REMARKS ON NONLINEAR FUNCTIONAL EQUATIONS, II

BY
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Introduction

In a series of recent papers, the writer [1]–[13] and G. J. Minty [15]–[17] have studied nonlinear functional equations in Banach spaces involving monotone operators, i.e. operators T from a Banach space X to its dual X^* for which

$$\text{Re}\,(Tu - Tv, u - v) \geq 0 \tag{1}$$

for all u and v in X. A recent theorem of Zarantonello [18] for continuous bounded operators in Hilbert space obtains similar results for operators T satisfying the condition

$$|\,(Tu - Tv, u - v)\,| \geq c\,\|\,u - v\,\|^2. \tag{2}$$

In a preceding paper under the same title [14], the writer generalized and sharpened Zarantonello's result to obtain the following theorem:

THEOREM [14]. *Let X be a reflexive complex Banach space, X^* its dual, (w, u) the pairing between w in X^* and u in X. Let T be a mapping from X to X^* which is demicontinuous* [2] *(i.e. T is continuous from the strong topology of X to the weak topology of X^*). Suppose that T satisfies both of the following conditions:*

(i) *There exists a continuous real-valued function $c(r)$ on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that*

$$|\,(Tu, u)\,| \geq c(\|\,u\,\|)\|\,u\,\| \tag{3}$$

for all u in X.

(ii) *For each $N > 0$, there exists a continuous increasing real-valued function $k_N(r)$ on R^1 with $k_N(0) = 0$ such that*

$$|\,(Tu - Tv, u - v)\,| \geq k_N(\|\,u - v\,\|)\|\,u - v\,\| \tag{4}$$

for all u and v in X with $\|\,u\,\| \leq N$, $\|\,v\,\| \leq N$.

Then T is a one-to-one mapping of X onto X^* and has a continuous inverse.

The serious part of the conclusion of this theorem, is of course, that the range of T is all of X^*.

In the present paper, it is our object to extend this result in two significant directions already considered by the writer in [1]–[13] for monotone operators.

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These directions of extensions consist of the following: First, to admit completely continuous perturbations of Condition (ii); Second, to allow the addition to T of a suitably restricted densely defined closed linear operator. These extensions are based on the following result in finite-dimensional spaces which is interesting in its own right:

THEOREM 1. *Let F be a finite-dimensional complex Banach space of dimension >1, F^* its dual space. Suppose T is a continuous mapping of F into F^* such that for a given $R > 0$, $(Tu, u) \neq 0$ for all u in F with $\| u \| = R$.*
Then there exists u_0 with $\| u_0 \| < R$ such that $Tu_0 = 0$.

A mapping C of X into X^* is said to be completely continuous if C is continuous from the weak topology of X to the strong topology of X^*. The first of our basic results is the following:

THEOREM 2. *Let X be a reflexive complex Banach space of dimension >1, X^* its dual space, T a demicontinuous mapping of X into X^*. Suppose that both of the following conditions are satisfied:*

(i) *There exists a continuous real-valued function $c(r)$ on R^1 with $c(r) \to +\infty$ as $r \to +\infty$ such that*

$$|(Tu, u)| \geq c(\| u \|)\| u \| \tag{3}$$

for all u in X.

(ii) *For each $N > 0$, there exist a continuous increasing real-valued function $k_N(r)$ on R^1 with $k_N(0) = 0$ and a completely continuous mapping C_N of X into X^* such that for all u, v in X with $\| u \| \leq N$, $\| v \| \leq N$,*

$$|(Tu - Tv, u - v)| \geq k_N(\| u - v \|)\| u - v \| - |(C_N u - C_N v, u - v)|. \tag{4}$$

Then T maps X onto X^*.

ADDENDUM TO THEOREMS 1 AND 2. *If X or F are of dimension 1, Theorems 1 and 2 are no longer true.*

For our second general theorem on mappings in Banach spaces, we shal consider mappings T (possibly nonlinear) whose domain $D(T)$ is a dense linear subset of X and with range $R(T)$ in X^*.

THEOREM 3. *Let X be a reflexive complex Banach space of dimension >1, T a densely defined mapping from X to X^* such that $T = L + G$, where*

(a) *G is a demicontinuous function from X to X^* which carries bounded sets of X into bounded sets of X^*;*

(b) *L is a closed densely defined linear mapping from X to X^* such that if L^* is its adjoint (which is also a closed densely defined linear map from X to X^*), then L^* is the closure of its restriction to $D(L) \cap D(L^*)$.*

Suppose that both of the following conditions hold:

(i) *There exists a continuous real-valued function $c(r)$ on R with $c(r) \to +\infty$ as $r \to +\infty$ such that*

$$(5) \qquad |(Tu, u)| \geq c(\| u \|)\| u \|;$$

for all u in $D(T)$.

(ii) *For each $N > 0$, there exist a continuous increasing function $k_N(r)$ on R^1 with $k_N(0) = 0$ and a completely continuous mapping C_N of X into X^* such that*

$$(6) \quad |(Tu - Tv, u - v)| \geq k_N(\| u - v \|)\| u - v \| - |(C_N u - C_N v, u - v)|$$

for all u and v in $D(T)$ with $\| u \| \leq N$, $\| v \| \leq N$.

Then if $R(T)$ is the range of T, $R(T) = X^*$.

Section 1 is devoted to the proof of Theorem 1, Section 2 contains the proof of Theorem 2, and Section 3 gives the proof of Theorem 3.

1. Proof of Theorem 1

Let F be a Banach space of finite dimension $m \geq 1$. For $R > 0$, let

$$S_R = \{u \mid u \in F, \| u \| = R\}, \qquad B_R = \{u \mid u \in F, \| u \| < R\},$$

S_R is homeomorphic to a sphere of real dimension $(2m - 1)$ while B_R is homeomorphic to a real ball of dimension $2m$.

Lemma 1. *Let S^1 be the unit circle in C^1, i.e. $S^1 = \{\lambda \mid \lambda \in C^1, |\lambda| = 1\}$. Let s be a continuous mapping of S_R into S^1 where $R > 0$. If $m > 1$, s is homotopic to the constant map s_0; ($s_0(u) = 1$ for all u in S_R).*

Proof of Lemma 1. If $m > 1$, S_R is homeomorphic to S^{2m-1} where $2m - 1 \geq 3$. The lemma follows from the well-known fact that $\pi_j(S^1) = 0$ for $j > 1$. (Indeed, $\pi_j(S^1) = \pi_j(R^1)$ where R^1, the universal covering space of S^1, is contractible.)

Definition. *Let f be a continuous mapping of S_R into F such that $f(u) \neq 0$ for all u in S_R. Let $f^\#$ be the mapping of S_R into S_R given by*

$$f^\#(u) = Rf(u)/\| f(u) \|.$$

The degree of f on S_R over 0 is defined to be the degree of $f^\#$ as a mapping of the $(2m - 1)$-dimensional sphere S_R into itself.

Lemma 2. (a) *If f is the restriction to S_R of a mapping f_0 of F into F and if the degree of f on S_R over 0 is different from zero, then there exists a point u_0 in B_R such that $f_0(u_0) = 0$.*

(b) *If f is homotopic to g as mappings of S_R into $F - \{0\}$, then the degree of f on S_R over 0 is equal to the degree of g on S_R over 0.*

Proof of Lemma 2. *Proof of* (b). $f^\#$ is homotopic to $g^\#$ as maps of S_R into S_R.

Proof of (a). If $f_0(u) \neq 0$ for all u in B_R, define

$$f^t(u) = f_0(tu), \qquad 0 \leq t \leq 1.$$

Then $f^1 = f$, f^0 is the constant map and f is homotopic to f^0. Hence the degree of f on S_R over 0 is zero, contradicting the assumption of (a), Q.E.D.

Proof of Theorem 1. We may assume without loss of generality that F is a Hilbert space and that $F^* = F$. Hence T is a mapping of F into F such that $(Tu, u) \neq 0$ for $u \in S_R$.

Let s be the mapping of S_R into S^1 given by

$$s(u) = (Tu, u)^* \, |(Tu, u)|^{-1}$$

(λ^* = the conjugate of λ). Since $m > 1$, it follows from Lemma 1 that there exists a homotopy

$$\sigma : S_R \times I \to S^1$$

where $I = \{t \mid 0 \leq t \leq 1\}$, such that $\sigma(u, 0) = 1$, $u \in S_R$; $\sigma(u, 1) = s(u)$.

We define the homotopy

$$\zeta : S_R \times I \to F - \{0\}$$

by $\zeta(u, t) = \sigma(u, t)Tu$. Then $\zeta(u, 0) = Tu$, $\zeta(u, 1) = s(u)Tu$.

By Lemma 2(b), the degree of T on S_R over 0 is equal to the degree of T_1 on S_R over 0 where $T_1 u = s(u)Tu$. For T_1, we know that

$$(T_1 u, u) = s(u)(Tu, u) = |(Tu, u)| \geq C_0 > 0$$

for $u \in S_R$.

We now define the homotopy α of T_1 with the injection map J of S_R into F by

$$\alpha : S_R \times I \to F - \{0\}$$

$$\alpha(u, t) = (1 - t)T_1 u + tu.$$

Indeed for $u \in S_R$,

$$(\alpha(u, t), u) = (1 - t)(T_1 u, u) + t \, \| u \|^2 \geq c_0(1 - t) + tR^2 > 0$$

so that $\alpha(u, t) \neq 0$.

Hence the degree of T_1 on S_R over 0 equals the degree of J on S_R over 0, and the latter degree equals 1. Hence the degree of T on S_R over 0 equals 1. By Lemma 2(a), there exists u_0 in B_R such that $Tu_0 = 0$.

Proof of the Addendum. If $m = 1$, let $F = C^1$ and $(w, v) = wv^*$. If f is a continuous map of C^1 into C^1, then

$$|(f(\lambda), \lambda)| = |f(\lambda)\lambda^*| = |f(\lambda)| \cdot R$$

for $|\lambda| = R$. Hence

$$|(f(\lambda), \lambda)| \neq 0$$

if and only if $f(\lambda) \neq 0$, $\lambda \in S_R$.

We can easily construct an f of this sort violating both Theorems 1 and 2, namely

$$f(\lambda) = 1, \quad |\lambda| \leq 1$$

$$= |\lambda|^2, \quad |\lambda| \geq 1.$$

2. Proof of Theorem 2

If Λ is a directed set, $\{u_F ; F \in \Lambda\}$ a function from Λ to X or X^*, we shall denote strong convergence of this function on Λ to u_0 by

$$u_F \longrightarrow u_0$$

and weak convergence on Λ to u_0 by

$$u_F \rightharpoonup u_0 .$$

Proof of Theorem 2. It suffices to show that $0 \in R(T)$.

We assume that $\dim (X) \geq 2$. Let Λ be the directed set of finite-dimensional subspaces F of X of dimension ≥ 2, with Λ ordered by inclusion.

For each $F \in \Lambda$, let j_F be the injection mapping of F into X, j_F^* the dual mapping of X^* onto F^*. We define the continuous mapping T_F of F into F^* by

$$T_F = j_F^* T j_F .$$

For $u \in F$,

$$(T_F u, u) = (j_F^* T_1 u, u) = (Tu, u).$$

Hence, for all u in F

$$|(T_F u, u)| \geq c(\| u \|)\| u \| > 0 \tag{2.1}$$

for $\| u \| = R$ with R sufficiently large but independent of F in Λ.

Applying Theorem 1, there exists u_F in F with $\| u_F \| < R$ such that

$$T_F u_F = 0.$$

We choose one such u_F for each F in Λ. Since X is reflexive, each closed ball in X is weakly compact. Therefore the function $\{u_F : F \in \Lambda\}$ on the directed set Λ has at least one limit point u_0 in X in the weak topology. We shall show that $Tu_0 = 0$.

Let u be an arbitrary element of X. Since $\dim X \geq 2$, there exists an element F_0 of Λ such that $u \in F_0$. Let F be any element of Λ such that $F_0 \subset F$. Then:

$$(Tu_F , u) = (Tu_F , j_F u) = (T_F u_F , u) = 0.$$

Thus

$$Tu_F \rightharpoonup 0.$$

Let F_1 be an arbitrary element of Λ. For any F in Λ such that $F_1 \subset F$, we have

$$\begin{aligned} k_R(\| u_F - u_{F_1} \|)\| u_F - u_{F_1} \| \leq &\ |(Tu_F - Tu_{F_1} , u_F - u_{F_1})| \\ &+ |(Cu_F - Cu_{F_1} , u_F - u_{F_1})|. \end{aligned} \tag{2.2}$$

For the first term on the right side of (2.2),

$$(Tu_F - Tu_{F_1} , u_F - u_{F_1}) = (Tu_F , u_F - u_{F_1}) + (Tu_{F_1} , u_{F_1}) - (Tu_{F_1} , u_F)$$

where

$$(Tu_F, u_F - u_{F_1}) = (T_F u_F, u_F - u_{F_1}) = 0,$$

and

$$(Tu_{F_1}, u_{F_1}) = (T_{F_1} u_{F_1}, u_{F_1}) = 0.$$

Hence

$$\tag{2.3} \begin{aligned} k_R(\| u_F - u_{F_1} \|)\| u_F - u_{F_1} \| \\ \leq |(Tu_{F_1}, u_F)| + |(Cu_F - Cu_{F_1}, u_F - u_{F_1})|. \end{aligned}$$

Let $q_R(r)$ be the continuous increasing function of r on R^1 which is the inverse function of $k_R(r)r$. Then the inequality (2.3) may be written

$$\tag{2.4} \| u_F - u_{F_1} \| \leq q_R(|(Tu_{F_1}, u_F)| + |(Cu_F - Cu_{F_1}, u_F - u_{F_1})|).$$

Consider the function s on X given by

$$\tag{2.5} s(v) = \| v - u_{F_1} \| - q_R(|(Tu_{F_1}, v)| + |(Cv - Cu_{F_1}, v - u_{F_1})|).$$

We consider s restricted to B, the closed ball $\{u \mid \| u \| \leq R\}$ in X taken in the weak topology. We know that $|(Tu_{F_1}, v)|$ is continuous in v on B, the mapping

$$v \rightarrow Cv - Cu_{F_1}$$

is continuous from B to the strong topology on X^*, and since B is bounded,

$$|(Cv - Cu_{F_1}, v - u_{F_1})|$$

is continuous on B. Since q_R is continuous on R^1, it follows that

$$q_R(|(Tu_{F_1}, v)| + |(Cv - Cu_{F_1}, v - u_{F_1})|)$$

is continuous in v on B. On the other hand, the function $\| v - u_{F_1} \|$ is lower semicontinuous in v on B. Hence $s(v)$ is lower semicontinuous in v on B. On the set $\{u_F ; F \in \Lambda, F_1 \subset F\}$, $s(v) \leq 0$. Hence $s(v) \leq 0$ on the closure of this set in B. Since u_0 lies in this closure, $s(u_0) \leq 0$, i.e.

$$\tag{2.6} \| u_0 - u_{F_1} \| \leq q_R(|(Tu_{F_1}, u_0)| + |(Cu_0 - Cu_{F_1}, u_0 - u_{F_1})|).$$

Given $\varepsilon > 0$, there exists $\delta > 0$ such that $q(r) < \varepsilon$ for $r < \delta$. We may choose $F_2 \in \Lambda$ so that for $F_1 \supset F_2$ we have

$$|(Tu_{F_1}, u_0)| < \delta/2.$$

We may choose such an F_1 with

$$|(Cu_0 - Cu_{F_1}, u_0 - u_{F_1})| < \delta/2.$$

Then

$$\| u_0 - u_{F_1} \| < \varepsilon.$$

Thus u_0 lies in the strong closure of the set $\{u_F ; F \in \Lambda, F_2 \subset F\}$ for every F_2 in Λ. Since T is demicontinuous, Tu_0 lies in the closure of the set

$$\{Tu_F ; F \in \Lambda, F_2 \subset F\}$$

for every F_2 in Λ. Since $Tu_F \rightharpoonup 0$, the only point in the intersection of these closures for all F_2 in Λ is 0. Hence $Tu_0 = 0$, Q.E.D.

3. Proof of Theorem 3

It suffices to show that $0 \in R(T)$.

By assumption, $\dim X \geq 2$. Hence $\dim (D(L)) \geq 2$.

Let Λ be the directed set consisting of the finite-dimensional subspaces F of $D(L)$ with $\dim F \geq 2$, Λ being ordered by inclusion. As in Section 2, we define

$$T_F = j_F^* T j_F,$$

mapping F into F^* for each F in Λ. There exists $R > 0$ independent of F in Λ such that there exists u_F in F with $\| u_F \| \leq R$ and

$$T_F u_F = 0.$$

We again let u_0 be a weak limit point of the directed set $\{u_F ; F \in \Lambda\}$, i.e.

$$u_0 \in \bigcap_F \text{cl} \{u_F : F \in \Lambda, F_2 \subset F\}.$$

Let $F_1 \subset F$; $F, F_1 \in \Lambda$. Then as in Section 2,

$$(3.1) \quad \| u_F - u_{F_1} \| \leq q_R(|(Tu_{F_1}, u_F)| + |(Cu_F - Cu_{F_1}, u_F - u_{F_1})|)$$

and by the same argument as before

$$(3.2) \quad \| u_0 - u_{F_1} \| \leq q_R(|(Tu_{F_1}, u_0)| + |(Cu_0 - Cu_{F_1}, u_0 - u_{F_1})|).$$

We wish to show that for every F_2 in Λ, u_0 lies in the strong closure of the set

$$K_{F_2} = \{u_F \mid F \in \Lambda, F_2 \subset F\}.$$

We may find F_1 in K_{F_2} such that

$$|(Cu_0 - Cu_{F_1}, u_0 - u_{F_1})| < \delta/2.$$

Hence it suffices to show that for a suitable F_2, for every F_1 in Λ with $F_2 \subset F_1$ we have

$$|(Tu_{F_1}, u_0)| < \delta/2.$$

This will be true if $u_0 \in D(L)$ since then $u_0 \in F_2$ for some F_2 in Λ and then for $F_2 \subset F_1$

$$(Tu_{F_1}, u_0) = (T_{F_1} u_{F_1}, u_0) = 0.$$

Thus we must show that $u_0 \in D(L)$.

For every v in $D(L^*) \cap D(L)$, we have $v \in F_2$ for some F_2 in Λ and for F in Λ with $F_2 \subset F$, we have

$$0 = (Tu_F, v) = (Lu_F, v) + (Gu_F, v)$$

while

$$(Lu_F, v) = (u_F, L^*v).$$

Thus

$$|(u_F, L^*v)| = |(Gu_F, v)| \leq c_1 \| v \|.$$

Since L^* is the closure of its restriction to $D(L) \cap D(L^*)$, it follows that

$$|(u_F, L^*v)| \leq c_1 \| v \| \tag{3.3}$$

for all $v \in D(L^*)$. Since

$$|(w, L^*v)| - c_1 \| v \|$$

is weakly continuous in w, it follows from the inequality (3.3) that

$$|(u_0, L^*v)| \leq c_1 \| v \|, \qquad v \in D(L^*). \tag{3.4}$$

Since L is a closed linear operator from X to X^*, $u_0 \in D(L)$. Hence we have shown that for every F_2 in Λ, u_0 lies in the strong closure of the set

$$\{u_F \mid F \in \Lambda, F_2 \subset F\}.$$

Let v be any element of $D(L) \cap D(L^*)$, F_2 an element of Λ containing v. For F in Λ with $F_2 \subset F$,

$$(u_F, L^*v) = -(Gu_F, v)$$

as above. Given $\varepsilon > 0$, we may choose F in Λ with $F_2 \subset F$ such that

$$\| u_F - u_0 \| < \varepsilon$$

and by the demi-continuity of G,

$$|(Gu_F - Gu_0, v)| < \varepsilon.$$

Hence

$$|(u_0, L^*v) + (Gu_0, v)| \leq \varepsilon(1 + \| L^*v \|).$$

Since ε is arbitrary, it follows that

$$(u_0, L^*v) = -(Gu_0, v) \tag{3.5}$$

for all v in $D(L) \cap D(L^*)$. Since L^* is the closure of its restriction to $D(L) \cap D(L^*)$ it follows that (3.5) holds for all v in $D(L^*)$, and $Lu_0 = -Gu_0$, i.e. $Tu_0 = 0$, Q.E.D.

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