A MARTINGALE CONVERGENCE THEOREM OF WARD'S TYPE

BY

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Introduction

The martingale convergence theorems were first utilized by Doob [2, p. 343] in giving a new proof of the Lebesgue differentiation theorem of functions of bounded variation on a real line. Later Chow [1] gave a proof of the Lebesgue differentiation theorem of interval functions of bounded variation by applying convergence theorems of partially ordered martingales. In 1959, Ward's differentiation theorem [8, p. 137, p. 141], among other things, has been generalized by Rutowitz [7] to cell functions by introducing the concept of the p-bordering property. In this paper, by following Doob's approach in [3], we are able to obtain a convergence theorem (Theorem 1), which includes some martingale convergence theorems and extends a theorem of Rutowitz [7, Theorem II] to the case of a non-atomic basis. Theorem 4 puts the above cited Ward's theorem into martingale setting.

1. Definitions and notation

Suppose that $(\Omega, \mathfrak{F}, P)$ is a complete measure space with $P(\Omega) = 1$. A stochastic basis $(\mathfrak{F}_\delta, \Delta)$ is a net, where Δ is a directed set, $\mathfrak{F}_\delta$ is a sub-σ-algebra of $\mathfrak{F}$ for each $\delta \in \Delta$, and $\mathfrak{F}_\delta \subset \mathfrak{F}_{\delta'}$ if $\delta < \delta'$. A stochastic process $(x_\delta, \mathfrak{F}_\delta, \Delta)$ is a triple, where $(\mathfrak{F}_\delta, \Delta)$ is a stochastic basis and x_δ is an $\mathfrak{F}_\delta$-measurable function. P^* is the outer measure induced by P and the integral $\int_A x$ will mean $\int_A x\, dP$. For a set A, the $\mathfrak{F}_\delta$-cover of A is denoted by A_δ^* and the $\mathfrak{F}$-cover by A^*. $A - B$ will be the proper difference of sets A and B, and $I(A)$ the indicator (or characteristic) function of the set A. The function x_δ is sometimes written as $x(\delta)$. $\| x \|_q$ is the L_q-norm of x. For sets A and B, $A \in \mathfrak{F}_\delta B$, if $A \subset B$ and $A \in \mathfrak{F}_\delta$.

DEFINITION 1. A stochastic basis is said to satisfy the Vitali condition V_q for $1 \leqq q \leqq \infty$, if for every $\varepsilon > 0$, every set A and every net (K_δ, Δ) of $\mathfrak{F}_\delta$-sets such that $\limsup_\Delta K_\delta \supset A$ a.e., there exist $\delta_i > \delta$ for any given δ, and $\mathfrak{F}_{\delta_i}$-sets $L_i \subset K_{\delta_i}$ so that

$$P^*(A - B) < \varepsilon \tag{1.1}$$

where $B = \bigcup_1^n L_i$, and so that

$$\left\| \textstyle\sum_1^n I(L_i) - I(B) \right\|_q < \varepsilon. \tag{1.2}$$

The conditions V_1 and V_∞ are called respectively the weak and the strong

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Vitali conditions. If Δ is a countable linearly ordered set, then any stochastic basis $(\mathfrak{F}_\delta, \Delta)$ satisfies V_∞. The ordinary differentiation basis satisfies the strong Vitali condition V_∞ (see [1] or [4, p. 209]; in [1] V_∞ has been denoted by V_0), and the strong differentiation basis has the property V_1 (see [4, p. 210]).

A stochastic basis is said to satisfy the Vitali condition V_q^*, if it satisfies the conditions of Definition 1, replacing lim sup by ess lim sup and A by A^*. Both definitions of V_q and V_q^* are due to Krickeberg [5], [6]. (He denotes V_q and V_q^* by V_q^* and V_q.)

Definition 2. Let $b > 0$, $1 \leqq q \leqq \infty$ and $W = [\sup_\Delta |x(\delta)| < b]$. $(x_\delta, \mathfrak{F}_\delta, \Delta)$ is said to satisfy the condition $(A, b)_q$, if for every $\delta_0 \in \Delta$ there exists $0 < c < \infty$ such that for any given $\delta_1, \cdots, \delta_m$ in $\Delta(\delta > \delta_0)$ and $L_i \in W_{\delta_i}{}^*\mathfrak{F}_{\delta_i}$, there are $\eta \geq \delta_i$ $(i = 1, 2, \cdots, m)$ and $\mathfrak{F}_\eta$-measurable functions $y' = y'(\eta)$, $y'' = y''(\eta)$ with $\| y' \|_q \leqq c$, $\| y'' \|_q \leq c$ so that there exist $\eta_{i,1} = \delta_i \leqq \eta_{i,2} \leqq \cdots \leqq \eta_{i,k_i} = \eta$ and $\mathfrak{F}_\eta$-measurable functions $x_i' = x_i'(\eta)$, $x_i'' = x_i''(\eta)$ satisfying for $i = 1, 2, \cdots, n$ and $j = n+1, \cdots, m$

$$(1.3) \quad x_i' = x(\eta) = x_j'' \text{ in } W, \qquad x_i' \leqq c \text{ in } L_i, \qquad x_j'' \geqq -c \text{ in } L_j,$$

$$(1.4) \qquad \int_{L_i} x(\delta_i) \leqq \int_{L_iA_i} y' + \int_{L_i - A_i} x_i',$$

$$(1.5) \qquad \int_{L_j} x(\delta_j) \geqq \int_{L_jB_j} y'' + \int_{L_j - B_j} x_j'',$$

where $A_i = [\max_{k \leqq k_i} x(\eta_{i,k}) \geqq b]$ and $B_j = [\min_{k \leqq k_j} x(\eta_{j,k}) \leqq -b]$.

Definition 3. A stochastic process $(x_\delta, \mathfrak{F}_\delta, \Delta)$ is a martingale, if x_δ is integrable, and if for $\delta' \leq \delta$ $E(x_\delta \mid \mathfrak{F}_{\delta'}) = x_\delta$, a.e., where $E(x_\delta \mid \mathfrak{F}_{\delta'})$ is the Radon-Nikodym derivative of the integral of x_δ relative to $\mathfrak{F}_{\delta'}$.

If $(x_\delta, \mathfrak{F}_\delta, \Delta)$ is a martingale and $\sup_\Delta \| x_\delta \|_q \leq K < \infty$, then the condition $(A, b)_q$ is satisfied for every $b > 0$, by taking $\eta \geqq \delta_i$ $(i = 1, 2, \cdots, m)$, $y' = y'' = x(\eta)$, $\eta_{i,2} = \eta$, $x_i' = \min[x(\eta), b]$, $x_i'' = \max[x(\eta), -b]$, and $c > \max[b, K]$.

2. Martingale convergence theorems

Theorem 1. *If $1 \leqq q < \infty$, $p^{-1} + q^{-1} = 1$ and $(x_\delta, \mathfrak{F}_\delta, \Delta)$ is a stochastic process satisfying the Vitali condition V_q, then x_δ converges a.e. where*

$$\sup_\Delta |x_\delta| < b,$$

provided $(A, b)_p$ is satisfied for some $b > 0$.

Proof. Suppose that it is false and $\delta_0 \in \Delta$. Then there exist two real numbers $a < d$ and a set W with $P^*(W) > 0$ such that

$$(2.1) \qquad \sup_\Delta |x_\delta| < b, \qquad \limsup_\Delta x_\delta > d > a > \liminf_\Delta x_\delta$$

on W. Put

$$K_\delta = W_\delta^* \ (x_\delta > d). \tag{2.2}$$

Then $\limsup K_\delta \supset W$. By the Vitali condition V_q, for $1 > \varepsilon > 0$ there exist $\delta_i > \delta_0$ and $L_i \in K_{\delta_i} \mathfrak{F}_{\delta_i}$, $i = 1, \cdots, n$, such that

$$P^*(W - A) < \varepsilon, \qquad \left\| \textstyle\sum_1^n I(L_i) - I(A) \right\|_q < \varepsilon, \tag{2.3}$$

where $A = \bigcup_1^n L_i$. Put

$$H_\delta = AW_\delta^* \ (x_\delta < a). \tag{2.4}$$

By V_q again, for $\delta_0' > \delta_i$, $i = 1, \cdots, n$, there exist $\delta_j > \delta_0'$ and $L_j \in H_{\delta_j} \mathfrak{F}_{\delta_j}$, $j = n + 1, \cdots, m$, such that

$$P^*(AW - B) < \varepsilon, \qquad \left\| \textstyle\sum_{n+1}^m I(L_j) - I(B) \right\|_q < \varepsilon, \tag{2.5}$$

where $B = \bigcup_{n+1}^m L_j$. By the condition $(A, b)_p$, there exist c, η, y', y'', x_i', x_i'' and $\eta_{i,k}$ ($i = 1, \cdots, n, \cdots, m$; $k = 1, \cdots, k_i$) satisfying the conditions in $(A, b)_p$. For each $i = 1, \cdots, n$, let s_i be the first $k \leqq k_i$ such that $x(\eta_{i,k}) \geqq b$ if there is one, and $s_i = \infty$ otherwise. Then for $i = 1, \cdots, n$

$$\begin{aligned} &\int_{L_i} x(\delta_i) \leqq \int_{L_i(s_i<\infty)} y' + \int_{L_i(s_i=\infty)} x_i', \\ &d \sum_1^n P(L_i) \leqq \sum_1^n \int_{L_i(s_i<\infty)} y' + \sum_1^n \int_{L_i(s_i=\infty)} x_i'. \end{aligned} \tag{2.6}$$

Choose δ_0 so large such that $P(W_{\delta_0}^* - W_\delta^*) < \varepsilon$ for $\delta > \delta_0$. Then

$$\begin{aligned} \textstyle\sum_1^n \int_{L_i[(s_i=\infty)-W_\eta^*]} x_i' \leqq c \textstyle\sum_1^n P(L_i[(s_i = \infty) - W_\eta^*] \\ - cP(\textstyle\bigcup_1^n L_i[(s_i = \infty) - W_\eta^*]) \\ + cP(\textstyle\bigcup_1^n L_i[(s_i = \infty) - W_\eta^*]) \\ \leqq c[\textstyle\sum_1^n P(L_i) - P(A)] \\ + cP(\textstyle\bigcup_1^n L_i[(s_i = \infty) - W_\eta^*]) \\ < c\varepsilon + cP(A - W_\eta^*) \\ \leqq c\varepsilon + cP(W_{\delta_0}^* - W_\eta^*) < 2c\varepsilon. \end{aligned}$$

Hence

$$\sum_1^n \int_{L_i[(s_i=\infty)-W_\eta^*]} x_i' < 2c\varepsilon. \tag{2.7}$$

Since $q < \infty$, we can assume that δ_0 is so large that $P(W_{\delta_0}^* - W_\delta^*) < \varepsilon^q$ for every $\delta > \delta_0$. Then

$$\int_{W_{\delta_0}^* - W_\eta^*} | y' | \leqq \| y' \|_p \, \varepsilon \leqq c\varepsilon. \tag{2.8}$$

Put $D = \bigcup_1^n L_i(s_i < \infty)$. Since $W_\eta^* \subset (s_i = \infty)$ for each i and $D \subset A \subset W_{\delta_0}^*$,

$$\int_D y' \leqq \int_{A-W_\eta^*} |y'| \leqq \int_{W_{\delta_0}^*-W_\eta^*} |y'| \leqq c\varepsilon.$$

By (2.3),

$$\sum_1^n \int_{L_i(s_i<\infty)} y' - \int_D y' \leq \left\| \sum_1^n I(L_i) - I(A) \right\|_q \| y' \|_p < c\varepsilon.$$

Hence

$$(2.9) \qquad \sum_1^n \int_{L_i(s_i<\infty)} y' < 2c\varepsilon.$$

From (2.9), (2.7) and (2.6),

$$(2.10) \qquad d\sum_1^n P(L_i) \leqq 4c\varepsilon + \sum_1^n \int_{L_iW_\eta^*} x_i'.$$

Similarly,

$$(2.11) \qquad a\sum_{n+1}^m P(L_j) \geqq -4c\varepsilon + \sum_{n+1}^m \int_{L_jW_\eta^*} x_j''.$$

Put $L_1' = L_1$ and $L_i' = L_i - \bigcup_1^{i-1} L_k'$ for $i = 2, \cdots, n$ and $L_{n+1}' = L_{n+1}$ and $L_j' = L_j - \bigcup_{n+1}^{j-1} L_k'$ for $j = n+2, \cdots, m$. Define $z' = x_i'$ on each L_i' and $z'' = x_j''$ on each L_j'. Then

$$(2.12) \quad \sum_1^n \int_{L_iW_\eta^*} x_i' \leqq \int_{AW_\eta^*} z' + c\left[\sum_1^n P(L_i) - P(A)\right] \leqq \int_{AW_\eta^*} z' + c\varepsilon.$$

Similarly,

$$(2.13) \qquad \sum_{n+1}^m \int_{L_jW_\eta^*} x_j'' \geqq \int_{BW_\eta^*} z'' - c\varepsilon.$$

Hence

$$(2.14) \qquad \begin{aligned} \int_{AW_\eta^*} z' - \int_{BW_\eta^*} z'' &\leqq cP[(A-B)W_\eta^*] \\ &\leqq cP(A-B) \\ &\leqq P^*(AW-B) + P^*(A-W) < 2\varepsilon. \end{aligned}$$

From (2.10)–(2.14), we have

$$(2.15) \qquad d\textstyle\sum_1^n P(L_i) - a\sum_{n+1}^m P(L_j) < 12\, c\varepsilon.$$

Thus we completed the proof.

THEOREM 2. *Let $(\mathfrak{F}_\delta, \Delta)$ satisfy the Vitali condition V_q and $(x_\delta, \mathfrak{F}_\delta, \Delta)$ be a martingale with $\sup_\Delta \| x_\delta \|_p < \infty$, where $p \geqq 1$ and $p^{-1} + q^{-1} = 1$. Then x_δ converges a.e.*

Proof. For $p = 1$, it follows immediately from Theorem 4.2 of [1] that $\lim_\Delta x_\delta$ exists a.e., and for $p > 1$ states that $\lim_\Delta x_\delta$ exists a.e. where both $\limsup_\Delta x_\delta$ and $\liminf_\Delta x_\delta$ are finite. Hence we need only to prove that under the conditions of Theorem 2, both $\limsup_\Delta x_\delta$ and $\liminf_\Delta x_\delta$ are finite a.e.

Assume that $W = (\limsup_\Delta x_\delta = \infty)$ and $P^*(W) > a > 0$. Then by V_q, for any $0 < K < \infty$, $\varepsilon > 0$, and $\delta_0 \in \Delta$, there exist $\delta_1, \delta_2, \cdots, \delta_m$ and $\mathfrak{F}_{\delta_i}$ – sets $L_i \subset [x(\delta_i) > K]$ such that $\delta_i > \delta_0$ and

$$(2.16) \qquad P(A) > a, \qquad \left\| \sum_1^m I(L_i) - I(A) \right\|_q < \varepsilon,$$

where $A = \bigcup_1^m L_i$. Take $\eta > \delta_i$ $(i = 1, 2, \cdots, m)$. Then

$$\begin{aligned} KA &\leqq \sum_1^m \int_{L_i} x(\delta_i) = \sum_1^m \int_{L_i} x(\eta) \\ &\leqq \left\| \sum_1^m I(L_i) - I(A) \right\|_q \| x(\eta) \|_p + \| x(\eta) \|_p \\ &\leqq (1 + \varepsilon) \| x(\eta) \|_p . \end{aligned}$$

Hence we arrive at a contradiction and $P(W) = 0$. Similarly,

$$P(\liminf_\Delta x_\delta = -\infty) = 0.$$

From the previous proofs, immediately we have

COROLLARY 1. *Both Theorems* 1 *and* 2 *hold, if we replace* V_q *by* V_q^*, sup *by* ess sup *and convergence by essential convergence.*

Corollary 1 completes a theorem due to Krickeberg [5, Theorem 3.5] on essential convergence of martingales of decreasing stochastic basis.

Added in proof. In a recent paper of K. Krickeberg and C. Pauc (Bull. Soc. Math. France, vol. 91 (1963), pp. 455–543), the essential convengence part of Corollary 1 was proved by a different method.

3. A convergence theorem for martingales generated by cell functions

Let $\mathcal{I}$ be a family of $\mathfrak{F}$-sets with positive measures. Each element in $\mathcal{I}$ is called a cell. Two cells are said to be non-overlapping, if their intersection is a null set. A partition of a set $X \subset \Omega$ is a sequence of non-overlapping cells I_n with $\bigcup_1^\infty I_n = X$ and any cell meets at most a finite number of I_n. For a family $\mathcal{G}$ of cells, each cell in $\mathcal{G}$ is called a $\mathcal{G}$-cell. $A(\mathcal{G})$ will be the union of all $\mathcal{G}$-cells, $\mathcal{G}^u$ the family of cells which are finite unions of $\mathcal{G}$-cells, and for a set X, $\mathcal{G}X$ is the family of all $\mathcal{G}$-cells which are subsets of X. A complex $\mathcal{K}$ is a finite family of non-overlapping cells. For a complex $\mathcal{K}$, define $P(\mathcal{K}) = P(A(\mathcal{K}))$. For two families $\mathcal{G}$ and $\mathcal{H}$ of cells, if $\mathcal{G} \subset \mathcal{H}^u$, we say that $\mathcal{H}$ refines $\mathcal{G}$, or $\mathcal{H}$ is $\mathcal{G}$-fine, denoted by $\mathcal{G} < \mathcal{H}$. For two complexes $\mathcal{K}$

and $\mathcal{K}^b$, $\mathcal{K}^b$ is said to be a bordering complex of $\mathcal{K}$, if every $\mathcal{K}$-cell is contained in some $\mathcal{K}^b$-cell and no $\mathcal{K}^b$-cell is included in $\mathcal{K}^u$. For a cell I, a partition η of I is said to be p-bordering ($p > 1$), if for each cell $J \in \eta^u$ and each complex $\mathcal{K} \subset \eta^u J$ with $A(\mathcal{K}) \neq J$, there exists a bordering complex $\mathcal{K}^b$ of $\mathcal{K}$ with $\mathcal{K}^b \subset \eta^u J$ and $P(\mathcal{K}^b) \leqq pP(\mathcal{K})$. $\mathcal{I}$ will be said to have the p-bordering property, if to every cell I and every complex $\mathcal{K}$ of subcells of I, there corresponds a $\mathcal{K}$-fine p-bordering partition of I.

Assume that the family Λ of all partitions λ of Ω forms a directed set with respect to the order $>$ (refinement). For each $\lambda \in \Lambda$, let $\mathfrak{F}_\lambda$ be the σ-algebra generated by the λ-cells.

THEOREM 3. *Let $(x_\lambda, \mathfrak{F}_\lambda, \Lambda)$ be a martingale and $\mathcal{I}$ have the p-bordering property with $1 < p < \infty$. Let B be an $\mathfrak{F}_{\lambda_0}$-cell, $W = [\sup_{\lambda > \lambda_0} | x_\lambda | < b]$ for $0 < b < \infty$, and $c = 2pb$. For any given $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots, \lambda_m$ in*

$$\Lambda(\lambda > \lambda_0)$$

and $\mathfrak{F}_{\lambda_i}$-sets $L_i \subset BW^*_{\lambda_i}$, there exists $\eta > \lambda_i$, $i = 1, 2, \cdots, m$ in Λ such that

$$(3.1) \quad \int_{L_i} x(\lambda_i) \leqq cP[L_i(x(\eta) \geqq b)] + \int_{L_i[x(\eta)<b]} x^{(i)}(\eta), \qquad i = 1, \cdots, n$$

$$(3.2) \quad \int_{L_j} x(\lambda_j) \geqq -cP[L_j(x(\eta) \leqq -b)] + \int_{L_j[x(\eta)>-b]} x^{(j)}(\eta), \qquad j = n+1, \cdots, m,$$

where

$$(3.3) \quad x_\eta^{(i)}(\omega) = x_\eta(\omega) = x_\eta^{(j)}(\omega) \quad \text{if} \quad \omega \in I \in \eta, IW \neq \emptyset, \qquad i = 1, \cdots, n; \quad j = n+1, \cdots, m,$$

$$(3.4) \quad x_\eta^{(i)}(\omega) = c = -x_\eta^{(j)}(\omega) \quad \text{if} \quad \omega \in I \in \eta, IW = \emptyset, \qquad i = 1, \cdots, n; \quad j = n+1, \cdots, m.$$

Proof. We can and will assume that each L_i is an $\mathfrak{F}_{\lambda_i}$-cell. Let η' be a partition of Ω such that $\eta' > \lambda_i$, $i = 1, \cdots, n, \cdots, m$. Let $\mathcal{J} = \eta' B$. Then $\mathcal{J}$ is a complex and $L_i \in \mathcal{J}^u$ for each $i = 1, \cdots, m$. By the p-bordering property of $\mathcal{I}$, there exists a $\mathcal{J}$-fine, p-bordering partition δ of B. Put

$$\eta = \eta'(\Omega - B) \cup \delta.$$

Then, $\eta \in \Lambda$ and $\eta > \eta' > \lambda_i$, $i = 1, \cdots, m$. For each $i = 1, \cdots, n$, let $\mathcal{K}_i = [I \mid I \in \eta L_i, IW = \emptyset]$. If $\mathcal{K}_i = \emptyset$, then

$$\int_{L_i} x(\lambda_i) = \int_{L_i} x(\eta) = \int_{L_i[|x(\eta)|<b]} x(\eta) = \int_{L_i[|x(\eta)|<b]} x^{(i)}(\eta)$$
$$= \int_{L_i[x(\eta)<b]} x^{(i)}(\eta).$$

If $A(\mathcal{K}_i) = L_i$, then since $L_i \subset W^*_{\lambda_i}$,

$$\int_{L_i} x(\lambda_i) \leqq cP(L_i) = cP[L_i(x(\eta) \geqq b)] + cP[L_i(x(\eta) \leqq -b)]$$

$$\leqq cP[L_i(x(\eta) \leqq b)] + \int_{L_i[x(\eta)\leqq -b]} x^{(i)}(\eta)$$

$$= cP[L_i(x(\eta) \geqq b)] + \int_{L_i[x(\eta)<b]} x^{(i)}(\eta).$$

Now assume that $\mathcal{K}_i \neq \emptyset$ and $\mathcal{K}_i \neq \eta L_i$. Since $A(\mathcal{K}_i) \neq L_i \in \delta^u$ and $\mathcal{K}_i \subset \delta^u L_i$, by the p-bordering property of δ, there exists a complex $\mathcal{K}_i^b \subset \delta^u L_i$ such that every $\mathcal{K}_i$-cell is contained in some $\mathcal{K}_i^b$-cell, $IW \neq \emptyset$ for every $\mathcal{K}_i^b$-cell I, and $P(\mathcal{K}_i^b) \leqq pP(\mathcal{K}_i)$. Hence

$$\int_{A(\mathcal{K}_i)} x(\eta) = \int_{A(\mathcal{K}_i{}^b)} x(\eta) - \int_{A(\mathcal{K}_i{}^b)-A(\mathcal{K}_i)} x(\eta)$$

$$\leqq bP(\mathcal{K}_i^b) + b[P(\mathcal{K}_i^b) - P(\mathcal{K}_i)] \leqq 2bP(\mathcal{K}_i^b) \leqq cP(\mathcal{K}_i).$$

Therefore

$$\int_{L_i} x(\lambda_i) = \int_{L_i} x(\eta) = \int_{A(\mathcal{K}_i)} x(\eta) + \int_{L_i - A(\mathcal{K}_i)} x(\eta)$$

$$\leqq cP(\mathcal{K}_i) + \int_{L_i - A(\mathcal{K}_i)} x(\eta)$$

$$\leqq cP[A(\mathcal{K}_i)(|\, x(\eta)\, | \geqq b)] + cP[A(\mathcal{K}_i)(|\, x(\eta)\, | < b)]$$

$$+ \int_{[L_i - A(\mathcal{K}_i)][|x(\eta)|<b]} x(\eta)$$

$$\leqq cP[L_i(|\, x(\eta)\, | < b)] + \int_{L_i[|(\eta)|<b]} x^{(i)}(\eta).$$

Since by (3.4)

$$\int_{L_i[x(\eta)\geqq -b]} x^{(i)}(\eta) = cP[L_i(x(\eta) \leqq -b)],$$

$$\int_{L_i} x(\lambda) \leqq cP[L_i(x(\eta) \geqq b)] + \int_{L_i[x(\eta)<b]} x^{(i)}(\eta).$$

Similarly we can prove (3.2).

THEOREM 4. *Let $(x_\lambda, \mathfrak{F}_\lambda, \Lambda)$ be a martingale satisfying the weak Vitali condition V_1 and $\mathcal{J}$ have the p-bordering property with* $1 < p < \infty$. *Then x_λ converges a.e. where* $\sup |\, x_\lambda\, | < \infty$.

Proof. Theorem 3 states that $(x_\lambda, \mathfrak{F}_\lambda, \Lambda)$ satisfies the condition $(A, b)_\infty$ for every $b > 0$. Therefore, Theorem 4 follows from Theorem 1 immediately.

Theorem 3 includes Theorem II of Rutowitz' [7], which in turn (See [7, p. 29]) includes a theorem of Ward [8, p. 141].

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