mathfrak{N}.$$

Then $\mathfrak{N}^*$ *is the* 1-*dimensional radical of* $\mathfrak{G}^*$ *and* $\mathfrak{H}^*$ *is a semisimple ideal.*

Proof. Obviously $\dim \mathfrak{N}^* = 1$. The adjoint representation of the semisimple algebra $\mathfrak{H}^*$ on $\mathfrak{N}^*$ is trivial since the algebra of derivations on $\mathfrak{N}^*$ is abelian. Hence $\mathfrak{H}^*$ centralizes $\mathfrak{N}^*$ and is therefore an ideal.

LEMMA 7. *With the previous notation, let* $\mathfrak{N} \subset \mathfrak{H}$. *Denote* $\mathfrak{G}/\mathfrak{N}$ *with* $\tilde{\mathfrak{G}}$ *and* $\mathfrak{H}/\mathfrak{N}$ *with* $\tilde{\mathfrak{H}}$. *Then* $\mathfrak{G}$ *is a direct sum of a semisimple ideal* $\tilde{\mathfrak{S}}$ *and a simple ideal* $\tilde{\mathfrak{T}}$ *isomorphic to* $sl(2)$, *and* $\tilde{\mathfrak{S}} \subset \tilde{\mathfrak{H}}$, $\dim \tilde{\mathfrak{T}} \cap \tilde{\mathfrak{H}} = 2$.

Proof. Let π be any homomorphism of $\tilde{\mathfrak{G}}$ onto a simple Lie algebra. Then $\pi(\tilde{\mathfrak{H}}) = \pi(\tilde{\mathfrak{G}})$ or $\dim \pi(\tilde{\mathfrak{H}}) = \dim \pi(\tilde{\mathfrak{G}}) - 1$; in the latter case $\pi(\tilde{\mathfrak{G}}) = sl(2)$ according to Lemma 2. Since all these homomorphisms separate points of $\tilde{\mathfrak{G}}$ there is at least one π with $\dim \pi(\tilde{\mathfrak{H}}) = 2$ Let $\tilde{\mathfrak{S}} = \ker \pi$. Then $\tilde{\mathfrak{S}} \subset \tilde{\mathfrak{H}}$ because otherwise $\tilde{\mathfrak{S}} + \tilde{\mathfrak{H}} = \tilde{\mathfrak{G}}$ contradicting $\pi(\tilde{\mathfrak{H}}) \neq \pi(\tilde{\mathfrak{G}})$. In view of the structure of semisimple Lie algebras there is a simple ideal $\tilde{\mathfrak{T}}$ such that $\tilde{\mathfrak{G}} = \tilde{\mathfrak{S}} \oplus \tilde{\mathfrak{T}}$ and that $\tilde{\mathfrak{T}} \cong sl(2)$, $\dim \tilde{\mathfrak{T}} \cap \tilde{\mathfrak{H}} = 2$.

It is now easy to prove the following:

THEOREM I. *Let* $\mathfrak{G}$ *be an n-dimensional real Lie algebra and* $\mathfrak{H}$ *an* $(n - 1)$-*dimensional subalgebra. Then one and only one of the following cases occurs:*

(i) $\mathfrak{H}$ *is an ideal.*

(ii) $\mathfrak{H}$ *contains an ideal* $\mathfrak{J}$ *of* $\mathfrak{G}$ *such that* $\mathfrak{G}/\mathfrak{J}$ *is isomorphic to the* 2-*dimensional solvable non-commutative Lie algebra and* $\mathfrak{H}/\mathfrak{J}$ *is a* 1-*dimensional subalgebra which is not an ideal.*

(iii) *$\mathfrak{H}$ contains an ideal $\mathfrak{J}$ of $\mathfrak{G}$ such that $\mathfrak{G}/\mathfrak{J}$ is isomorphic to $sl(2)$ and $\mathfrak{H}/\mathfrak{J}$ is a 2-dimensional solvable subalgebra.*

Proof. If $\mathfrak{R} \subset \mathfrak{H}$ then Lemma 7 yields (iii) with the counterimage of $\bar{\bar{\mathfrak{S}}}$ in $\mathfrak{G}$ as $\mathfrak{J}$. If $\mathfrak{R} \not\subset \mathfrak{H}$, $\mathfrak{N} \subset \mathfrak{H}$, then Lemma 6 gives (i), since the full counterimage of $\mathfrak{H}^*$ is $\mathfrak{H}$ and $\mathfrak{H}^*$ is an ideal. If $\mathfrak{N} \not\subset \mathfrak{H}$, then, by Lemma 4 we have either (ii) with the full counterimage of $\bar{\mathfrak{B}} + \bar{\bar{\mathfrak{S}}}$ as $\mathfrak{J}$ provided that $\dim \bar{\mathfrak{A}} = 1$, or else we have case (i) again if all of $\bar{\mathfrak{H}}$ is in the centralizer of $\bar{\mathfrak{N}}$.

As the referee points out Theorem I holds for more general ground fields than the reals. Indeed, from Lemma 3 on we have only used that the ground field has characteristic 0, and in Lemma 1 we used the orderability of the ground field. In Lemma 2, however, we used the fact that every complex semisimple Lie algebra has a compact real form. A closer inspection of Weyl's proof of this fact as given e.g. in [4, p. 147 ff.] shows that this result remains valid for split simple Lie algebras over a field $K(i)$, where K is an ordered field in which every positive element has a square root and where $i^2 = -1$; more specifically, if $\mathfrak{G}^*$ is a split simple Lie algebra over $K(i)$ then there exists a compact simple Lie algebra $\mathfrak{G}$ over K such that $\mathfrak{G}^* \cong \mathfrak{G} \oplus K(i)$. Thus Lemma 2 remains valid for Lie algebras over an ordered field K in which every positive element has a square root—provided that $K(i)$ is split simple (which means that the characteristic values of all elements of some Cartan subalgebra acting and adjoint operation are in $K(i)$). This is certainly so if $K(i)$ is algebraically closed which is the same as saying that K is formally real (i.e. every polynomial in $K[x]$ of odd order has a root in K) or is itself algebraically closed. Thus Theorem I remains valid if "real Lie algebra" is replaced by "Lie algebra over a formally real or algebraically closed field of characteristic zero". It remains an open question whether or not Theorem I is true for Lie algebras over a field of characteristic 0.

In the sequel let $\mathfrak{G}$ be the Lie algebra of an n-dimensional real connected Lie group G and $\mathfrak{H}$ the Lie algebra of a (not necessarily closed) $(n-1)$-dimensional Lie subgroup H. Let I be the Lie subgroup of the ideal $\mathfrak{J}$ mentioned in Theorem I. We denote with $(X, g) \to X \cdot g$ the adjoint representation of G on $\mathfrak{G}$ defined by $g^{-1}(\exp X)g = \exp(X \cdot g)$. We define $\mathfrak{E}$ to be the set of all $X \in \mathfrak{G}$ such that $X \cdot g \notin \mathfrak{H}$ for all $g \in G$ and let $\mathfrak{E}^*$ be its closure in $\mathfrak{G}$ with respect to the natural vector space topology on $\mathfrak{G}$. In other words, $\mathfrak{E}$ is the complement of $\bigcup \{\mathfrak{H} \cdot g : g \in G\}$. We want to exploit Theorem I in order to describe the set $\mathfrak{E}$.

LEMMA 8. *If $\mathfrak{J}$ is any ideal of $\mathfrak{G}$ contained in $\mathfrak{H}$, then $\mathfrak{E} + \mathfrak{J} = \mathfrak{E}$. Conversely, if $\mathfrak{E}_{\mathfrak{J}} \subset \mathfrak{G}/\mathfrak{J}$ is the set of all $X + \mathfrak{J}$ such that $(X + \mathfrak{J}) \cdot g \notin \mathfrak{H}/\mathfrak{J}$ for all $g \in G$, then $\mathfrak{E}$ is the full counterimage of $\mathfrak{E}_{\mathfrak{J}}$ in $\mathfrak{G}$.*

Proof. Let $X \in \mathfrak{G}$, $Y \in \mathfrak{J}$ *and* $g \in G$. Then $(X + Y) \cdot g \in \mathfrak{H}$ is equivalent to $X \cdot g \in \mathfrak{H}$ since $Y \cdot g \in \mathfrak{J} \subset \mathfrak{H}$. Thus $X \in \mathfrak{E}$ iff $x + \mathfrak{J} \in \mathfrak{E}_{\mathfrak{J}}$.

LEMMA 9. *If H is normal, i.e. if $\mathfrak{H}$ is an ideal, then $\mathfrak{E} = \mathfrak{G} \backslash \mathfrak{H}$, $\mathfrak{E}^* = \mathfrak{G}$.*

Proof. $X \in \mathfrak{G}$ and $X \cdot g \in \mathfrak{H}$ implies $X \in \mathfrak{H} \cdot g^{-1} = \mathfrak{H}$.

LEMMA 10. *If G is the nonabelian 2-dimensional solvable connected Lie group then $\mathfrak{C}^* = \mathfrak{C} \cup \{0\}$ is the derived algebra of $\mathfrak{G}$.*

Proof. The group G is the split extension of $\mathbf{R}$ by $\mathbf{R}^{\times}$, the multiplicative group of positive reals under the natural action. Since $H^1(\mathbf{R}^{\times}, \mathbf{R}) = 0$ with the natural action, all complementary subgroups to the normal subgroup are conjugate. One of these is H. Thus the elements different from 1 on the commutator subgroup (which is the unique nontrivial closed normal subgroup) are the only ones which have no conjugate in H. This proves the assertion.

LEMMA 11. *If $\mathfrak{G}/\mathfrak{J}$ is the 2-dimensional solvable nonabelian algebra, then $\mathfrak{C}^* = \mathfrak{C} \cup \mathfrak{J}$ is an $(n - 1)$-dimensional ideal and is equal to $\mathfrak{G}' + \mathfrak{J}$, where $\mathfrak{G}'$ denotes the derived algebra of $\mathfrak{G}$.*

Proof. Since the adjoint representations of the universal covering group of G and of G have the same effect on $\mathfrak{G}$ we may as well assume that G is simply connected. Then I is closed as a normal connected Lie subgroup of a simply connected Lie group and G/I is the 2-dimensional solvable nonabelian connected Lie group. The assertion then follows from Lemmas 8 and 10.

LEMMA 12. *If $G = Sl(2)$ then $\mathfrak{C}^* = \mathfrak{C} \cup (\mathfrak{C}^* \cap \mathfrak{H})$ where $\mathfrak{C}^* \cap \mathfrak{H}$ is a 1-dimensional linear subspace of $\mathfrak{H}$, and $\mathfrak{C}^*$ is a 2-dimensional solid double cone bounded by a quadratic surface whose singular point is the origin of $\mathfrak{G}$. Moreover X is in $\mathfrak{C}$ iff* $\exp \mathbf{R}X$ *is a circle group.*

Proof. (a) We show first that H, a 2-dimensional solvable subgroup of G is conjugate to the subgroup of all matrices

$$\begin{bmatrix} r & s \\ 0 & 1/r \end{bmatrix}, \qquad r > 0, r, s \in \mathbf{R}.$$

For this purpose it is sufficient to exhibit at least one irreducible 1-dimensional invariant subspace of $\mathbf{R}^2$ under the action of H; then we can pick a coordinate system such that all elements of H have triangular matrices with respect to this coordinate system; since the determinant of every element $h \in H$ is 1, the product of the diagonal elements must be 1; thus H is conjugate to the the group described within $Gl(2)$, but then also in $Sl(2)$ since some scalar multiple of any element of $Gl(2)$ is in $Sl(2)$. Now let us assume that $\mathbf{R}^2$ is irreducible under H. Let $\mathfrak{V}$ be an irreducible invariant subspace under H', the abelian commutator group of H, which is isomorphic to $\mathbf{R}$. If $\mathfrak{V} = \mathbf{R}^2$ then H' acts as a rotation group on $\mathbf{R}^2$, but this is impossible because then an infinite cyclic subgroup of H' would act trivially on $\mathbf{R}^2$ contradicting the fact that H acts faithfully on $\mathbf{R}^2$. Hence $\mathfrak{V}$ is 1-dimensional. Since $\mathfrak{V}$ is invariant under H' and H' is normal in H, the subspace $\mathfrak{V} \cdot h$ is invariant under H' for all $h \in H$. Since $\mathfrak{V}$ is not invariant under H and H is connected there are at least three different subspaces $\mathfrak{B}$, $\mathfrak{B}\cdot h$, $\mathfrak{B}\cdot h'$ invariant under H', which implies that H' consists of scalar multiplications; but since all elements of H' have determinant 1 and H' is connected and therefore can only contain multiplications with positive scalars, H' must act trivially, again contradicting the fact that H acts faithfully on $\mathbf{R}^2$.

(b) $\mathfrak{G} = sl(2)$ is isomorphic to the Lie algebra of all endomorphisms of the vector space $\mathbf{R}^2$ with trace 0. It has a basis of the form

$$U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad W = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with $[U, W] = 2U$, $[W, V] = 2V$, $[V, U] = W$, and $\mathfrak{H}$ is spanned by U and W. By straightforward computation we find

$X\cdot \exp tY = Xe^{t\,\mathrm{ad}Y}$	$Y = U$	$Y = V$	$Y = W$
$X = U$	U	$U - tW - t^2V$	$e^{2t}U$
$X = V$	$V + tW - t^2U$	V	$e^{-2t}V$
$X = W$	$W - 2tU$	$W + 2tV$	W

Every element of $G = Sl(2)$ is representable in the form

$$\exp vV \exp wW \exp uU \in (\exp \mathbf{R}V)H, \qquad \exp vV = \begin{bmatrix} \cos v & \sin v \\ -\sin v & \cos v \end{bmatrix}.$$

The following are congruences modulo the vector space $\mathfrak{H}$:

$$\begin{aligned}(V - aU + bW)\cdot \exp vV \exp wW \exp uU & \\ &\equiv (1 + 2bu + au^2)V\cdot \exp wW \exp uU \\ &= (1 + 2bu + au^2)e^{-2w}V\cdot \exp uU \\ &\equiv (1 + 2bu + au^2)e^{-2w}V.\end{aligned}$$

Thus the element $V - aU + W$ has a conjugate in $\mathfrak{H}$ iff the equation

$$au^2 + 2bu + 1 = 0$$

is solvable in $\mathbf{R}$. This is the case iff $b^2 + a \geq 0$ and not $a = b = 0$. The set $\mathfrak{C}$ therefore contains exactly the points $c(V - aU + bW)$ with $c \in \mathbf{R}$, $b^2 + a < 0$ and all conjugates of cV. Now

$$\begin{aligned}V\cdot \exp vV \exp wW \exp bU &= V\cdot \exp wW \exp bU \\ &= e^{-2w}V\cdot \exp bU = e^{-2w}(V - b^2U + bW).\end{aligned}$$

Thus $\mathfrak{C}$ is the collection of all points $c(V - aU + bW)$ with $a, b, c \in \mathbf{R}$, $b^2 + a \leq 0$. Now $\mathfrak{C}^* = \mathfrak{C} \cup \mathbf{R}U$ is a solid double cone mapped into itself by all scalar multiplications and bounded by a nondegenerate quadratic surface. Moreover, if X is in $\mathfrak{C}$ i.e. if X is a scalar multiple of $V - aU + bW$, $a + b^2 \leq 0$, then $Y = 1/sX \in \mathfrak{C}$ with $s = (-\det X)^{1/2} = (1 - a - b^2)^{1/2}$. Direct calculation shows $Y^2 = -E$, E = unit matrix, and $\exp tY = E \cos t + Y \sin t$, i.e. $\exp \mathbf{R}X$ is a circle group. Conversely, $X \notin \mathfrak{C}$ implies that a conjugate of $\exp \mathbf{R}X$ is in H which does not contain any circle group.

LEMMA 13. *If* $\mathfrak{G}/\mathfrak{J} = sl(2)$ *then* $\mathfrak{C}^*$ *is an n-dimensional solid double cone bounded by a quadratic hypersurface and such that* $r\mathfrak{C}^* = \mathfrak{C}^*$ *for all* $r \in \mathbf{R}$, $r \neq 0$, *and such that* $X \in \mathfrak{C}^*$ *implies* $X + \mathfrak{J} \subset \mathfrak{C}^*$. *Moreover* $\mathfrak{C} = \mathfrak{C}^* \backslash (\mathfrak{C}^* \cap \mathfrak{H})$, *and* $\mathfrak{C}^* \cap \mathfrak{H}$ *is an* $(n-2)$*-dimensional linear space.*

Proof. This follows from Lemma 8 and Lemma 11 along the same lines given in the proof of Lemma 11.

We have now proved

THEOREM II. *Let G be a connected real n-dimensional Lie group and* $\mathfrak{G}$ *its Lie algebra, let* $\mathfrak{H}$ *be an* $(n-1)$*-dimensional subalgebra and denote with* $\mathfrak{C} \subset \mathfrak{G}$ *the set of all* $X \in \mathfrak{G}$ *such that* $X \cdot g \notin \mathfrak{H}$ *for all* $g \in G$ *(where* $g^{-1}(\exp X)g = \exp(X \cdot g)$*). Let* $\mathfrak{C}^*$ *be the closure of* $\mathfrak{C}$ *in* $\mathfrak{G}$. *Then* $\mathfrak{C} = \mathfrak{C}^* \backslash (\mathfrak{C}^* \cap \mathfrak{H})$ *and one of the following cases occurs (in accordance with Theorem I):*

(i) $\mathfrak{C}^* = \mathfrak{G}$.

(ii) $\mathfrak{C}^*$ *is an* $(n-1)$*-dimensional ideal and is equal to* $\mathfrak{G}' + \mathfrak{J}$, *where* $\mathfrak{G}'$ *is the derived algebra of* $\mathfrak{G}$.

(iii) $\mathfrak{C}^*$ *is a solid n-dimensional double cone bounded by a quadratic hypersurface.* $\mathfrak{C}^*$ *is invariant under all scalar multiplications and contains* $X + \mathfrak{J}$ *with* $X \in \mathfrak{C}^*$.

COROLLARY 1 TO THEOREM II. *There are* $n - 1$ *linearly independent vectors* $X_1, \cdots, X_{n-1}$ *of* $\mathfrak{G}$ *such that* $X_i \cdot g \notin \mathfrak{H}$, $i = 1, \cdots, n-1$, $g \in G$. *In case* (i) *and case* (iii) *there is even an n-th vector* X_n *linear independent of* X_i, $i = 1, \cdots, n-1$ *such that* $X_n \cdot g \notin \mathfrak{H}$, *for all* $g \in G$. *In case* (ii) *for any vector* Y *linearly independent of* $X_1, \cdots, X_{n-1}$ *there is an element* $c \in C$, C = *Lie subgroup with Lie algebra* $\mathfrak{C}^*$, *such that* $Y \cdot c \in \mathfrak{H}$.

The corollary follows directly from the fact that in case (i) and (iii) the set $\mathfrak{C}$ contains a basis of $\mathfrak{G}$, and from the fact that in the 2-dimensional solvable nonabelian connected Lie group two nonnormal one-parameter groups are conjugate under an element of the commutator group (see Lemma 10).

Remark. For any given dimension ≥ 3 there are Lie groups and Lie algebras of type (i), (ii) and (iii); case (i) is realized by abelian algebras, case (ii) (resp. (iii)) by an appropriate direct product of the 2-dimensional solvable nonabelian algebra (resp. by $sl(2)$) with some abelian algebra.

COROLLARY 2 TO THEOREM II. *If everything is as in Theorem II, then* $X \in \mathfrak{C}$ *implies* $G = H \exp \mathbf{R}X$.

Proof. We may again assume that G is simply connected so that I is closed. If the assertion is true for the homomorphic image G/I, then it is true for G itself. We may therefore assume that $I = 1$. If G is the 2-dimensional solvable group, then exp $\mathbf{R}X$ is the commutator group G' and $G = HG'$. In order to prove the assertion for the covering group of $Sl(2)$ it is sufficient to prove it for $Sl(2)$. But in this case exp $\mathbf{R}X = C$ is a circle group by Lemma 12 and it is known that $G = HC$.

References

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[1] In pursuit of his investigations about semigroups on manifolds with boundary, Professor Horne has suggested the present problem to the author.