A NOTE ON HOMOTOPY IN HOMEOMORPHISM SPACES[1]

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1. INTRODUCTION

In some papers soon to be published or under preparation ([7] and [8]) concerning homotopy properties of the space of homeomorphisms on a 2-manifold, the following situation occurs. There is a mapping f of S^k into the space H of homeomorphisms of the compact metric space K, usually a 2-manifold, onto itself. A certain subset, L, of K, usually a point or simple closed curve, has the property that if $f_L(x)$ is the restriction to L of $f(x)$, then there is a homotopy θ_t such that for each x in S^k, $\theta_0(x) = f_L(x)$, $\theta_1(x)$ is the identity map of L onto itself, and $\theta_t(x)$ is a homeomorphism of L into K. It is then desirable to extend each $\theta_t(x)$ to $\theta_t^*(x)$ to obtain the homotopy θ_t^* in H such that for each x in S^k, $\theta_0^*(x) = f(x)$, $\theta_1^*(x)$ is the identity map of K onto itself, and $\theta_t^*(x)$ is a homeomorphism of K onto itself. Such an extension can sometimes be obtained by looking at $S^k \times K \times I$ and using various theorems about regular mappings. In Section 3 of this note is proved a general elementary theorem that permits these homotopy extensions to be constructed. In fact H is proved, under certain conditions, to be a fiber space (in the sense of Hu, [9, p. 63]). Section 2 contains the necessary theorem about regular mappings and some related theorems, useful in [7] and [8], and Section 4 contains an application of the theorem of Section 3 to the space of homeomorphisms of S^1 into an annulus. This application and variations of it are useful in [7] and [8].

2. Regular mappings

A mapping z of a metric space X onto a metric space Y is *completely regular* if for each positive number ε and point y of Y, there is a positive number δ such that $d(y, y') < \delta$, $y, y' \in Y$, implies that there is a homeomorphism of $z^{-1}(y)$ onto $z^{-1}(y')$ that moves no point as much as ε (i.e. an ε-homeomorphism). In [2]–[6] were proved theorems giving conditions that imply that (X, f, Y) is a locally trivial fiber space or a direct product. In this section are given other such conditions, easily deduced from the earlier ones.

Suppose that K is a compact metric space, L is a closed subset of K and z is a completely regular mapping of a complete metric space X onto a finite (covering) dimensional space Y such that for each point y of Y, there is a homeomorphism z_y of K onto $z^{-1}(y)$. Suppose, further, that there is a homeomorphism h of $\bigcup z_y(L)$ onto $Y \times L$ such that the diagram

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$$\begin{tikzcd} \bigcup z_y(L) \arrow[r, "h"] \arrow[dr, "z"'] & Y \times L \arrow[d, "\pi"] \\ & Y, \end{tikzcd}$$

where π is the projection map, is commutative and that z is completely regular in the strong sense that there is an ε-homeomorphism g of $z^{-1}(y)$ onto $z^{-1}(y')$ such that if $x \in z_y(L)$, then $h(x)$ and $hg(x)$ have the same L-coordinate in $Y \times L$. Finally, denote by $H(K, L)$ the space of homeomorphisms of K onto itself leaving L pointwise fixed, this space being topologized by the uniform convergence metric.

Note 1. A metric space is LC^k if for each point y and each positive number ε, there is a positive number δ such that every mapping of an n-sphere, S^n, $n \leq k$, into the δ-neighborhood of y is homotopic to 0 in the ε-neighborhood of y.

Note 2. The n-sphere S^n is considered as the boundary of the n-ball R^{n+1}.

Note 3. A mapping f of X into Y is homotopy n-regular if for each point x of $f^{-1}(y)$ and positive number ε, there is a positive number δ such that every mapping of the k-sphere, $k \leq n$, into $S(x, \delta) \cap f^{-1}(y')$, $y' \in Y$, is homotopic to 0 in $S(x, \varepsilon) \cap f^{-1}(y')$, where $S(x, \varepsilon)$ is the (open) ε-neighborhood of x. In [2]–[6] it was proved that under most circumstances, if K and L are manifolds of low dimension and f is homotopy n-regular, then f is completely regular.

Note 4. If M and N are topological spaces with subsets M_1 and N_1 respectively, the notations $M_1 \times N_1$ and (M_1, N_1) will be used to denote the subspace of $M \times N$ consisting of those points (x, y) such that $x \in M_1$ and $y \in N_1$.

THEOREM 2.1. *If $H(K, L)$ is locally connected and Y is a cell, then h may be extended to a homeomorphism h^* of X onto $Y \times K$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "h^*"] \arrow[dr, "z"'] & Y \times K \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative.

Proof. This is the lemma of [6].

COROLLARY. *This theorem remains true if h^* is already defined on $Y' \times 0$ (where $Y = Y' \times I$ and Y' is a cell of dimension one less than that of Y).*

THEOREM 2.2. *If $H(K, L)$ is LC^k and $Y = S^k \times I$, then h may be extended to the required h^*.*

Proof. It follows from Lemma 5.6 of [5] and its proof that the space of homeomorphisms of $S^k \times K$ onto itself leaving each (x, K) invariant and $S^k \times L$ pointwise fixed is locally connected. The proof of Theorem 5 of [2] or of the lemma of [6] now applies.

COROLLARY. *This remains true if h^* is already defined on $z^{-1}(x, I)$ for some point x and/or it is already defined on $(S^k, 0)$.*

THEOREM 2.3. *If $Y = R^k \times I$, $H(K, L)$ is LC^k, and h extends to a homeomorphism h_1 of $\bigcup z_y(K)$, $y \in S^{k-1} \times I$, onto $S^{k-1} \times I \times K$ such that the diagram*

$$\begin{tikzcd}
\bigcup\limits_{y \in S^{k-1}\times I} z_y(K) \arrow[r, "h_1"] \arrow[dr, "z"'] & S^{k-1} \times I \times K \arrow[d, "\pi"] \\
& S^{k-1} \times I
\end{tikzcd}$$

is commutative, then h_1 may be extended to the required h^*.

Proof. It also follows from the proof of Lemma 5.6 of [5] that the space of homeomorphisms of $R^k \times K$ onto itself leaving each (x, K) invariant and $(R^k \times L) \cup (S^{k-1} \times K)$ pointwise fixed is locally connected, so the theorem follows as above.

COROLLARY. *This remains true if h^* is already defined on $z^{-1}(x, I)$ for some point x and/or it is already defined on $(R^k, 0)$.*

THEOREM 2.4. *If $H(K, L)$ is LC^k and its homotopy groups vanish in dimensions $\leq k$ and Y has covering dimension $\leq k + 1$, then h^* exists as above. If h^* is already defined on the inverse under z of a closed subset of Y, then the theorem still holds.*

Proof. This is a consequence of Theorem 2 of [2]. The second sentence above does not follow directly from the statement of that theorem, but the selection theorem of Michael [12] used in its proof and stated in [2] as Theorem M makes the truth of that sentence evident.

3. Homotopy in homeomorphism spaces

In this section, let H denote the space of homeomorphisms of K onto itself and H_L the space of homeomorphisms of L into K that are extendable to elements of H. *Suppose that these homeomorphism spaces have the properties that* (1) $H(K, L)$ *is locally connected and* (2) *for each* $\varepsilon > 0$ *and* $f \in H_L$, *there exists a* $\delta > 0$ *such that if* g *is an element of* H_L *and* $d(f, g) < \delta$, *then if* f^* *is an element of* H *and* $f = f^* \mid L$, *then there is an element* g^* *of* H *such that* $d(f^*, g^*) < \varepsilon$ *and* $g^* \mid L = g$. If K is a 2-manifold and L is a simple closed curve or point or if K is a 3-manifold and L is a tamely imbedded 2-manifold, then these conditions are satisfied ([3], [4] and [5]), so Theorem 3.1 below is applicable to the study of the space of homeomorphisms on a 2 or 3-manifold.

Let p denote the mapping of H into H_L such that $p(f^*) = f^* \mid L$. Then for each element f of H_L, $p^{-1}(f)$ is homeomorphic to $H(K, L)$.

THEOREM 3.1. *The space* H *is a fiber space over the base space* H_L *with fiber* $H(K, L)$ *and projection map* p.

Proof. Let ϕ be a mapping of R^k into H and θ a mapping of $R^k \times I$ into H_L such that $\theta(x, 0) = p\phi(x)$ for each x in R^k. Consider $R^k \times I \times K$ as the X of Section 2 and the projection map π of this on $R^k \times I$ as z. Let $z_{y,t}$ be a homeomorphism of K into $\pi^{-1}(y, t) = (y, t, K)$ taking the point x of L into $(y, t, \theta(y, t)(x))$. Such a homeomorphism exists as a consequence of the definition of H_L. Let h be the mapping of $\bigcup z_{y,t}(L)$ into $R^k \times I \times L$ carrying $(y, t, \theta(y, t)(x))$ onto (y, t, x). Let h_1 be the homeomorphism of $R^k \times 0 \times K$ onto itself taking $(y, 0, \phi(y)(x))$ onto $(y, 0, x)$. Conditions (1) and (2) above guarantee that the requirements in the hypotheses of Theorem 2.1 and its corollary are met, so h and h_1 extend to a homeomorphism h^* of $R^k \times I \times K$ onto itself leaving each (y, t, K) invariant.

For each x in K and y in R^k, let $\theta^*(y, t)(x) = x^1$, where $h^*(y, t, x^1) = (y, t, x)$. Then, since $h^*(y, 0, \phi(y)(x)) = (y, 0, x)$, $\theta^*(y, 0) = \phi(y)$. If $x \in L$, $h^*(y, t, \theta(y, t)(x)) = (y, t, x)$; hence $\theta^*(y, t)(x) = \theta(y, t)(x)$—i.e. $p\theta^*(y, t) = \theta(y, t)$. Therefore the covering homotopy property holds for simplices and the theorem is proved.

COROLLARY 1. *Let* f^* *be a mapping of* S^k *into* H *and* θ_t *a homotopy of* pf^* *in* H_L *such that* $\theta_0(x) = pf^*(x)$ *for each* x *in* S^k. *Then there is a homotopy* θ_t^* *of* f^* *in* H *such that* $p\theta_t^*(x) = \theta_t(x)$ *for each* x *in* S^k.

COROLLARY 2. *Let* f^* *be a mapping of* R^{k+1} *into* H, θ_t *a homotopy of* pf^* *in* H_L *such that* $\theta_0(x) = pf^*(x)$ *for each* x *in* R^{k+1} *and* ϕ_t^* *a homotopy of* $f^* \mid S^k$ *in* H *such that for each* x *in* S^k, $\phi_0^*(x) = f^*(x)$ *and* $p\phi_t^*(x) = \theta_t(x)$ *for each* t. *Then* ϕ_t^* *can be extended to a homotopy* θ_t^* *of* f^* *in* H *such that for each* t *and each* x *in* R^{k+1}, $\theta_0^*(x) = f^*(x)$ *and* $p\theta_t^*(x) = \theta_t(x)$ *and for* x *in* S^k, $\theta_t^*(x) = \phi_t^*(x)$.

Proof of Corollaries 1 *and* 2. These are just alternative forms of the requirement that H be a fiber space with base H_L and projection p. (See Hu, p. 63.)

COROLLARY 3. *If M is a closed subset of L and all mappings in H and H_L have the further properties that they leave M pointwise fixed* (*conditions* (1) *and* (2) *remain satisfied*), *then H is a fiber space with base space H_L, projection p, and fiber $H(K, L)$.*

Proof. The proof follows that for Theorem 3.1.

4. Application to annuli

The above theorems can be used to prove that if K is a 2-sphere, then (1) if L consists of only one or two points, $\pi_1(H(K, L)) = Z$ and $\pi_i H(K, L) = 0$ for $i > 1$ and (2) if L is finite and contains more than two points, then $\pi_i H(K, L) = 0$ for each i. These statements were proved by McCarty in [11]. In [6], it is proved that if K is a disc with holes and L is its boundary, then $\pi_i(H(K, L)) = 0$ for each i. The techniques of [6] can be used to prove without difficulty that is K has n holes, then H has the same homotopy groups as the space of homeomorphisms on S^2 leaving n points fixed. The following theorems, whose proof uses these facts, is useful in [7] and variations of it are useful in [8], where surfaces more complicated than the annulus must be considered.

THEOREM 4.1. *Let K be an annulus, L a simple closed curve in the interior of K that separates the boundary curves of K, and x a point of L. Then $\pi_i(H_L) = 0$ for $i > 1$, $\pi_1(H_L) = Z$, and the identity component of the space of homeomorphisms of L into* int K *that leave x fixed is homotopically trivial.*

Proof. It follows from [3, Theorem 2] that $H(K, L)$ is homotopically trivial. In particular, it is connected ($\pi_0(H(K, L)) = 0$). The spaces H, H_L and $H(K, L)$ satisfy the requirements for Theorem 3.1, as noted earlier. Hence H is a fiber space over H_L with fiber $H(K, L)$. Since $\pi_i(H(K, L)) = 0$ for $i \geqq 0$, the exactness of the homotopy sequence of this fibering (see Hu, p. 152) yields $\pi_i(H) = \pi_i(H_L)$ for $i \geqq 1$. But $\pi_i(H) = 0$ for $i > 1$ and $\pi_1(H) = Z$, as noted above, so the first two conclusions of the theorem follow.

Let $H_{L,x}$ denote the space of homeomorphisms of L into int K that leave x fixed. Then it follows from Corollary 3 to Theorem 3.1 that $H(K, x)$ is a fiber space with base space $H_{L,x}$ and fiber $H(K, L)$. Since $\pi_i(H(K, L)) = 0$ for $i \geqq 0$, $\pi_i(H(K, x)) = \pi_i(H_{L,x})$ for $i \geqq 1$. But $H(K, x)$ has the homotopy groups of the space of homeomorphisms on a disc with three holes. Therefore $\pi_i(H_{L,x}) = 0$ for $i \geqq 1$.

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