REMARKS ON NONLINEAR FUNCTIONAL EQUATIONS, III

BY
Felix E. Browder[1]

Introduction

In two preceding papers under the same title [2], [3], the writer has studied nonlinear functional equations in reflexive complex Banach spaces involving operators T mapping such a Banach space X into its dual X^* and satisfying inequalities of the type

$$(1) \qquad |(Tu - Tv, u - v)| \geq k(u, v)h(\| u - v \|)$$

where (w, v) denotes the pairing between w of X^* and v of X.

A representative result of this type is Theorem 1 of [2] which asserts that if T is demicontinuous and saeisfies the two conditions:

(i) There exists a real function $c(r)$ on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that for all u of X,

$$(2) \qquad |(Tu, u)| \geq c(\| u \|)\| u \|;$$

(ii) For each $N > 0$, there exists a continuous strictly increasing real function $k_N(r)$ on R^1 with $k_N(0) = 0$ such that for $\| u \| \leq N$, $\| v \| \leq N$

$$(3) \qquad |(Tu - Tv, u - v)| \geq k_N(\| u - v \|)\| u - v \|;$$

then T maps X onto X^*.

This theorem is an extension and generalization of a theorem of Zarantonello [7] which asserts that if T is a continuous map of a Hilbert space H into H which carries bounded sets into bounded sets and such that for a suitable constant $c > 0$

$$|(Tu - Tv, u - v)| \geq c \| u - v \|^2$$

then T maps H onto H.

Further extensions were given by the writer in [3] in which on the one hand the inequality (3) of (ii) was modified to

$$(4) \quad |(Tu - Tv, u - v)| \geq k_N(\| u - v \|) - |(C_N u - C_N v, u - v)|$$

where for each $N > 0$, C_N is some completely continuous map of X into X^*, and on the other, the demicontinuity of T was replaced by the condition that $T = L + G$ where G is demicontinuous and maps bounded sets of X into bounded sets of X^* while L is a closed densely defined linear map of X into X^* such that its adjoint L^* is the closure of its restriction to $D(L) \cap D(L^*)$.

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It is our object in the present paper to establish a different sort of extension of Theorem 1 of [2], namely one in which the hypotheses are completely local in character.

Our principal results are the following:

THEOREM 1. *Let X be a reflexive complex Banach space, X^* its dual, T a mapping of X into X^* which is demicontinuous (i.e. continuous from the strong topology of X to the weak topology of X^*). Let U be an open subset of X and suppose that there exists a continuous strictly increasing real function $k(r)$ on R^1 with $k(0) = 0$ such that*

$$|(Tu - Tv, u - v)| \geq k(\| u - v \|)\| u - v \| \tag{5}$$

for all u and v in U.

Then $T(U)$ is open in X^* and T is an open, one-to-one mapping of U into X^*.

THEOREM 2. *Let X be a reflexive complex Banach space, T a continuous mapping of X into X^*. Suppose that both of the following conditions are satisfied:*

(1) *For each point u_0 in X, there exists an open neighborhood U of u_0 in X and a continuous increasing real function $k_U(r)$ on R^1 with $k_U(0) = 0$ such that*

$$|(Tu - Tv, u - v)| \geq k_U(\| u - v \|)\| u - v \| \tag{6}$$

for all u and v in U.

(2) *There exists a positive continuous decreasing function $h(r)$ on R^1 with*

$$\int_0^\infty h(r)\, dr = +\infty$$

and for each point u_0 in X, an open neighborhood V of u_0 in X such that

$$\| Tu - Tv \| \geq h(\| u_0 \|)\| u - v \| \tag{7}$$

for all u and v in V.

Then T is a one-to-one bicontinuous map of X onto X^*.

Section 1 is devoted to the proof of Theorem 1. Section 2 contains the proof of Theorem 2.

Using the results of the present paper, we shall present elsewhere an extension to the elliptic, non-strongly elliptic case of the existence and uniqueness theorems given by the writer in [1] for nonlinear strongly elliptic systems of partial differential equations in generalized divergence form.

1. Proof of Theorem 1

It suffices to show that $T(U)$ contains a neighborhood of $T(u_0)$ in X^*. If we replace T by T_1 where $T_1 u = T(u + u_0) - Tu_0$, T_1 satisfies the same hypotheses as T and we may assume without loss of generality that $u_0 = 0$ and $Tu_0 = 0$.

It follows from the hypothesis of Theorem 1 that there exists $r > 0$ and a continuous increasing function $k(r)$ with $k(0) = 0$ such that

$$|(Tu - Tv, u - v)| \geq k(\| u - v \|)\| u - v \| \tag{8}$$

for $\| u \| \leq r$, $\| v \| \leq r$. In particular, setting $v = 0$, we have

$$|(Tu, u)| \geq k(\| u \|)\| u \| \tag{9}$$

for $\| u \| \leq r$.

Let w be any element of X^* with $\| w \| < k(r)$. Then for $\| u \| = r$, we have

$$|(Tu - w, u)| \geq |(Tu, u)| - \| w \| \cdot \| u \| \geq k(r)r - \| w \| \cdot r > 0.$$

If we replace T by T_2, where $T_2 u = Tu - w$, we have for T_2,

$$|(T_2 u - T_2 v, u - v)| \geq k(\| u - v \|)\| u - v \| \tag{10}$$

for $\| u \| \leq r$, $\| v \| \leq r$, and

$$|(T_2 u, u)| > 0 \tag{11}$$

for $\| u \| = r$. To show that there exists u_1 with $\| u_1 \| < r$ such that $Tu_1 = w$, it suffices to show that there exists u_1 with $\| u_1 \| < r$ with $T_2 u_1 = 0$. We may replace T_2 by T, assume that T satisfies the inequalities (10) and (11) imposed on T_2, and we need to prove under these assumptions that there exists u_0 with $\| u_0 \| < r$ such that $Tu_0 = 0$.

If X is of finite dimension, Theorem 1 follows immediately from the Brouwer theorem on invariance of domain since the inequality (5) implies that T is locally one-to-one. Hence we may assume without loss of generality that X is of infinite dimension.

We proceed as in the proof of Theorem 2 of [3]. Let Λ be the directed set of finite-dimensional subspaces F of X of dimension ≥ 2. For each $F \in \Lambda$, let j_F be the injection map of F into X, j_F^* the dual projection map of X^* onto F^*. We form the continuous mapping T_F of F into F^* by setting

$$T_F = j_F^* T j_F .$$

For $u \in F$, $(T_F u, u) = (Tu, u)$. In particular, if $\| u \| = r$,

$$|(T_F u, u)| = |(Tu, u)| > 0.$$

Since F is of dimension ≥ 2, it follows from Theorem 1 of [3] that there exists u_F in F with $\| u_F \| \leq r$ such that $T_F u_F = 0$.

By the weak compactness of closed balls in the reflexive B-space X, the directed set $\{u_F ; F \in \Lambda\}$ has at least one weak limit point u_0 in X with $\| u_0 \| \leq r$. As in the proof of Theorem 2 of [3], if $F_1 \subset F$ with F_1, $F \in \Lambda$, we have

$$k(\| u_F - u_{F_1} \|)\| u_F - u_{F_1} \| \leq |(Tu_{F_1}, u_F)|$$

or if q is the continuous increasing function which is the inverse function of $k(r)r$,

$$\| u_F - u_{F_1} \| \leq q(|(Tu_{F_1}, u_F)|).$$

Since u_0 lies in the closure of the set $\{u_F \mid F \in \Lambda, F_1 \subset F\}$ and since the function of v on the ball $\{v \mid \| v \| \leq r\}$ given by

$$\| v - u_{F_1} \| - q(|(Tu_{F_1}, v)|)$$

is lower semi-continuous in the weak topology, it follows that

$$\| u_0 - u_{F_1} \| \leq q(|(Tu_{F_1}, u_0)|).$$

On the other hand, if $u_0 \in F_1$ we have

$$(Tu_{F_1}, u_0) = (Tu_F, j_{F_1} u_0) = (j_{F_1}^* Tu_{F_1}, u_0) = (T_{F_1} u_{F_1}, u_0) = 0.$$

Hence for such F_1, $u_0 = u_{F_1}$.

Let v be any element of X and let F_1 be an element of Λ containing u_0 and v. Then

$$(Tu_0, v) = (Tu_0, j_{F_1} v) = (j_{F_1}^* Tu_0, v) = (j_{F_1}^* Tu_{F_1}, v) = (T_{F_1} u_{F_1}, v) = 0.$$

Hence $Tu_0 = 0$, Q.E.D.

2. Proof of Theorem 2

We begin with the following useful lemma:

LEMMA 1. *Let f be a local homeomorphism of the Banach space X into the Banach space Y. Suppose that if C is an open curve in X mapped homeomorphically by f on an open segment in Y of the form*

$$S = \{v \mid v \in Y, v = tv_0, 0 \leq t < t_0\},$$

C must be of finite length.

Then f is a homeomorphism of X onto Y.

Proof of Lemma 1. The proof was given by Paul Lévy in [5], using an idea applied for finite dimensional spaces by Hadamard ([4]).

Proof of Theorem 2. By hypothesis (1) and Theorem 1, T is a local homeomorphism of X into X^*. Suppose T is not a homeomorphism. Then by Lemma 1, there exists a curve C of infinite length, an element v_0 of X^*, and $t_0 > 0$ such that f maps C homeomorphically onto S. Let C be given parametrically by

$$u = u(t), \qquad 0 \leq t < t_0$$

where $T(u(t)) = tv_0$.

We consider two cases: (a) C is bounded; (b) C is unbounded. In case (a), it follows from hypothesis (2) that there exists a constant $c > 0$ and for each point u_0 of C a neighborhood V of u_0 in X such that for u and v in U

$$\| Tu - Tv \| \geq c \| u - v \|.$$

Since C is an open curve of infinite length, for a given $M > 0$, we may choose $t_1 < t_0$ such that the compact curve C_1 given by $\{u = u(t); 0 \leq t \leq t_1\}$ has length greater than M. In particular there must exist a sequence of parameter values $0 < s_1 < s_r < \cdots < s_r = t_1$ with each pair $u_{s_{j-1}}$, u_{s_j} lying in some neighborhood V as above such that

$$\sum_{j=1}^{r} \| u_{s_j} - u_{s_{j-1}} \| \geq M.$$

Hence

$$\sum_{j=1}^{r} \| Tv_{s_j} - Tu_{s_{j-1}} \| \geq cM.$$

However, $Tu_{s_j} - Tu_{s_{j-1}} = (s_j - s_{j-1})v_0$. Hence

$$\sum_{j=1}^{r} (s_j - s_{j-1}) \| v_0 \| \geq cM$$

which implies that

$$t_1 \| v_0 \| \geq cM$$

or $M \leq c^{-1}t_1 \| v_0 \|$. Since we may choose M arbitrarily large, this yields a contradiction for case (a).

In case (b), we may find an infinite sequence of parameter values

$$0 = s_0 < s_1 < s_2 < s_3 < \cdots < t_0$$

such that for each j

$$\| u_{s_j} \| = j; \quad \| u_s \| \leq j \qquad \text{for} \quad s \leq s_j$$

We may then choose for each j, a finite sequence of parameter values

$$s_j = t_0^{(j)} < t_1^{(j)} < t_2^{(j)} < \cdots < t_{r(j)}^{(j)} = s_{j+1}$$

such that for every k, the pair $u_{t_k^{(j)}}$, $u_{t_{k+1}^{(j)}}$ lies in a single neighborhood V as above. Then

$$\begin{aligned}
\| v_0 \| t_0 \geq \sum_{j=1}^{\infty} \| v_0 \| (s_{j+1} - s_j) \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{r(j)} \| v_0 \| (t_k^{(j)} - t_{k-1}^{(j)}) \\
&= \sum_{j=1}^{\infty} \sum_{k=1}^{r(j)} \| Tu_{t_k^{(j)}} - Tu_{t_{k-1}^{(j)}} \| \\
&\geq \sum_{j=1}^{\infty} \sum_{k=1}^{r(j)} h(j+1) \| u_{t_k^{(j)}} - u_{t_{k+1}^{(j)}} \| \\
&\geq \sum_{j=1}^{\infty} h(j+1) \{ \sum_{k=1}^{r(j)} \| u_{t_k^{(j)}} - u_{t_{k-1}^{(j)}} \| \} \\
&\geq \sum_{j=1}^{\infty} h(j+1) \| u_{s_{j+1}} - u_{s_j} \| \\
&\geq \sum_{j=1}^{\infty} h(j+1) \\
&\geq \int_2^{\infty} h(r)\, dr = +\infty
\end{aligned}$$

which is a contradiction for case (b). Thus the proof of Theorem 2 is complete.

Remark. After the present paper was submitted for publication, the writer received a mimeographed manuscript from George Minty entitled *Postcript to Zarantonello's Theorem* in which he obtains results in Hilbert space of the type considered here under local hypotheses. His hypotheses are considerably more restrictive than those imposed above, since Minty assumes the existence of positive constants $c > 0$, $r > 0$ such that

$$|(Tu - Tv, u - v)| \geq c \| u - v \|^2$$

in each ball of radius r in the Hilbert space H. It follows easily from the latter hypothesis that T is a covering mapping of H onto H and hence a homeomorphism. In the present paper, we show that T is a local homeomorphism and use Lemma 1 of Section 2 to prove that T is a homeomorphism. For detailed studies of local homeomorphisms and covering mappings and their interrelations, see the writer's papers [8] and [9].

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Institute for Advanced Study
Princeton, New Jersey
University of Chicago
Chicago, Illinois