ON HOMOTOPY 3-SPHERES[1]

BY

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A homotopy 3-sphere M^3 is a compact, simply connected 3-manifold without boundary. After the work of Moise [6] and Bing [1] M^3 possesses a triangulation. The Poincaré conjecture [9] states that every homotopy 3-sphere M^3 is a 3-sphere. In this paper we prove three theorems, related to the Poincaré conjecture, about maps of a 3-sphere S^3 onto M^3 and about 1- and 2-spheres in M^3.

1. Theorems 1 and 2, concerning maps $S^3 \to M^3$ and closed curves in M^3. From the work of Hurewicz [5], Part III, it follows that there exists a continuous map $\varphi : S^3 \to M^3$ of degree 1 (where S^3 means a 3-sphere). We shall prove that there exists an especially simple map of this kind.[2]

THEOREM 1. *If M^3 is a homotopy 3-sphere then there exists a simplicial map $\gamma : S^3 \to M^3$ of degree 1 such that the singularities of γ (i.e. the closure of the set of those points $p \in M^3$ for which $\gamma^{-1}(p)$ consists of more than one point) lie in a (polyhedral, compact) handlebody in M^3.*

One might consider this result as a step towards a proof of the Poincaré conjecture. Indeed, if it were possible to restrict the singularities of γ to a 3-cell in M^3 instead of a handlebody the existence of a homeomorphism $S^3 \to M^3$ would follow.

From Theorem 1 we may derive another aspect of the Poincaré problem by considering simple closed curves in M^3.

From the definition of simple connectedness it follows that every closed curve $C^1 \subset M^3$ bounds a singular disk $D^2 \subset M^3$. If C^1 is a tame, simple closed curve then one can find a D^2 which is also tame and possesses only "normal" singularities (see [7], [8]), i.e. double curves in which two sheets of D^2 pierce each other, triple points in which three sheets pierce each other, and branch points from each of which one or more double arcs originate; the triple points, the branch points, and the interiors of the double curves are disjoint from the boundary $\dot{D}^2$ of D^2, but the double curves may have end points in $\dot{D}^2$.

As Bing [2] has proved, M^3 is a 3-sphere if (and only if) every tame, simple closed curve $C^1 \subset M^3$ lies in a (compact) 3-cell in M^3. The statement that C^1 lies in a 3-cell $D^3 \subset M^3$ is equivalent to the statement that C^1 bounds a "knot projection cone" D^2 in M^3, i.e. a (tame) singular disk whose singularities are one branch point P and double arcs originating from P, being pairwise

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[2] Theorem 1 is a consequence of a "monotonic mapping theorem" announced by Moise in [6a]; however the proof is different from Moise' proof.

disjoint otherwise, and terminating in $\dot{}D^2$. (A small neighborhood of a knot projection cone in M^3 is always a 3-cell.) Hence one would prove the Poincaré conjecture if one could prove that every tame, simple closed curve $C^1 \subset M^3$ bounds a knot projection cone in M^3. Theorem 2 of this paper (which may be considered as a corollary of Theorem 1) is a first step in this direction: it states that C^1 always bounds a knot projection cone D^2 with additional singularities that do not touch $\dot{}D^2 = C^1$.

THEOREM 2. *If C^1 is a tame, simple closed curve in a homotopy 3-sphere M^3 then there is a (tame) singular disk $D^2 \subset M^3$ with $\dot{}D^2 = C^1$ such that D^2 has the following singularities:*

(a) *One branch point P of multiplicity g (g may be zero) and g double arcs $Q_1^1, \cdots, Q_g^1$ (in each of which two sheets of D^2 pierce each other), starting from P and ending at $\dot{}D^2$ with ${}^0Q_i^1 \subset {}^0D^2$ such*[3] *that the $Q_i^1 - P$'s are pairwise disjoint.*

(b) *Closed double curves $R_1^1, \cdots, R_h^1$ (h may be zero) which may pierce themselves and the Q_i^1's in triple points of D^2, but which are disjoint from $\dot{}D^2$.*

In the special case $h = 0$, D^2 is a knot projection cone; in the case $g = 0$, D^2 is a so called Dehn disk (see [8]). In the latter case it follows from Dehn's lemma (stated by Dehn [3] and proved by Papakyriakopoulos [8]) that there exists a (tame) disk D^{*2} with $\dot{}D^{*2} = C^1$ and $h^* = 0$ (and also $g^* = 0$). Now the question arises whether it follows in the general case ($g \neq 0$) that there exists a (tame, singular) disk D^{*2} with $\dot{}D^{*2} = C^1$ and $h^* = 0$ (and g^* arbitrary, not necessarily equal to g). An affirmative answer to this question would imply the Poincaré conjecture.

If one applies the methods for proving Dehn's lemma, as developed by Papakyriakopoulos [8] and later simplified by Shapiro and Whitehead [12], to this problem then one has to consider a small neighborhood $D^3 \subset M^3$ of D^2, a covering of D^3, etc. Then all conclusions of the proof of Dehn's lemma in [12] apply to our problem as well, except in case (1) wherein the boundary $\dot{}D^3$ of D^3 (or that of one of the neighborhoods in the coverings) consists of 2-spheres only: for case (1) it follows easily in dealing with Dehn's lemma that C^1 bounds a nonsingular disk; however it seems to be difficult to prove for case (1) in dealing with our problem, $g \neq 0$, that C^1 bounds a knot projection cone. Nevertheless I hope that someone will be able to fill this gap in the proof of the Poincaré conjecture.

2. Theorem 3, concerning 2-spheres in M^3.

We obtain another aspect of the Poincaré problem if we consider 2-spheres in M^3 instead of closed curves. If we remove the interior of a 3-cell C^3 from M^3 we get a so called homotopy 3-cell M_*^3. It follows from the Hurewicz theorem [5], Part II, that every 2-sphere in M_*^3 may be homotopically deformed into one point.

Let us consider a 2-sphere $F_0^2 \subset M_*^3$, "topologically parallel" to the bound-

[3] We denote the interior of a (tame) point set X by ${}^0X$, the boundary by $\dot{}X$, and the closure by $\bar{X}$ or ${}^-X$.

ary of M^3_*, i.e. such that $F^2_0 + {}^{\cdot}M^3_*$ bounds a 3-annulus $F^3_0 \subset M^3_*$. If one could prove that F^2_0 can be deformed[4] into a 3-cell $H^3 \subset M^3_*$ not only by a homotopy but also by an isotopy whose image is tame at each level then the Poincaré conjecture would follow (since it would follow that M^3_* is a 3-cell). It follows from the work of Smale [13] on regular homotopy that F^2_0 can be deformed onto the boundary of a 3-cell in H^3 in such a way that no branch points occur at any stage of the deformation. In order to go one step further in this direction we shall show that F^2_0 can be deformed into H^3 by especially simple homotopic deformations that take place in a special order.

First we have to define some special homotopic deformations. Let

$$\alpha : F'^2 \to M^3_*,$$

with the image $\alpha(F'^2) \subset {}^0M^3_*$ denoted by F^2, be a continuous map, defining a (tame) 2-sphere with canonical singularities (i.e. normal double curves and triple points, but without branch points, see [8]). Let A'^2 be a disk in F'^2 whose image $\alpha(A'^2)$ is also a (nonsingular) disk A^2. Let

$$A^{*2} \subset {}^0M^3_*$$

be another tame disk with $A^{*2} \cap A^2 = {}^{\cdot}A^2 = {}^{\cdot}A^{*2}$ such that $A^2 + A^{*2}$ bounds a 3-cell $K^3 \subset M^3_*$. Now we consider a deformation δ that changes α into α^* such that

$$\alpha^* \,|(F'^2 - {}^0A'^2) = \alpha\,|(F'^2 - {}^0A'^2)$$

and $\alpha^* \mid A'^2$ is a homeomorphism onto A^{*2}. We call such a deformation *non-essential* if there exists an epi-homeomorphism

$$\zeta : M^3_* \to M^3_* \quad \text{with} \quad \zeta(F^2) = \alpha^*(F'^2)$$

that is the identity outside a small neighborhood of K^3. We call δ an *elementary deformation of type* 1, 2, *or* 3, respectively, if the surface defined by α^* has only normal singularities and one of the following conditions holds (see Fig. 1):

Type 1. Either case (a) ${}^-({}^0K^3 \cap F^2)$ is a disk B^2 with ${}^{\cdot}B^2 \subset {}^0A^{*2}$; or case (b) ${}^-({}^0K^3 \cap F^2)$ consists of two disks B^2, C^2 such that

$${}^{\cdot}B^2,\ {}^{\cdot}C^2 \subset {}^0A^{*2}$$

and $B^2 \cap C^2$ is an arc with

$${}^0(B^2 \cap C^2) \subset {}^0K^3.$$

[4] For convenience we shall use the word "deformation" not only for deformations of maps but also for deformations of polyhedra $X \subset M^3$ (i.e. for changes of X into X^* such that there can be found homotopic maps $\xi, \xi^* : X' \to M^3$ with $\xi(X') = X$, $\xi^*(X') = X^*$). This is convenient since a surface with normal singularities, defined by a map

$$\xi : X'^2 \to M^3,$$

is essentially determined by the image polyhedron $\xi(X'^2)$.

Type 2. $^-(^0K^3 \cap F^2)$ is a disk B^2 such that each of the intersections $\dot{}B^2 \cap A^2$ and $\dot{}B^2 \cap A^{*2}$ consists of two disjoint arcs with

$$^0(\dot{}B^2 \cap A^2) \subset {}^0A^2 \quad \text{and} \quad ^0(\dot{}B^2 \cap A^{*2}) \subset {}^0A^{*2}.$$

Type 3. Either case (a) $^-(^0K^3 \cap F^2)$ is a disk B^2 with $\dot{}B^2 \subset {}^0A^2$; or case (b) $^-(^0K^3 \cap F^2)$ consists of two disks B^2, C^2 such that $\dot{}B^2 \subset {}^0A^2$ and each of the intersections $\dot{}C^2 \cap A^2$, $\dot{}C^2 \cap A^{*2}$, $C^2 \cap B^2$ is an arc with

$$^0(\dot{}C^2 \cap A^2) \subset {}^0A^2, \qquad ^0(\dot{}C^2 \cap A^{*2}) \subset {}^0A^{*2}, \qquad ^0(C^2 \cap B^2) \subset {}^0C^2, {}^0B^2.$$

We remark that an elementary deformation of type 1 (a or b) changes the image sphere F^2 only in a small neighborhood (small with respect to F^2) of an arc (connecting a point in $^0A^2$ to a point in $^0B^2$ or in $^0B^2 \cap {}^0C^2$, respectively); a deformation[4] of type 2 changes F^2 in a small neighborhood of a disk (whose boundary intersects each A^2 and B^2 in one arc). According to this one might say that a deformation of type i ($i = 1, 2, 3$) is essentially i-dimensional.

THEOREM 3. *Let M^3_* be a homotopy 3-cell and $\alpha_0 : F'^2 \to M^3_*$ an embedding of a 2-sphere, topologically parallel to $\dot{}M^3_*$. Then α_0 can be deformed step by step into maps α_1, α_2, α_3 of F'^2 into M^3_* such that the following holds:*

(a) *α_i ($i = 1, 2, 3$) is obtained from α_{i-1} by a finite sequence of elementary deformations of type i and non-essential deformations.*

(b) *The image $\alpha_3(F'^2)$ lies in a 3-cell $H^3 \subset {}^0M^3_*$.*

The two essential points of this theorem (which are not immediate consequences of Smale's results [13]) are (1) the order in which the deformations take place and (2) that no deformations are used that move the surface over a triple point.

We remark without proof: If it were possible to avoid the deformations of type 1b (i.e. to avoid triple points) *or* to avoid the deformations of type 2 then this would imply the Poincaré conjecture; this would hold even if H^3 were not a 3-cell, but homeomorphic to any compact subset of euclidean 3-space with connected boundary.

3. Sketch of the proofs. The theorems are proved by considering deformations of singular 2-spheres in a homotopy 3-cell M^3_*. We start with an embedding

$$\beta_0 : F_0'^3 \to M^3_*$$

of a 3-annulus $F_0'^3$ into M^3_* such that one boundary sphere S'^2 of $F_0'^3$ is mapped onto $\dot{}M^3_*$ and the other boundary sphere $F_0'^2$ onto the 2-sphere $F_0^2 = \alpha_0(F'^2)$. Now we deform F_0^2 into a 3-cell $H^3 \subset {}^0M^3_*$ in the simplest way we can find. To do this we choose a simple cell-decomposition Γ of the homotopy 3-sphere $M^3 = M^3_* + C^3$ (C^3 being a 3-cell with $C^3 \cap M^3_* = \dot{}C^3 = \dot{}M^3_*$) into one vertex E^0, r elements E^1_i, E^2_i ($i = 1, \cdots, r$) of each dimension 1 and 2, and one open 3-cell E^3 containing C^3. Then we choose a neighborhood J^3 of the 2-skeleton G^2 of Γ, and we may assume that our initial 3-annulus $\beta_0(F_0'^3)$ is $M^3_* - {}^0J^3$, hence $F_0^2 = {}^{\cdot}J^3$. Now we use the fact that M_*^3 is simply connected by taking a collection of r singular disks, bounded by the 1-skeleton G^1 of Γ (that consists of the r loops $\bar{E}_i^1$ with the common vertex E^0); these disks with the boundary point E^0 in common form a "fan" V^2 with singularities. We can choose V^2 such that its only singularities are pairwise disjoint double arcs A_j^1 ($j = 1, \cdots, s$, as depicted in Fig. 2). Now we contract V^2, changing it only within small neighborhoods A_j^3 of the A_j^1's, onto a nonsingular fan V_*^2, a small neighborhood H^3 of which is a 3-cell; that means we deform the 1-skeleton G^1 into the 3-cell H^3. We carry out corresponding deformations (see footnote 4) of the 2-skeleton G^2 onto a "singular 2-skeleton" $G_\#^2$ and of its neighborhood J^3 onto a singular polyhedron $J_\#^3$; and we change the map β_0 correspondingly into a map $\beta_I : F_I'^3 \rightarrow M_*^3$ with $\beta_I \mid S'^2 = \beta_0 \mid S'^2$. All the deformations of G^2, J^3 take place in the A_j^3's. $H^3 + \bigcup_{j=1}^s A_j^3$ is a handlebody K^3. The corresponding deformations of F_0^2 onto F_I^2 are of type 1a only.

Now we have to deform the rest of F_I^2 into H^3. First we remark that $J_\#^3$ may be decomposed into a neighborhood $T_\#^3$ of the deformed 1-skeleton ${}^{\cdot}V_*^2$ and into r "prismatic", singular 3-cells $P_{\#i}^3$ (being prismatic neighborhoods of middle parts of the deformed E_i^2's), such that $T_\#^3 \subset {}^0H^3$. That means, that part of F_I^2 lying outside of H^3 lies in the "top" and "bottom" disks of the $P_{\#i}^3$'s. The boundaries of the top and bottom disks of $P_{\#i}^3$ may be joined by an arc $W_i^1 \subset F_I^2 \cap {}^0H^3$ and by an arc $W_{Pi}^3 \subset {}^{\cdot}P_{\#i}^3$; the so obtained 1-spheres $W_i^1 + W_{Pi}^1$ bound singular disks $W_i^2 \subset {}^0H^3$. We can choose these W_i^2's such that their only singularities are double arcs and that singular, prismatic neighborhoods W_i^3 of them fit properly to F_I^2 and to the $P_{\#i}^3$'s. Then we expand the singular 3-annulus, defined by β_I, over these singular prisms W_i^3 (denoting the changed β_I by β_{II}); the corresponding deformation of F_I^2 onto a singular 2-sphere F_{II}^2 may be decomposed into deformations of type 1 (a and b) yielding a singular 2-sphere F_1^2 (and a map α_1 according to Theorem 3) and after them deformations of type 2 yielding F_{II}^2. Now F_{II}^2 contains "folds" around the $P_{\#i}^3$'s consisting of the top and bottom disks and joining disks (containing the W_{Pi}^1's); so we can expand the singular 3-annulus over the $P_{\#i}^3$'s (denoting the changed β_{II} by $\beta : F'^3 \rightarrow M_*^3$ with $\beta \mid S'^2 = \beta_0 \mid S'^2$). The corresponding deformation of F_{II}^2 yields $F_3^2 \subset {}^0H^3$ (and α_3) and may be decomposed into deformations of type 2, yielding F_2^2 (and α_2), and after them deformations of type 3 (a and b); this completes the proof of Theorem 3.

To prove Theorem 2 we observe that the complement $M_*^3 - {}^0K^3$ of the handlebody K^3 is covered one-to-one by β. So we deform the given curve C^1 isotopically into a curve $C_0^1 \subset M_*^3 - K^3$; then we choose a knot projection cone D'^2 bounded by the knot $\beta^{-1}(C_0^1)$ in the 3-annulus F'^3; we bring about by small deformations the situation in which $\beta(D'^2)$ has only normal singularities. Then $D^2 = \beta(D'^2)$ has the demanded properties. Theorem 1 is proved by extending β to a 3-sphere $S^3 \supset F'^3$.

We remark: If it were possible to find the map

$$\beta : F'^3 \rightarrow M_*^3$$

(with $\beta(\dot{F}'^3 - S'^2) \subset {}^0H^3$) such that $\beta \mid \beta^{-1}(M^3_* - H^3)$ is locally one-to-one then the Poincaré conjecture would follow by an easy conclusion. We would obtain such a map β if it were possible to deform the 3-annulus $\beta_0(F_0'^3)$ onto $\beta(F'^3)$ by "expansions" only. But in our procedure some of the very first deformations in the A^3_j's (and only these) are not expansions, so we get certain surfaces in F'^3 such that β is not locally one-to-one at (and only at) the points of these surfaces. (β maps these surfaces homeomorphically into K^3. Moreover it is possible to arrange our procedure such that these exceptional surfaces become disks.)

I. Proof of Theorems 1 and 2

We prove Theorem 1 and 2 first. After this we shall prove Theorem 3 by consideration of some more details.

4. Preliminaries. Let M^3 be a homotopy 3-sphere. After Moise [6] and Bing [1] there exists a triangulation of M^3. This means there exists a homotopy 3-sphere, homeomorphic to M^3, that is a (straight-lined, finite) polyhedron in a euclidean space $\mathfrak{E}^n$ of sufficiently high dimension n. So we may assume for convenience and without loss of generality that M^3 itself is a polyhedron in $\mathfrak{E}^n$. All point sets considered in the subsequent part of this paper are *polyhedral in* $\mathfrak{E}^n$ in the sense of [10] (i.e. finite unions of straight-lined, finite, convex, open cells in $\mathfrak{E}^n$); they are denoted by capital roman letters, and their dimensions by upper indices. We use the notation $\dot{X}$, $\bar{X}$, ${}^0X$ for the *boundary*, *closure*, *interior* of X, respectively, and $X - Y = X - (X \cap Y)$ for the *difference*.

By a *decomposition* of X we mean always a collection of *finitely many* pairwise disjoint point sets whose union is X. A decomposition Δ is called a *cell-decomposition*, if the elements of Δ are open cells such that for every two cells $A, B \in \Delta$ either $A \cap \dot{B} = \emptyset$ or $A \subset \dot{B}$ holds. We call a cell-decomposition Δ a *straight-lined triangulation* if its elements are open, straight-lined simplices in $\mathfrak{E}^n$ such that the open faces of each element are also elements of Δ; we call a cell-decomposition Θ a *triangulation* in general if for each element $A \in \Theta$ the decomposition $\Theta(\bar{A})$ of $\bar{A}$, consisting of all those elements of Θ that lie in $\bar{A}$, is isomorphic to the decomposition of a simplex (of the same dimension as A) into its interior and its open faces.

By a (polyhedral) *neighborhood of* X *in* Y (as defined in [14]) we mean the closure of the simplex star of X in a second barycentric subdivision Δ^{**} of a (general) triangulation Δ of Y such that X is the union of elements of Δ; the neighborhood is called *small with respect to* $Z| V | \cdots | W$ (see [4, Kap. I,2]) if $Z \cap Y$, $V \cap Y$, $\cdots$, $W \cap Y$ are unions of elements of Δ.

By an *arc*, *disk*, or 3-*cell* we mean, if not stated otherwise, a compact, nonsingular 1-, 2-, or 3-cell, respectively.

All maps considered in the subsequent part of this paper are *simplicial* maps in the sense of [11, p. 114]: a continous map $\alpha : A' \to B$ is called simplicial if there exist straight-lined triangulations Δ' of A' and Δ of B such that α maps each element of Δ' linearly onto an element of Δ.

Let C^3 be a 3-cell in M^3 and denote the homotopy 3-cell $M^3 - {}^0C^3$ by M^3_*.

5. A simple cell-decomposition Γ of M^3. We can find a cell-decomposition Γ of M^3 with the following properties:

(i) Γ contains just one 0-dimensional element, say E^0, and just one 3-dimensional element, say E^3.

(ii) $C^3 \subset E^3$.

(iii) Γ contains r elements, say $E^1_1, \cdots, E^1_r$, of dimension 1 and r elements, say $E^2_1, \cdots, E^2_r$, of dimension 2.

(iv) Each element E^1_i lies at least 2 times in the boundary of $\bigcup_{j=1}^r E^2_j$ (i.e.: if U^3 is a neighborhood of a point of E^1_i in M^3, which is small with respect to

$$E^1_1| \cdots |E^1_r E^2_1 \cdots |E^2_r,$$

then ${}^0U^3 \cap \bigcup_{j=1}^r E^2_j$ consists of at least 2 pairwise disjoint open disks).

Proof of the assertion. Γ may be found as follows:

Step 0. We take an arbitrary decomposition Γ_0 of M^3 into open cells.

Step 1. We delete, step by step, such 2-dimensional elements of Γ_0 that separate two different 3-dimensional elements; this yields finally a decomposition Γ_1 with only one 3-dimensional element (see [11]).

Step 2. Now we contract a maximal tree in the 1-skeleton of Γ_1 into one point; this yields a decomposition Γ_2 with property (i).

Step 3. If a 1-dimensional element $E^1 \in \Gamma_2$ lies just once in the boundary of a 2-dimensional element $E^2 \in \Gamma_2$ and does not lie in the boundary of any other 2-dimensional element of Γ_2 then we delete both E^1 and E^2; repeating this operation as often as possible, we obtain a decomposition Γ_3 with properties (i) and (iv). Γ_3 possesses also property (iii) since the Euler characteristic of M^3 is zero (see [11]).

Step 4. To obtain Γ we deform the 2-skeleton of Γ_3 isotopically such that the deformed 2-skeleton lies in $M^3 - C^3$.

Remark. In the case $r = 0$, M^3 is obviously a 3-sphere and we have nothing to prove. Therefore we may assume for the subsequent sections of this paper that $r \neq 0$. We denote the 1-skeleton $\bigcup_{i=1}^r \bar{E}^1_i$ and the 2-skeleton $\bigcup_{i=1}^r \bar{E}^2_i$ of Γ by G^1, G^2, respectively.

6. The 1-skeleton G^1 of Γ bounds a singular fan V^2. We assert: There exists a map

$$\zeta : V'^2 \to M^3_*,$$

with the image $\zeta(V'^2) \subset {}^0M^3_*$ denoted by V^2, and with the following properties (see Fig. 2):

(i) V'^2 consists of r disks $V'^2_1, \cdots, V'^2_r$, possessing one common boundary point E'^0, and otherwise being pairwise disjoint; V'^2 is disjoint from M^3, F'^2.

(ii) $\dot{V}^2 = G^1$.

(iii) The only singularities of V^2 are pairwise disjoint, normal, double arcs $A_1^1, \cdots, A_s^1$ (s may be zero) such that each of the two connected components $A_j'^1, A_j''^1$ of $\zeta^{-1}(A_j^1)$ possesses just one boundary point in $\dot{V}'^2 - E'^0$ and otherwise lies in ${}^0V'^2$ (for all $j = 1, \cdots, s$).

(iv) The arcs A_j^1 ($j = 1, \cdots, s$) intersect $G^2 - G^1$ at most in isolated piercing points, V^2 intersects $G^2 - G^1$ at most in piercing curves whose intersection and self-intersection points are the piercing points $A_j^1 \cap (G^2 - G^1)$.

(v) $\zeta^{-1}(\bar{\ }\{V^2 \cap [G^2 - G^1]\})$ is disjoint from $\dot{V}'^2 - E'^0$, i.e. a connected component of

$$\zeta^{-1}(V^2 \cap [G^2 - G^1])$$

is either a 1-sphere or an open arc whose boundary lies in

$$E'^0 + \bigcup_{j=1}^{s}[(\dot{A}_j'^1 + \dot{A}_j''^1) \cap {}^0V'^2]$$

(see Fig. 3).

Proof of the assertion. Step 0. Since M_*^3 is simply connected there exists a map $\zeta_0 : V'^2 \rightarrow M_*^3$ with properties (i) and (ii).

Step 1. From ζ_0 we can obtain by small deformations (by a similar procedure as described in [7]) a map $\zeta_I : V'^2 \rightarrow M_*^3$, also with properties (i), (ii), such that the only singularities of $V_I^2 = \zeta_I(V'^2)$ are normal double curves, triple points, and branch points of multiplicity 1 (see [8]), and such that the triple points, the branch points, and the interiors of the double curves lie in ${}^0V_I^2$, and that E^0 is no double point.

Step 2. Now we consider the set D_I of all double points (not including the triple points) of V_I^2, and we remove, step by step, all those connected components $D_{I1}^1, \cdots, D_{Id}^1$ of D_I that are disjoint from $\dot{V}_I^2$. To do this we can find an arc $C_k^1 \subset V_I^2$ that joins a point of $\dot{V}_I^2 - (E^0 + \dot{D}_I)$ to a point of a component D_{Ik}^1 (provided that $d \neq 0$) such that ${}^0C_k^1 \cap \bar{D}_I$, ${}^0C_k^1 \cap \dot{V}_I^2 = \emptyset$; then we remove D_{Ik}^1 (without introducing a new component of that kind) by a deformation of ζ_I (see Fig. 4) that changes V_I^2 only in a neighborhood of C_k^1, and so on. In this way we obtain finally after d deformations a map $\zeta_{II} : V'^2 \rightarrow M_*^3$.

Step 3. Now we can remove the triple points of $V_{II}^2 = \zeta_{II}(V'^2)$ by deformations of ζ_{II} that change V_{II}^2 only in neighborhoods of double arcs of V_{II}^2 that join the triple points to $\dot{V}_{II}^2 - E^0$. Further we can remove the branch points by cuts along those double arcs of V_{II}^2 that join the branch points to $\dot{V}_{II}^2 - E^0$. This yields a map

$$\zeta_{III} : V'^2 \rightarrow M_*^3,$$

with $\zeta_{III}(V'^2)$ denoted by V_{III}^2, such that the set D_{III} of double points of V_{III}^2 consists of pairwise disjoint arcs $D_{III1}^1, \cdots, D_{IIIe}^1$.

Step 4. If one of the components of the inverse image of D_{IIIk}^1—say $D_{IIIk}'^1$—is disjoint from $\dot{V}'^2$, then we choose an arc $C_k'^1 \subset V'^2$, joining a point of ${}^0D_{IIIk}'^1$ to a point of

$$\cdot V'^2 - [E'^0 + \zeta_{III}^{-1}(\cdot D_{III})],$$

with ${}^0C_k'^1 \cap \zeta_{III}^{-1}(D_{III})$, ${}^0C_k'^1 \cap \cdot V'^2 = \emptyset$, and we remove $D_{IIIk}'^1$ by a deformation of ζ_{III} (similar to Step 2) that changes V_{III}^2 only in a neighborhood of $\zeta_{III}(C_k'^1)$; and so on. This yields finally a map

$$\zeta_{IV} : V'^2 \to M_*^3$$

with the properties (i), (ii), and (iii).

Step 5. From ζ_{IV} we obtain by small deformations a map

$$\zeta_V : V'^2 \to M_*^3,$$

with $\zeta_V(V'^2)$ denoted by V_V^2, having the properties (i), $\cdots$, (iv).

Step 6. From ζ_V we obtain, by deformations that change V_V^2 only in a small neighborhood of $\cdot V_V^2 = G^1$, a map $\zeta : V'^2 \to M_3^*$ with the required properties.

7. Neighborhoods A_j^3 of the double arcs A_j^1 of V^2. Let $A_1^3, \ldots, A_s^3$ be pairwise disjoint neighborhoods of $A_1^1, \cdots, A_s^1$, respectively, in M_*^3, which are small with respect to $G^2 \mid V^2$ (see Fig. 5a).

$A_j^3 \cap G^1$ consists of two disjoint arcs; we denote them by K_j^1, L_j^1. The closures of the connected components of $(A_j^3 \cap V^2) - A_j^1$ are two disks; we denote them by V_{Kj}^2, V_{Lj}^2 such that

$$K_j^1 \subset \cdot V_{Kj}^2, \qquad L_j^1 \subset \cdot V_{Lj}^2.$$

We choose a neighborhood A_j^2 of A_j^1 in V_{Kj}^2, which is small with respect to G^2, and we denote the nonsingular fan ${}^-(V^2 - \bigcup_{j=1}^s A_j^2)$ by V_*^2.

We denote those connected components of $A_j^3 \cap G^2$ that contain K_j^1, L_j^1, respectively, by K_j^2, L_j^2. The closures of the connected components of $K_j^2 - K_j^1$ and $L_j^2 - L_j^1$ are disks $K_{j1}^2, \cdots, K_{jt_j}^2$ and $L_{j1}^2, \cdots, L_{ju_j}^2$, respectively. Those connected components of $A_j^3 \cap G^2$ that are different from K_j^2, L_j^2 are disks $N_{j1}^2, \cdots, N_{jv_j}^2$ (v_j may be zero). We arrange the notation such that the disks $K_{j1}^2, \cdots, K_{jt_j}^2$ lie around K_j^1 in the order of the enumeration and such that V_{Kj}^2 lies in this order between $K_{jt_j}^2$ and K_{j1}^2.

8. A small neighborhood J^3 of the 2-skeleton G^2 and its complementary 3-annulus F_0^3. Let T^3 be a neighborhood of G^1 in M_*^3, which is small with respect to

$$G^2 \mid V^2 \mid A_1^3 \mid \cdots \mid A_s^3 \mid A_1^2 \mid \cdots \mid A_s^2;$$

Let J^3 be a neighborhood of G^2 in M_*^3, which is small with respect to

$$T^3 \mid V^2 \mid A_1^3 \mid \cdots \mid A_s^3 \mid A_1^2 \mid \cdots \mid A_s^2.$$

Then $M_*^3 - {}^0J^3$ is a 3-annulus F_0^3.

We denote $T^3 \cap J^3$ by T_J^3, and the two connected components of $T_J^3 \cap A_j^3$ ($j = 1, \cdots, s$) by T_{Kj}^3, T_{Lj}^3 (see Fig. 5b) such that $K_j^1 \subset T_{Kj}^3$ and $L_j^1 \subset T_{Lj}^3$.

Further we denote the connected components of $J^3 \cap A_j^3$ by $K_j^3, L_j^3, N_{j1}^3, \cdots, N_{jv_j}^3$ where

$$K_j^2 \subset K_j^3, \quad L_j^2 \subset L_j^3, \quad N_{jm}^2 \subset N_{jm}^3 \quad (m = 1, \cdots, v_j),$$

and the connected components of $^-(K_j^3 - T_{Kj}^3)$ and $^-(L_j^3 - T_{Lj}^3)$ by $K_{j1}^3, \cdots, K_{jt_j}^3$ and $L_{j1}^3, \cdots, L_{ju_j}^3$, respectively, where

$$K_{jk}^2 \cap K_{jk}^3 \neq \emptyset \ (k = 1, \cdots, t_j) \quad \text{and} \quad L_{jl}^2 \cap L_{jl}^3 \neq \emptyset \ (1 = 1, \cdots, u_j).$$

Those $t_j - 1$ connected components of $^-(A_j^3 - K_j^3)$ that are disjoint from V_{Kj}^2 are 3-cells $F_{Kj1}^3, \cdots, F_{Kjt_j-1}^3$ in F_0^3 (see Fig. 5b).

The connected components of $^-(J^3 - T_J^3)$ are r 3-cells; we denote them by $P_1^3, \cdots, P_r^3$ where $E_i^2 \cap P_i^3 \neq \emptyset$ $(i = 1, \cdots, r)$, and we denote the disks $E_i^2 \cap P_i^3$ by P_i^2. Then P_i^3 can be represented as cartesian product $P_i^2 \times I^1$, where I^1 is the interval $-1 \leqq x \leqq +1$, such that

(i) P_i^2 is the central disk, i.e. $p \times 0 = p$ for all $p \in P_i^2$;

(ii) the top and bottom disks are the connected components of $^\cdot P_j^3 \cap {}^\cdot J^3$, i.e. $(P_i^2 \times 1) + (P_i^2 \times -1) = {}^\cdot P_i^3 \cap {}^\cdot J^3$;

(iii) the polyhedra A_j^3, V^2, A_j^2 intersect P_i^3 "prismatically", i.e.:

$$A_j^3 \cap P_i^3 = (A_j^3 \cap P_i^2) \times I^1, \ V^2 \cap P_i^3 = (V^2 \cap P_i^2) \times I^1, \ A_i^2 \cap P_i^3 = (A_i^2 \cap P_i^2) \times I^1.$$

Let $F_0'^3$ be a 3-annulus, disjoint from M^3, V'^2, F'^2, and let

$$\beta_0 : F_0'^3 \to M_*^3$$

be a homeomorphism with the image $\beta_0(F_0'^3) = F_0^3$. We denote the boundary 2-spheres $\beta_0^{-1}({}^\cdot J^3)$ and $\beta_0^{-1}({}^\cdot M_*^3)$ of $F_0'^3$ by $F_0'^2$ and S'^2, respectively. (We may bring about by isotopic deformations the situation in which $\beta_0(F_0'^2) = \alpha_0(F'^2)$ with α_0 the embedding given in Theorem 3.)

9. Deformations in the A_j^3's that take G^1 onto the boundary of the nonsingular fan V_*^2. We denote the 3-cell $K_j^3 + \bigcup_{k=1}^{t_j-1} F_{Kjk}^3$ (see Fig. 5b) by Q_j^3, and choose a neighborhood Q_{*j}^3 of $^-({}^\cdot A_j^3 - {}^\cdot Q_j^3)$ in $^-(A_j^3 - Q_j^3)$, which is small with respect to $G^2 \mid V^2 \mid A_j^2 \mid T^3 \mid T_J^3 \mid J^3$, such that (with respect to the product representation introduced in Sec. 8)

$$^-({}^0Q_{*j}^3 \cap P_i^3) = {}^-(Q_{*j}^3 \cap P_i^2) \times I^1 \qquad (i = 1, \cdots, r).$$

Then we denote the 3-cell $^-[A_j^3 - (Q_j^3 + Q_{*j}^3)]$ by O_j^3 and the disks ${}^\cdot O_j^3 \cap {}^\cdot Q_j^3$ and ${}^\cdot O_j^3 \cap {}^\cdot Q_{*j}^3$ by O_j^2 and O_{*j}^2, respectively.

Now we can find an epi-homeomorphism $\delta_j : Q_j^3 \to Q_j^3 + O_j^3$ with the following properties (see Fig. 5):

(i) $\delta_j \mid ({}^\cdot Q_j^3 - {}^0O_j^2) = \text{identity}$; $\delta_j(O_j^2) = O_{*j}^2$.

(ii) $\delta_j(K_j^1) = (K_j^1 - {}^\cdot A_j^2) + {}^-({}^\cdot A_j^2 - K_j^1)$.

(iii) $\delta_j(K_{jk}^2)$ intersects L_j^1 in just one point and intersects each disk O_j^2, $V_{Lj}^2, L_{j1}^2, \cdots, L_{ju_j}^2, N_{j1}^2, \cdots, N_{jv_j}^2$ in just one arc (for all $k = 1, \cdots, t_j$); $\delta_j({}^0K_{jk}^2)$ is disjoint from V_{Kj}^2.

(iv) The neighborhood $\delta_j(T^3_{Kj})$ of $\delta_j(K^1_j)$ in A^3_j is small with respect to $T^3_{Lj} \mid V^2 \mid L^3_{j1} \mid \cdots \mid L^3_{ju_j} \mid N^3_{j1} \mid \cdots \mid N^3_{jv_j}$ and intersects O^2_j in just two disjoint disks.

(v) The intersections of $\delta_j(K^3_{jk})$, $\delta_j(K^2_{jk})$ $(k = 1, \cdots, t_j)$, and $\delta_j(T^3_{Kj})$ with L^3_{jl} $(1 = 1, \cdots, u_j)$ and N^3_{jm} $(m = 1, \cdots, v_j)$ (see also Fig. 6) can be written as cartesian products, using the product representation of the P^3_i's introduced in Sec. 8; the same holds for the polyhedra

$$\delta_j^{-1}(L^3_{j1} \cap \delta_j(K^3_{jk})), \qquad \delta_j^{-1}(L^2_{j1} \cap \delta_j(K^3_{jk})), \qquad \delta_j^{-1}(N^3_{jm} \cap \delta_j(K^3_{jk})),$$

$$\delta_j^{-1}(N^2_{jm} \cap \delta_j(K^3_{jk})), \qquad \delta_j^{-1}(T^3_{Lj} \cap \delta_j(K^3_{jk})), \qquad \delta_j^{-1}(V^2_{Lj} \cap \delta_j(K^3_{jk})).$$

Let $\eta : J^3 \to M^3_*$ be the map defined by

(a) $\eta \mid {}^-(J^3 - \bigcup_{j=1}^s K^3_j) = \text{identity}$,
(b) $\eta \mid K^3_j = \delta_j \mid K^3_j$ (for all $j = 1, \cdots, s$),

and denote the images $\eta(J^3), \eta(G^1), \eta(G^2), \eta(T^3_J), \eta(P^3_i)$ by $J^3_\#, G^1_\#, G^2_\#, T^3_{\#J}, O^3_{\#i}$, respectively. Obviously we have $G^1_\# = {}^\cdot V^2_*$.

Now we denote $\beta_0^{-1}(O^2_j)$ by O'^2_j, and we choose s pairewise disjoint 3-cells $O'^3_1, \cdots, O'^3_s$ (see Fig. 7) that are disjoint from M^3, V'^2, F'^2, ${}^0F'^3_0$ such that ${}^\cdot O'^3_j \cap {}^\cdot F'^3_0 = O'^2_j$; then we denote $F'^3_0 + \bigcup_{j=1}^s O'^3_j$ by F'^3_I, and we choose a map

$$\beta_I : F'^3_I \to M^3_*$$

with the following properties:

(I) $\beta_I \mid {}^-[F'^3_0 - \bigcup_{j=1}^s \bigcup_{k=1}^{t_j-1} \beta_0^{-1}(F^3_{Kjk})] = \beta_0 \mid {}^-[F'^3_0 - \bigcup_{j=1}^s \bigcup_{k=1}^{t_j-1} \beta_0^{-1}(F^3_{Kjk})]$.

(II) $\beta_I \mid \beta_0^{-1}(F^3_{Kjk}) = [\delta_j \mid F^3_{Kjk}] \cdot [\beta_0 \mid \beta_0^{-1}(F^3_{Kjk})]$ for all $j = 1, \cdots, s$; $k = 1, \cdots, t_j$).

(III) $\beta_I \mid O'^3_j$ is an epi-homeomorphism of O'^3_j onto O^3_j.

We remark that the map β_I is locally one-to-one, except for the "reflection disks" O'^2_j, i.e. if p is a point of F'^3_I and if U'^3 is a sufficiently small neighborhood of p in F'^3_I then $\beta_I \mid U'^3$ is a homeomorisphm if and only if $p \notin \bigcup_{j=1}^s O'^2_j$.

10. $G^1_\#$ and its neighborhood $T^3_{\#J}$ lie in a 3-cell H^3. Let H^3 be a neighborhood of $V^2_* + T^3_{\#J}$ in M^3_*, which is small with respect to

$$G^2_\# \mid V^2 \mid J^3_\# \mid A^3_1 \mid \cdots \mid A^3_s \mid O^2_1 \mid \cdots \mid O^2_s,$$

that intersects the $P^3_{\#i}$'s prismatically, i.e.: $\eta^{-1}(H^3 \cap P^3_{\#i})$ $(i = 1, \cdots, r)$ can be written as cartesian product using the product representation of the P^3_i's introduced in Sec. 8 (compare Fig. 11a).

11. Arcs W^1_i in ${}^\cdot J^3 \cap {}^\cdot T^3_J$ joining top and bottom disks of the prisms P^3_i. T^3_J a handlebody of genus r. The intersection ${}^\cdot J^3 \cap {}^\cdot T^3_J$ is a 2-sphere with $2r$ holes, denoted by T^2.

We assert: There can be found r pairwise disjoint arcs $W^1_1, \cdots, W^1_r \subset T^2$ such that (for all $i = 1, \cdots, r$)

(i) ${}^0W_i^1 \subset {}^0T^2$; ${}^{\cdot}W_i^1 = p_i \times {}^{\cdot}I^1$ (using the product representation of the P_i^3's introduced in Sec. 8) with p_i an arbitrary point in ${}^{\cdot}P_i^2 - \bigcup_{j=1}^{s} A_j^3$; we denote the arc $p_i \times I^1$ by W_{Pi}^1 ;

(ii) if $S_i^1 \subset {}^0T_J^3$ is a 1-sphere, topologically parallel to $W_i^1 + W_{Pi}^1$, i.e.: such that there exists an annulus in T_J^3 with boundary curves S_i^1 and $W_i^1 + W_{Pi}^1$, then S_i^1 is homologous to 0 mod 2 in $M_*^3 - (W_i^1 + W_{Pi}^1)$.

We denote the arc $\eta(W_i^1)$ by $W_{\#i}^1$. There exists just one connected component of $\beta_I^{-1}(W_{\#i}^1)$—we denote it by $W_i'^1$—such that $\beta_I(W_i'^1) = W_{\#i}^1$; and $W_i'^1 \subset {}^{\cdot}F_I'^3$.

Proof of the assertion. First we remark that the 1-spheres ${}^{\cdot}P_1^2, \cdots, {}^{\cdot}P_r^2$ form a 1-dimensional homology basis mod 2 of T_J^3 (if we identify the chains mod 2 with the corresponding polyhedra). If ${}^{\cdot}P_1^2, \cdots, {}^{\cdot}P_r^2$ were homologously dependent mod 2 it would follow that there exists a surface in T_J^3 with boundary some of the ${}^{\cdot}P_i^2$'s; this surface could be completed by the corresponding disks P_i^2 to a closed surface, non-separating in M_*^3 ; but this is impossible since M_*^3 is a homotopy 3-cell.

We choose an arbitrary system of pairwise disjoint arcs

$$W_1^{*1}, \cdots, W_r^{*1} \subset T^2$$

fulfilling condition (i). Now $W_i^{*1} + W_{Pi}^1$ $(i = 1, \cdots, r)$ is homologous mod 2 in T_J^3 to a linear combination $\sum_{k=1}^{r} c_{ik}' {}^{\cdot}P_k^2$ with coefficients $c_{ik}' = 0$ or 1. If $c_{ii}' = 0$ then we take $W_i^1 = W_i^{*1}$. If $c_{ii}' \neq 0$ then to obtain W_i^1 we take a small neighborhood N_i^2 of ${}^{\cdot}P_i^2 \times 1$ in T^2 and replace the arc $W_i^{*1} \cap N_i^2$ by another arc in N_i^2 with the same boundary points such that $W_i^1 + W_{Pi}^1$ is homologous mod 2 to $W_i^{*1} + W_{Pi}^1 + {}^{\cdot}P_i^2$ in T_J^3. Now the W_i^1's fulfill condition (ii) also. For every $i = 1, \cdots, r$ there exists a surface in T_J^3 whose boundary consists of S_i^1 and some of the ${}^{\cdot}P_k^2$'s, except ${}^{\cdot}P_i^2$, and whose interior lies in ${}^0T_J^3$; this surface can be completed by the corresponding P_k^2's to a surface B_i^2 in $M_*^3 - (W_i^1 + W_{Pi}^1)$ that is bounded by S_i^1 only.

12. Singular disks $W_{\#i}^2$ in H^3 corresponding to the arcs $W_{\#i}^1$. Let $W_1'^2, \cdots, W_r'^2$ be r pairwise disjoint disks that are disjoint from M^3, ${}^0F_I'^3$, F'^2, V'^2 such that

$$W_i'^2 \cap {}^{\cdot}F_I'^3 = {}^{\cdot}W_i'^2 \cap {}^{\cdot}F_I'^3 = W_i'^1 \qquad (\text{for all } i = 1, \cdots, r).$$

We denote ${}^{\cdot}W_i'^2 - {}^0W_i'^1$ by $W_{Pi}'^1$, and $\bigcup_{i=1}^{r} W_i'^2$ by W'^2.

Now we assert: There exists a map $\vartheta : W'^2 \to H^3$, with the image $\vartheta(W'^2) \subset {}^0H^3$ denoted by $W_{\#}^2$, and with the following properties:

(i) $\vartheta \mid W_i'^1 = \beta_I \mid W_i'^1$ and $\vartheta(W_{Pi}'^1) = W_{Pi}^1$ (for all $i = 1, \cdots, r$).

(ii) The only singularities of $W_{\#}^2$ are pairwise disjoint, normal, double arcs $B_1^1, \cdots, B_b^1$ (b may be zero) such that each of the two connected components $B_f'^1$, $B_f''^1$ of $\vartheta^{-1}(B_f^1)$ possesses just one boundary point in $\bigcup_{i=1}^{r} {}^0W_i'^1$ and otherwise lies in ${}^0W'^2$ (for all $f = 1, \cdots, b$). W^2 intersects the $P_{\#i}^3$'s prismatically.

(iii) There exists a neighborhood U'^2 of ${}^{\cdot}W'^2$ in W'^2 such that $\vartheta({}^0U'^2) \subset {}^0T_{\#J}^3$.

Proof of the assertion. *Step* 0. Since $W^1_{\#i} + W^1_{Pi} \subset {}^0H^3$ (for all $i = 1, \cdots, r$) there exists a map $\vartheta_0 : W'^2 \to H^3$ with property (i).

Step 1. As in the proof of Sec. 6, steps 1 to 5, we can derive from ϑ_0 a map $\vartheta_I : W'^2 \to H^3$ with properties (i), (ii).

Step 2. We choose pairwise disjoint neighborhoods $N^3_1, \cdots, N^3_r$ of the 1-spheres $W^1_{\#i} + W^1_{Pi}$ in H^3, which are small with respect to $T^3_{\#J} \mid \vartheta_I(W'^2)$. The intersection ${}^{\cdot}N^3_i \cap \vartheta_I(W'^2_i)$ consists of a 1-sphere N^1_i, topologically parallel to $W^1_{\#i} + W^1_{Pi}$, and of an even number n_i of meridian circles of N^3_i each of which pierces N^1_i in just one point. Now we choose an oriented 1-sphere X^1_i in ${}^{\cdot}N^3_i \cap {}^0T^3_{\#J}$, topologically parallel to $W^1_{\#i} + W^1_{Pi}$, and an oriented meridian circle Y^1_i of N^3_i that intersects X^1_i in just one point; we denote the homology classes of X^1_i and Y^1_i in ${}^{\cdot}N^3_i$ by $\mathfrak{x}_i$ and $\mathfrak{y}_i$, respectively. Then the homology class $\mathfrak{n}_i$ of the properly oriented 1-sphere N^1_i is $\mathfrak{n}_i = \mathfrak{x}_i + w_i \mathfrak{y}_i$.

Now we need the fact that the coefficients w_i are even numbers. To prove this we show that both N^1_i and X^1_i are homologous 0 mod 2 in $M^3_* - (W^1_{\#i} + W^1_{Pi})$:

(1) N^1_i bounds a 2-dimensional polyhedron $D^2_i \subset \vartheta_I(W'^2_i)$ that intersects $W^1_{\#i} + W^1_{Pi}$ in the even number n_i of piercing points. From D^2_i we remove n_i disks, being the intersections of D^2_i with a small neighborhood U^3_i of $W^1_{\#i} + W^1_{Pi}$ in N^3_i, and replace them by $\frac{1}{2}n_i$ annuli in ${}^{\cdot}U^3_i$ such that we obtain a 2-dimensional polyhedron bounded by N^1_i and disjoint from $W^1_{\#i} + W^1_{Pi}$.

(2) $(\eta \mid T^3_J)^{-1}(X^1_i)$ is a 1-sphere $S^1_i \subset {}^0T^3_J$ and there exists an annulus B^{*2}_i with boundary curves S^1_i and $W^1_i + W^1_{Pi}$ and with ${}^0B^{*2}_i \subset {}^0T^3_J$. On the other hand S^1_i bounds a surface B^2_i in $J^3 - (W^1_i + W^1_{Pi})$ as constructed in the proof of Sec. 11 which can be chosen disjoint from ${}^0B^{*2}_i$. We can bring about by small deformations the situation in which $\eta(B^2_i + B^{*2}_i)$ has normal double curves but no branch points (since η is locally one-to-one). Therefore (and since $\eta \mid B^{*2}_i$ is one-to-one) $\eta(B^2_i)$ intersects the boundary curve $W^1_{\#i} + W^1_{Pi}$ of $\eta(B^2_i + B^{*2}_i)$ in an even number of piercing points. From $\eta(B^2_i)$ we obtain, as in (1), a 2-polyhedron disjoint from $W^1_{\#i} + W^1_{Pi}$ with boundary X^1_i.

If $w_i \neq 0$ (for some $i = 1, \cdots, r$) then we choose a point in ${}^0W^1_{\#i}$, which is no double point of $\vartheta_I(W'^2)$, and a neighborhood R^3_i of this point in N^3_i which is small with respect to $\vartheta_I(W'^2) \mid W^1_{\#i}$. We denote the disk $R^3_i \cap \vartheta_I(W'^2)$ by W^2_{Ri}. In ${}^0R^3_i$ we choose a disk R^2_i(*see* Fig. 8) such that ${}^{\cdot}R^2_i \cap W^1_{\#i}$ is one arc R^1_i, such that ${}^0R^2_i \cap {}^0W^2_{Ri}$ is an open arc one of whose boundary points lies in ${}^{\cdot}R^2_i - R^1_i$ and the other one in $W^1_{Ri} - R^1_i$, and such that ${}^-[{}^{\cdot}(W^2_{Ri} + R^2_i) \cap {}^0R^3_i]$ is an unknotted chord in R^3_i. Then we choose an epi-homeomorphism

$$\lambda_i : R^3_i \to R^3_i$$

with $\lambda_i \mid {}^{\cdot}R^3_i =$ identity and $\lambda({}^-[{}^{\cdot}(W^2_{Ri} + R^2_i) \cap {}^0R^3_i]) = W^1_{\#i} \cap R^3_i$ and a map

$$\vartheta_{II} : W'^2 \to H^3$$

with

$$\vartheta_{II} \mid {}^-[W'^2 - \vartheta_I^{-1}(W^2_{Ri})] = \vartheta_I \mid {}^-[W'^2 - \vartheta_I^{-1}(W^2_{Ri})]$$

and

$$\vartheta_{II}(\vartheta_I^{-1}(W^2_{Ri})) = \lambda_i(W^2_{Ri} + R_i^2).$$

Now let N^3_{IIi} be a neighborhood of $W^1_{\#i} + W^1_{Pi}$ in N^3_i, being small with respect to $\vartheta_{II}(W'^2) \mid T^3_{\#J}$. Then ${}^0N^3_{IIi} \cap \vartheta_{II}(W'^2)$ consists of a 1-sphere N^1_{IIi}, topologically parallel to $W^1_{\#i} + W^1_{Pi}$, and of $n_i + 2$ meridian circles of N^3_{IIi}. The homology class $\mathfrak{n}_{IIi}$ of the properly oriented N^1_{IIi} in $N^3_i - {}^0N^3_{IIi}$ is

$$\mathfrak{n}_{IIi} = \mathfrak{x}_{IIi} + (w_i \pm 2)\mathfrak{y}_{IIi}$$

with $\mathfrak{x}_{IIi}$, $\mathfrak{y}_{IIi}$ the homology classes of X^1_i, Y^1_i, respectively, in $N^3_i - {}^0N^3_{IIi}$. The sign in the coefficient $w_i \pm 2$ depends on the choice of R^2_i (see Fig. 8). So we can derive by $\frac{1}{2}\sum_{i=1}^{r} w_i$ operations of the kind described a map

$$\vartheta_* : W'^2 \to H^3$$

such that (under analogous notation) the curve N^1_{*i} is homologous to X^1_{*i} in $N^3_i - {}^0N^3_{*i}$ (for all $i = 1, \cdots, r$).

If $w_i = 0$ (for all $i = 1, \cdots, r$) then we choose $\vartheta_* = \vartheta_I$, etc.

Step 3. From ϑ_* we can obtain by deformations (that change $\vartheta_*(W'^2)$ only in the N^3_{*i}'s) a map $\vartheta : W'^2 \to H^3$ with the demanded properties (i), (ii), (iii).

13. Deformation over prismatic neighborhoods of the singular disks $W^2_{\#i}$. The map ϑ can be extended to a map $\tilde{\vartheta} : W'^3 \to H^3$, with $\tilde{\vartheta}(W'^3) \subset {}^0H^3$ denoted by $W^3_{\#}$, such that (see Fig. 9) the following hold:

(i) W'^3 may be represented as cartesian product $W'^2 \times I^1_*$ where I^1_* means an interval $-1 \leqq x_* \leqq 1$, with $p \times 0 = p$ for all $p \in W'^2$, and W'^3 is disjoint from M^3, F'^2, V'^2. We denote the components $W'^2_i \times I^1_*$ of W'^3 by W'^3_i.

(ii) $W'^3_i \cap F'^3_i = {}^{\cdot}W'^3_i \cap {}^{\cdot}F'^3_i = W'^1_i \times I^1_*$ with

$$\tilde{\vartheta} \mid ({}^{\cdot}W'^3_i \cap {}^{\cdot}F'^3_i) = \beta_I \mid ({}^{\cdot}W'^3_i \cap {}^{\cdot}F'^3_i).$$

(iii) $W^3_{\#}$ and the $P^3_{\#i}$'s intersect each other prismatically, i.e.:

$$\eta^{-1}(W^3_{\#} \cap P^3_{\#i}) = \{[\eta^{-1}(W^3_{\#} \cap P^3_{\#i})] \cap P^2_i\} \times I^1$$

and

$$\tilde{\vartheta}^{-1}(W^3_{\#} \cap P^3_{\#i}) = \{[\tilde{\vartheta}^{-1}(W^3_{\#} \cap P^3_{\#i})] \cap W\ ^2\} \times I^1_*$$

(using the product representations introduced in Sec.8 and in (i), respectively).

(iv) If p is a point of $W^3_{\#}$, $\vartheta^{-1}(p)$ is either one or two points. The set B of all double points of $W^3_{\#}$ is disjoint from the disks $\tilde{\vartheta}(W'^1_{Pi} \times I^1_*)$ $(i = 1, \cdots, r)$ and is prismatic, i.e.

$$\tilde{\vartheta}^{-1}(B) = [\tilde{\vartheta}^{-1}(B) \cap W'^2] \times I^1_*,$$

(using the same product representation as in (i)).

We denote the 3-annulus $F'^3_I + W'^3$ by F'^3_{II} and we define a map

$$\beta_{II} : F'^3_{II} \to M^3_*$$

such that $\beta_{II} \mid F'^3_I = \beta_I \mid F'^3_I$ and $\beta_{II} \mid W'^3 = \tilde{\vartheta}$.

14. Deformation over the prisms $P^3_{\#i}$. In $\dot{F}'^3_{II} - S'^2$ there are $2r$ pairwise disjoint disks P'^2_{+i}, P'^2_{-i} ($i = 1, \cdots, r$) mapping onto the top and bottom disks of the $P^3_{\#i}$'s, i.e. such that $\beta_{II}(P'^2_{\pm i}) = \eta(P^2_i \times \pm 1)$. Now we choose r pairwise disjoint 3-cells $P'^3_1, \cdots, P'^3_r$, disjoint from M^3, F'^2, V'^2, such that

$$P'^3_i \cap F'^3_{II} = \dot{P}'^3_i \cap \dot{F}'^3_{II} = P'^2_{+i} + P'^2_{-i} + (W'^1_{Pi} \times I^1_*)$$

(being a disk, for all $i = 1, \cdots, r$); and we choose epi-homeomorphisms

$$\varkappa_i : P'^3_i \to P^3_i$$

such that $\eta_i \cdot \varkappa_i \mid (\dot{P}'^3_i \cap \dot{F}'^3_{II}) = \beta_{II} \mid (\dot{P}'^3_i \cap \dot{F}'^3_{II})$. Finally we denote the 3-annulus $F'^3_{II} + \bigcup_{i=1}^r P'^3_i$ by F'^3 and we define a map

$$\beta : F'^3 \to M^3_*$$

such that $\beta \mid F'^3_{II} = \beta_{II}$ and $\beta \mid P'^3_i = \eta_i \cdot \varkappa_i$.

We denote the handlebody $H^3 + \bigcup_{j=1}^s A^3_j$ by K^3 and $\beta^{-1}(K^3 \cap \beta(F'^3))$ by K'^3. We remark that $\beta(\dot{F}'^3 - S'^2) \subset {}^0H^3$ and that

$$\beta \mid {}^-(F'^3 - K'^3) : {}^-(F'^3 - K'^3) \to {}^-(M^3_* - K^3)$$

is an epi-homeomorphism. Moreover β is locally one-to-one, except on the s surfaces ${}^-(\dot{O}'^3_j \cap {}^0F'^3)$; it is locally three-to-one on the arcs ${}^-(\dot{O}'^2_j \cap {}^0F'^3)$ and locally two-to-one otherwise on ${}^-(\dot{O}'^3_j \cap {}^0F'^3)$.

15. Conclusion. There can be found an epi-homeomorphism $\lambda : M^3 \to M^3$ such that the image $C^1_0 = \lambda(C^1)$ of the given curve C^1 lies in ${}^0M^3_* - K^3$. Then we choose a knot projection cone $D'^2 \subset F'^3$ with $\dot{D}'^2 = \beta^{-1}(C^1_0)$. We can choose D'^2 such that $\beta \mid D'^2$ is locally one-to-one. Further we can bring about by small deformations the situation in which the singularities of the image $\beta(D'^2)$ are normal. Then $D^2 = \lambda^{-1}(\beta(D'^2))$ possesses the demanded properties. This proves Theorem 2.

We choose two disjoint 3-cells C'^3, C''^3 with

$$C'^3 \cap F'^3 = S'^2 = \dot{C}'^3, \qquad C''^3 \cap F'^3 = \dot{F}'^3 - S'^2 = \dot{C}''^3,$$

an epi-homeomorphism

$$\beta' : C'^3 \to C^3$$

with $\beta' \mid S'^2 = \beta \mid S'^2$, and a map

$$\beta'' : C''^3 \to H^3$$

with $\beta'' \mid (\dot{F}'^3 - S'^2) = \beta \mid (\dot{F}'^3 - S'^2)$. Then $F'^3 + C'^3 + C''^3$ is a 3-sphere S^3 and the map $\gamma : S^3 \to M^3$, composed of β, β', β'', has the demanded properties. This proves Theorem 1.

II. Proof of Theorem 3

We bring about (by isotopic deformations) the situation in which the 2-sphere $\dot{J}^3 = \beta_0(F'^2_0)$ (see Sec. 8) is equal to the image $F^2_0 = \alpha_0(F'^2)$ under the given embedding α_0. We denote the 2-spheres

$$\cdot F_I'^3 - S'^2, \qquad \cdot F_{II}'^3 - S'^2, \qquad \cdot F'^3 - S'^2$$

by $F_I'^2$, $F_{II}'^2$, $F_{III}'^2$, respectively, and we choose epi-homeomorphisms μ_0, μ_I, μ_{II}, μ_{III} of F''^2 onto $F_0'^2$, $F_I'^2$, $F_{II}'^2$, $F_{III}'^2$, respectively, such that $\alpha_0 = (\beta_0 \mid F_0'^2)\cdot\mu_0$ and

$$\mu_{[i]}^{-1} \mid (F_{[i]}'^2 \cap F_{[i-1]}'^2) = \mu_{[i-1]}^{-1} \mid (F_{[i]}'^2 \cap F_{[i-1]}'^2) \qquad (\text{for } [i] = I, II, III)\cdot$$

We denote the maps

$$(\beta_I \mid F_I'^2)\cdot\mu_I, \qquad (\beta_{II} \mid F_{II}'^2)\cdot\mu_{II}, \qquad (\beta \mid F_{III}'^2)\cdot\mu_{III},$$

defining singular 2-spheres in M_*^3, by α_I, α_{II}, α_3, respectively. Now α_3 fulfills already the condition (b) of Theorem 3, and it remains to show that the deformation from α_0 to α_3, which may be derived from the proof of Theorem 1, 2, can be decomposed into a sequence of elementary deformations, according to condition (a).

16. Decomposing the deformations in the A_j^3's. The deformation from α_0 to α_I, changing the 2-sphere F_0^2 in the A_j^3's (see Sec. 9), can be decomposed into a sequence of $\sum_{j=1}^{s} t_j\cdot(u_j + 2v_j)$ elementary deformations of type 1a, intermixed with nonessential deformations, (see Fig. 5).

We denote the connected components of the (prismatic) intersections

$$\eta(K_{jk}^3) \cap L_{jl}^3 \qquad (j = 1, \cdots, s; k = 1, \cdots, t_j; l = 1, \cdots, u_j)$$

under current enumeration by $C_1^3, \cdots, C_c^3$ and the connected components of

$$\eta(K_{jk}^3) \cap N_{jm}^3 \qquad (m = 1, \cdots, v_j)$$

by $D_1^3, \cdots, D_d^3$. Further we denote that connected component of $\eta^{-1}(C_g^3)$ $(g = 1, \cdots, c)$ that is different from C_g^3 by $C_g'^3$, and that connected component of $\eta^{-1}(D_h^3)$ $(h = 1, \cdots, d)$ that is different from D_h^3 by $D_h'^3$. Finally we denote the intersections of the C_g^3, $C_g'^3$, D_h^3, $D_h'^3$'s with the P_i^2's (see Fig. 11a) by C_g^2, $C_g'^2$, D_h^2, $D^{2\prime}_h$, respectively, and the intersections of the K_{jk}^3, L_{jl}^3's with the P_i^2's by K_{Pjk}^2, L_{Pjl}^2, respectively.

17. Decomposing the deformations over $W_\#^3$. We can bring about by small deformations the situation in which the singular discs $W_{\#i}^2$ and their prismatic neighbourhood $W_\#^3$ (as constructed in Secs. 11, 12, 13) are in a "normal position" with respect to the singular 2-sphere $F_I^2 = \alpha_I(F'^2)$ and to the singular disks $P_{\#i}^2$, etc., i.e. such that the following conditions hold:

(i) F_I^2, H^3, the A_j^3's, and the $P_{\#i}^2$'s intersect $W_\#^3$ prismatically with respect to the product representation introduced in Sec. 13.

We denote $\tilde{\vartheta}(\vartheta^{-1}(p_i) \times I_*^1)$ by P_i^1 (Fig. 9).

(ii) $\eta^{-1}(W_\#^2 \cap P_{\#i}^2)$ $(i = 1, \cdots, r)$ is disjoint from those connected components of $K_{Pjk}^2 \cap \eta^{-1}(H^3 \cap P_{\#i}^3)$ and $L_{Pjl}^2 \cap \eta^{-1}(H^3 \cap P_{\#i}^3)$ $(j = 1, \cdots, s;$

$k = 1, \cdots, t_j$; $l = 1, \cdots, u_j$) that contain the arcs $\dot{K}^2_{Pjk} \cap \dot{P}^2_i$, $\dot{L}^2_{Pjl} \cap \dot{P}^2_i$, respectively, in their boundaries (see Fig. 11a).

Now we carry out the deformation of α_I into α_{II} in three steps:

Step 1. Let B'^3_f $(f = 1, \cdots, b)$ (see Fig. 10) be that connected component of $\tilde{\vartheta}^{-1}(B^3)$ that contains B'^1_f. We choose pairwise disjoint neighborhoods B'^3_{*f} of the B'^3_f's in W'^3, which are small with respect to $\tilde{\vartheta}^{-1}(F^2_I \cap W^3_{\#}) \mid \tilde{\vartheta}^{-1}(B^3)$ and which are cartesian products in the product representation introduced in Sec. 13. Now we deform F^2_I over the 3-cells $\tilde{\vartheta}(B'^3_{*f})$ which can be done by a sequence of elementary deformations of type 1a. We denote the map so obtained from α_I by α_{I*} and $^-(W'^2 - \bigcup_{f=1}^{b} B'^3_{*f})$ by W'^2_*. Now we have to deform $F^2_{I*} = \alpha_{I*}(F'^2)$ over the remaining nonsingular 3-cells $\tilde{\vartheta}(W'^2_* \times I^1_*)$.

Step 2. In W'^2_* we choose pairwise disjoint arcs $X^1_1, \cdots, X^1_x$ (see Fig. 10) with $^0X^1_m \subset {}^0W'^2_*$ that join points of

$$\dot{W}'^2_* - \bigcup_{i=1}^{r} W'^1_{Pi}$$

to points of

$$\vartheta^{-1}(F^2_{I*} \cap W^2_{\#}) \cap {}^0W'^2_*$$

such that

(a) every double point of $\vartheta^{-1}(F^2_{I*} \cap W^2_{\#}) \cap {}^0W'^2_*$ is end point of one arc X^1_m,

(b) every connected component of $\vartheta^{-1}(F^2_{I*} \cap W^2_{\#}) \cap {}^0W'^2_*$ contains at least one end point of an arc X^1_m,

(c) the X^1_m's intersect $\vartheta^{-1}(F^?_{I*} \cap W^2_{\#}) \cap {}^0W'^2_*$ in isolated piercing points that are no double points of $\vartheta^{-1}(F^2_{I*} \cap W^2_{\#}) \cap {}^0W'^2_*$,

(d) the points $\vartheta(\dot{X}^1_m \cap \dot{W}'^2_*)$ are no double points of F^2_{I*}.

Now we choose pairwise disjoint neighborhoods X^2_m of the X^1_m's in W'^2_*, which are small with respect to $\vartheta^{-1}(F^2_{I*} \cap W^2_{\#})$. Then we deform F^2_{I*} over the 3-cells $\tilde{\vartheta}(X^2_m \times I^1_*)$ which can be done by a sequence of elementary deformations of type 1a and 1b. According to the notation used in Theorem 3 we denote the map so obtained from α_{I*} by α_1 and $\alpha_1(F'^2)$ by F^2_1. Further we denote $^-(W'^2_* - \bigcup_{m=1}^{x} X^2_m)$ by W'^2_{**}.

Step 3. Finally we deform F^2_1 over the remaining 3-cells $\tilde{\vartheta}(W'^2_{**} \times I^1_*)$. This can be done by a sequence of elementary deformations of type 2 (and may be nonessential deformations) since the curves $\vartheta^{-1}(F^2_1 \cap W^2_{\#}) \cap {}^0W'^2_{**}$ are nonsingular, pairwise disjoint, open arcs with boundary points in

$$\dot{W}'^2_{**} - \bigcup_{i=1}^{r} W'^1_{Pi}.$$

By this we obtain from α_1 the map α_{II}.

18. Decomposing the deformations over the $P^3_{\# i}$'s. We carry out the deformation of α_{II} into α_3 in four steps (see Fig. 11).

Step 1. Let Q^1_i be a neighborhood of a point $\epsilon\, \dot{P}^1_i$ in $\dot{P}^2_i - {}^0P^1_i$ which is small with respect to $\eta^{-1}(F^2_{II} \cap P^2_{\#i})$ and let $Y^1_i = {}^{\cdot}P^2_i - {}^0Q^1_i$. Further we choose a neighborhood Y^2_i of Y^1_i in P^2_i, which is small with respect to

$$\eta^{-1}(H^3 \cap P^2_{\#i}) \mid \eta^{-1}(F^2_{II} \cap P^2_{\#i}) \mid \bigcup_{j,k=1}^{s,t_j} K^2_{Pjk}$$

and intersecting the disks C^2_g, C'^2_g, D'^2_h prismatically, i.e. such that

$$\eta^{-1}(\eta(Y^2_i \times I^1)) = [\eta^{-1}(\eta(Y^2_i \times I^1)) \cap P^2_i] \times I^1$$

(using the product representation introduced in Sec. 8). Then we deform $F^?_{II}$ over the 3-cells $\eta(Y^2_1 \times I^1)$ which can be done by a sequence of elementary deformations of type 2 (and may be nonessential deformations). We denote the map so obtained from α_{II} by α_{II*}, and $\alpha_{II*}(F'^2)$ by F^2_{II*}, further ${}^-(P^2_i - Y^2_i)$ by P^2_{*i} (see Fig. 11b), the image $\eta(P^2_{*i})$ by $P^2_{\#*i}$, and the intersections of K^2_{Pjk}, L^2_{Pjl} with the P^2_{*i}'s by K^2_{*jk}, L^2_{*jl}, respectively. Further we denote the set of double points of

$$\eta(\bigcup_{i=1}^r P^2_{*i} \times I^1)$$

by D_* and the connected components of

$$\eta^{-1}(D_*) \cap \bigcup_{i=1}^r P^2_{*i}$$

by C^2_{*g}, C'^2_{*g}, D^2_{*h}, D'^2_{*h} such that

$$C^2_{*g} \subset C^2_g, \qquad C'^2_{*g} \subset C'^2_g, \qquad D^2_{*h} \subset D^2_h,$$

$$D'^2_{*h} \subset D'_h \quad (g = 1, \cdots, c; h = 1, \cdots, d).$$

Step 2. We choose pairwise disjoint arcs $Y^1_{i1}, \cdots, Y^1_{iy_1}$ (see Fig. 11b) in P^2_{*i} with ${}^0Y^1_{if} \subset {}^0P^2_{*i}$ $(f = 1, \cdots, y_i)$ that join points of ${}^{\cdot}Y^2_1$ to points in ${}^0P^2_{*i} - \eta^{-1}(F^2_{II*} \cap P^2_{\#*i})$, and we choose pairwise disjoint neighborhoods Y^2_{if} of the Y^1_{if}'s in P^2_{*i}, which are small with respect to $\eta^{-1}(F^2_{II*} \cap P^2_{\#*i}) \mid \bigcup_{j,k=1}^{s,t_j} K^2_{*jk}$ such that, with the notation $P^2_{**i} = {}^-(P^2_{*i} - \bigcup_{f=1}^{y_i} Y^2_{if})$, the following hold:

(i) The arcs Y^1_{if} intersect the curves ${}^-[\eta^{-1}(F^2_{II*} \cap P^2_{\#*i}) \cap {}^0P^2_{*i}]$ in isolated piercing points that are no double points (and no boundary points) of that curves.

(ii) The arcs Y^1_{if} are disjoint from the disks C^2_{*g}, C'^2_{*g}, D^2_{*h} $(g = 1, \cdots, c;$ $h = 1, \cdots, d)$ and from the arcs ${}^-({}^{\cdot}K^2_{*jk} \cap {}^0P^2_{*i})$ $(j = 1, \cdots, s; k = 1, \cdots, t_j)$ and intersect the disks D'^2_{*h} prismatically, i.e. such that

$$\eta(Y^1_{if} \cap D'^2_{*h}) = [\eta(Y^1_{if} \cap D'^2_{*h}) \cap D^2_{*h}] \times I^1$$

using the product representation introduced in Sec. 8. The Y^2_{if}'s intersect the D'^2_{*h}'s also prismatically.

(iii) If Z^1 is a connected component of ${}^-[\eta^{-1}(F^2_{II*} \cap P^2_{\#*i}) \cap {}^0P^2_{**i}]$ then one of the following cases holds (see Fig. 12):

case a. Z^1 is an arc (that is either disjoint from the disks C^2_{*g}, C'^2_{*g}, D^2_{*h}, D'^2_{*h} or lies in the boundary of one disk C^2_{*g}, C'^2_{*g}, or D'^2_{*h}).

case b. Z^1 consists of two arcs, piercing each other in one point, and is disjoint from the disks C^2_{*g}, C'^2_{*g}, D^2_{*h}, D'^2_{*h}.
case c. Z^1 consists of two arcs Z^1_1, Z^1_2 lying in the boundary of one disk D'^2_{*h}, and of one arc Z^1_3 that pierces Z^1_1 and Z^1_2 each in one point.
case d. Z^1 consists of the boundary of one disk D^2_{*h} and of an arbitrary number of pairwise disjoint arcs that intersect D^2_{*h} each in one arc (and $\dot{D}^2_{*h}$ each in two points).

Then we deform F^2_{II*} over the 3-cells $\eta(Y^2_{if} \times I^1)$ $(i = 1, \cdots, r;$ $f = 1, \cdots, y_i)$ which can be done by a sequence of elementary deformations of type 2 (and may be nonessential deformations). According to the notation used in Theorem 3 we denote the map so obtained from α_{II*} by α_2 and $\alpha_2(F'^2)$ by F^2_2. Further we denote the intersections of the disks K^2_{*jk} with the P^2_{**i}'s by K^2_{**jk}.

Step 3. Now we deform F^2_2 over the 3-cells $\eta(K^2_{**jk} \times I^1)$ $(j = 1, \cdots, s;$ $k = 1, \cdots, t_j)$ which can be done by a sequence of elementary deformations of type 3a and 3b and nonessential deformations. We denote the map so obtained from α_2 by α_{2*} and $\alpha_{2*}(F'^2)$ by F^2_{2*}.

Step 4. The remaining parts $\eta(^-[P^2_{**i} - \bigcup_{j,k=1}^{s,t_j} K^2_{**jk}] \times I^1)$ of the $P^3_{\#i}$'s are nonsingular 3-cells, and we can deform F^2_{2*} over them by a sequence of elementary deformations of type 3a and 3b (and may be nonessential deformations). By this we obtain from α_{2*} the map α_3.

19. Conclusion. The maps α_1 and α_2, as obtained in Sec. 17, Step 2, and Sec. 18, Step 2, respectively, and the map α_3 possess the demanded properties, and Theorem 3 is proved.

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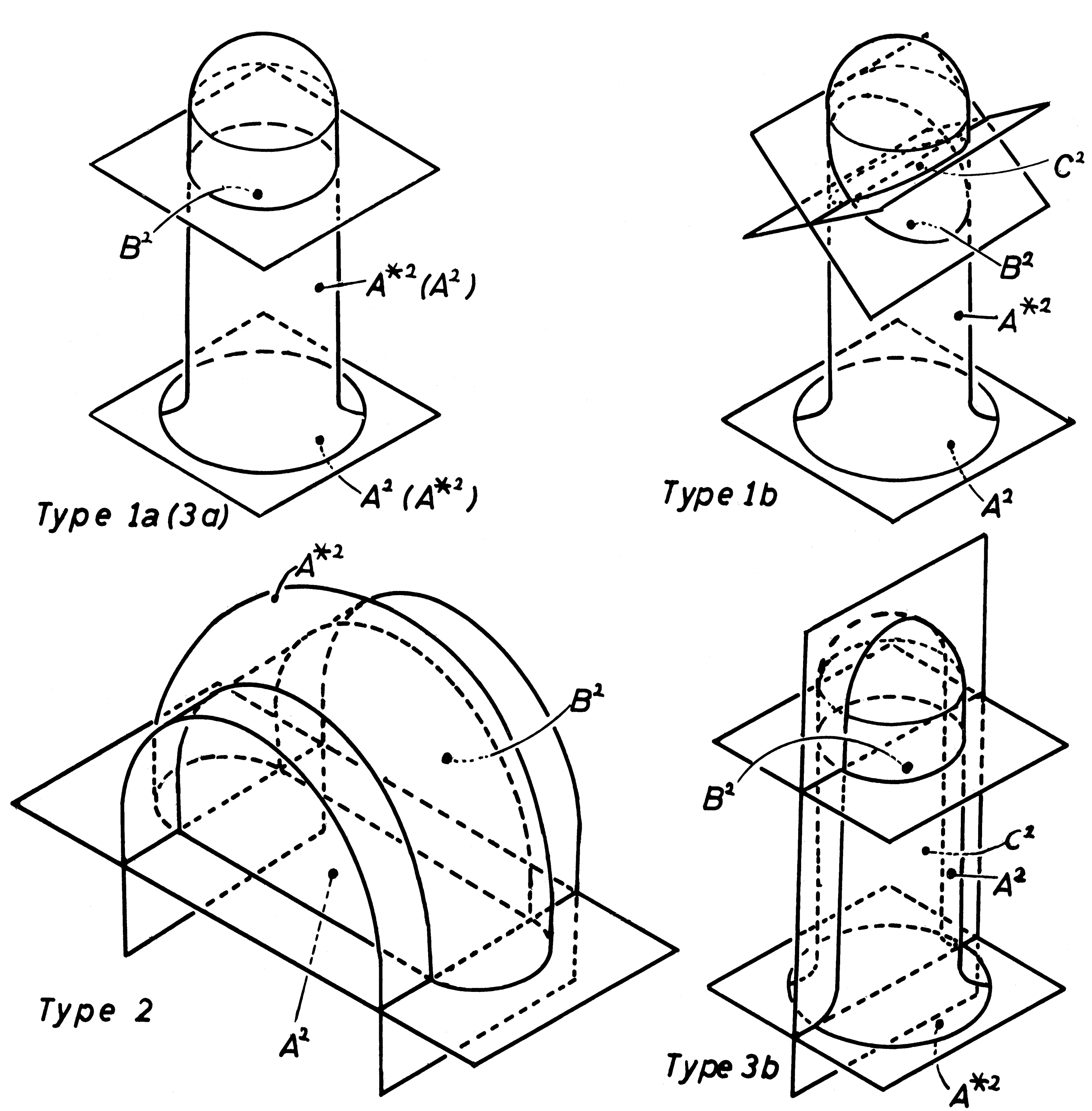

FIGURE 1

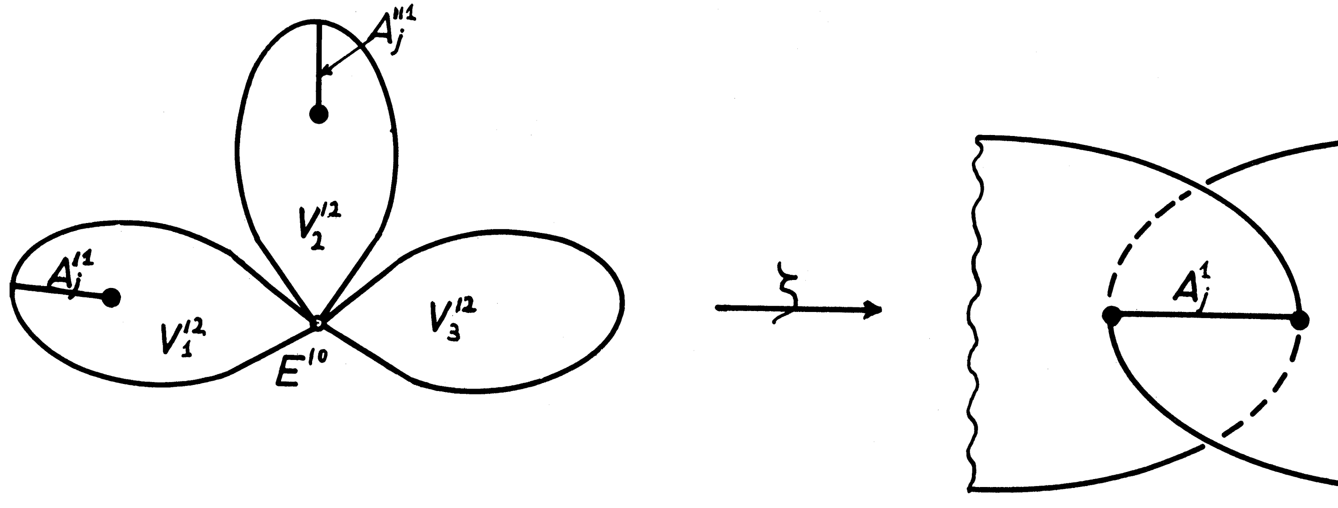

FIGURE 2

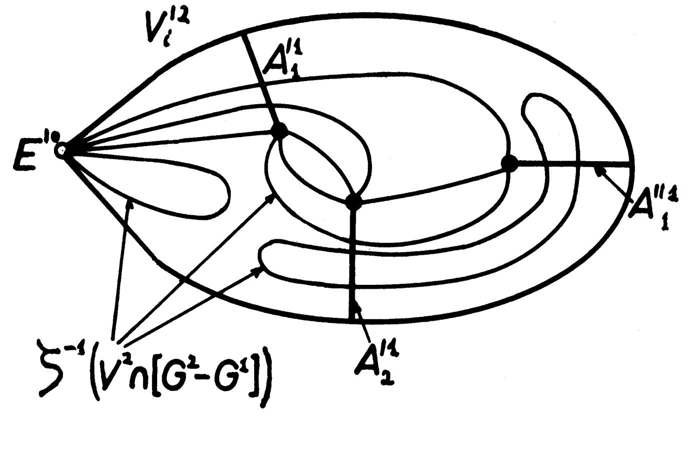

FIGURE 3

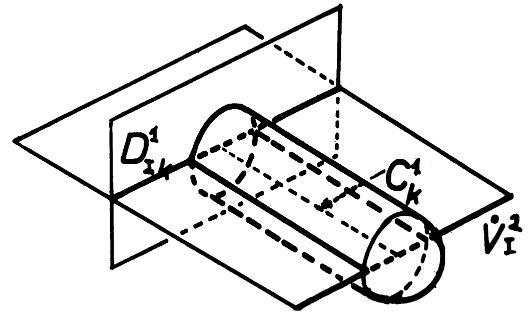

FIGURE 4

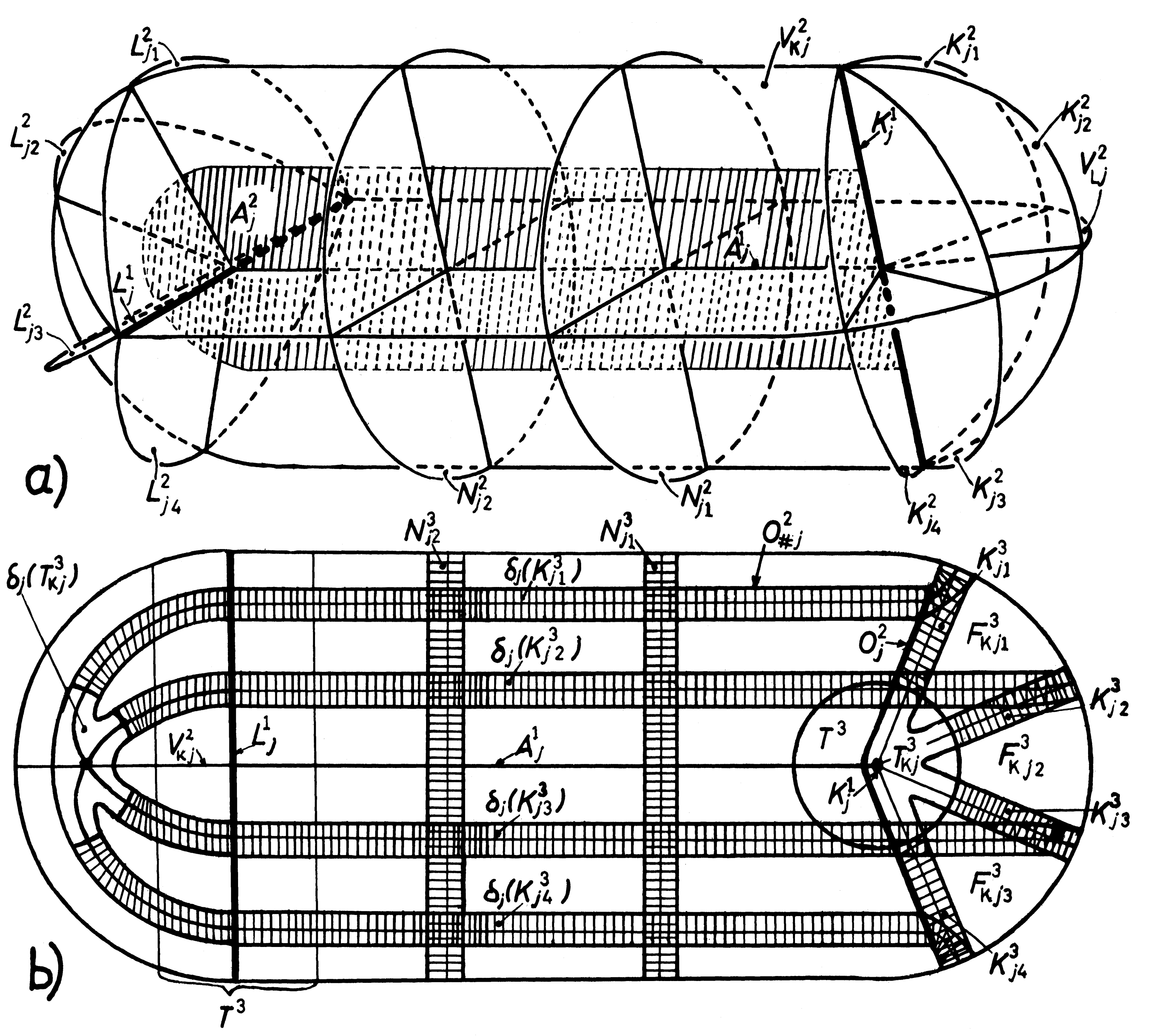

FIGURE 5

(a) depicts the intersection of V^2 and G^2 with A_j^3.

(b) depicts the intersection of (a), plus $T^3 \cap A_j^3$, $J^3 \cap A_j^3$, $J_*^3 \cap A_j^3$, with the V_{Lj}^2-plane.

The product representation of the P_i^3's and P_{*i}^3's is indicated by hatching.

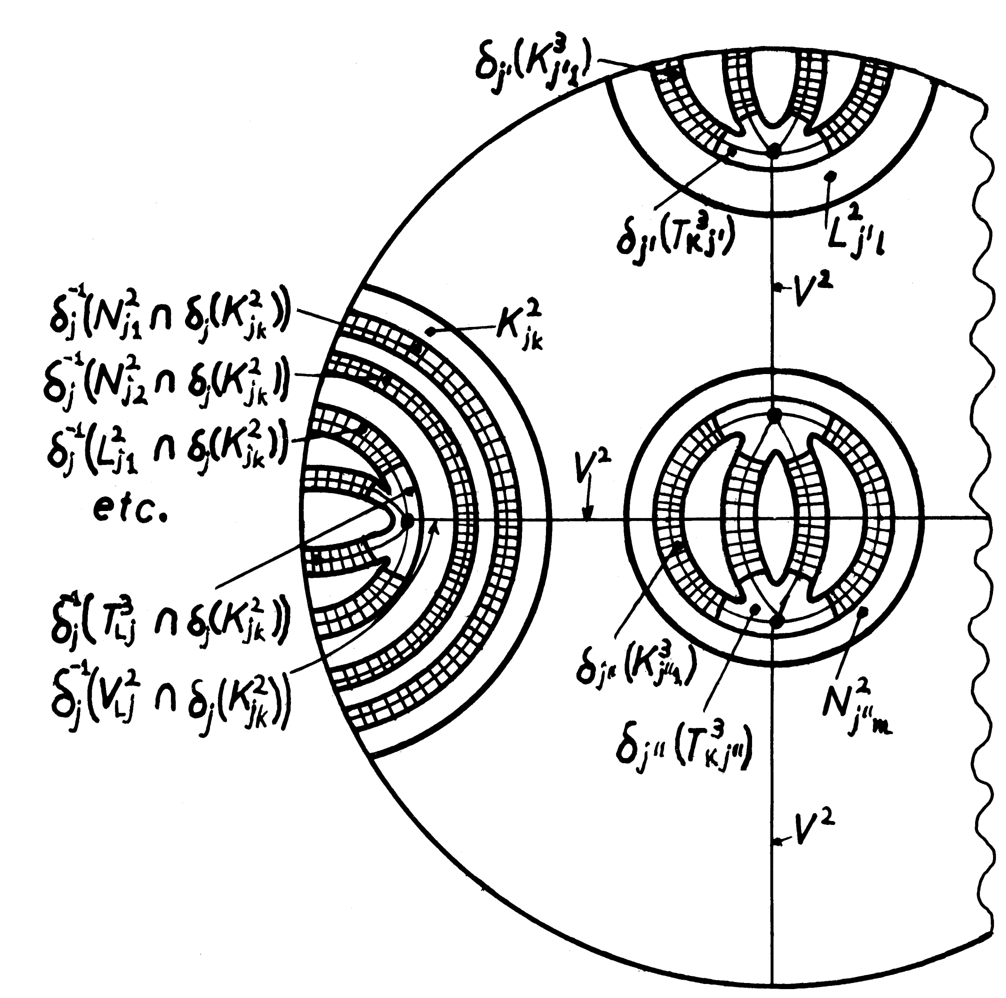

FIGURE 6

Intersections of K^2_{jk}, $L^2_{j'l}$, $N^2_{j''m}$ ($j, j', j'' = 1, \cdots, s$; $k = 1, \cdots, t_j$; $l = 1, \cdots, u_{j'}$; $m = 1, \cdots, v_{j''}$), etc. with P^2_i.

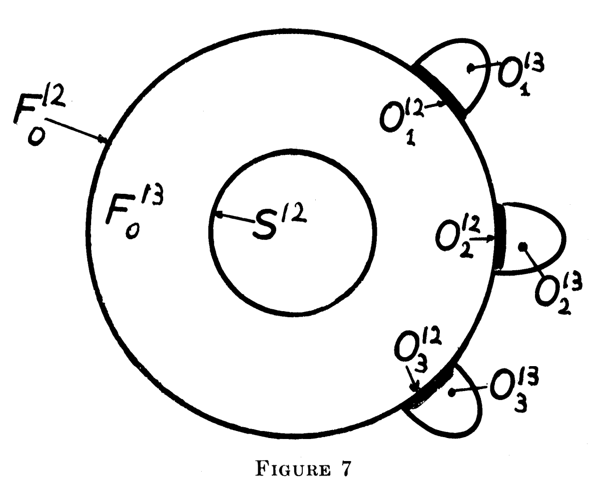

FIGURE 7

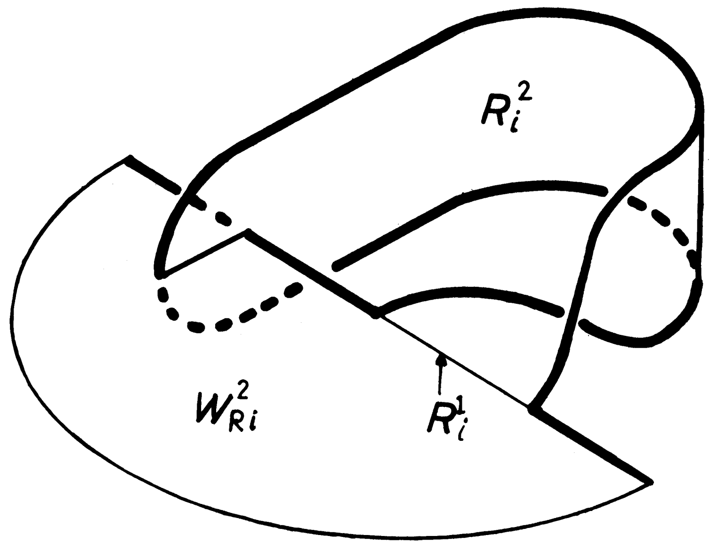

FIGURE 8

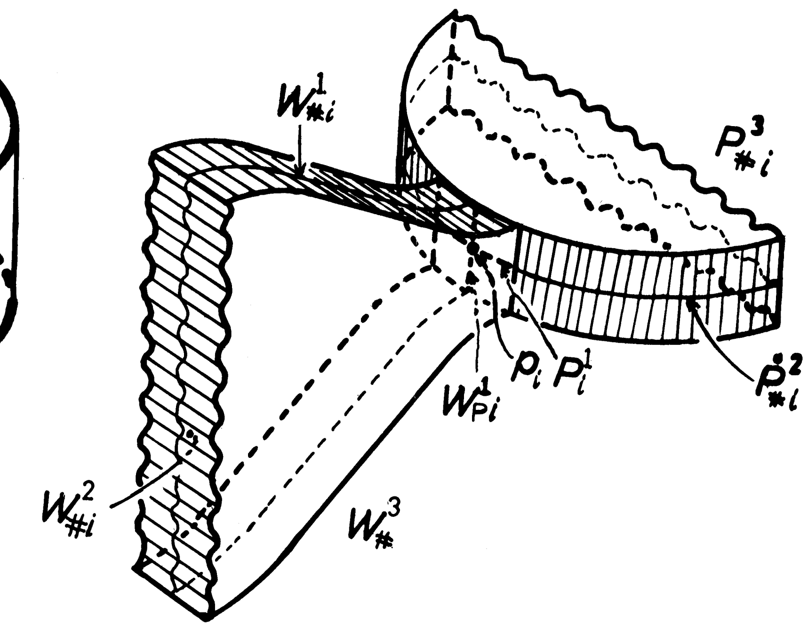

FIGURE 9

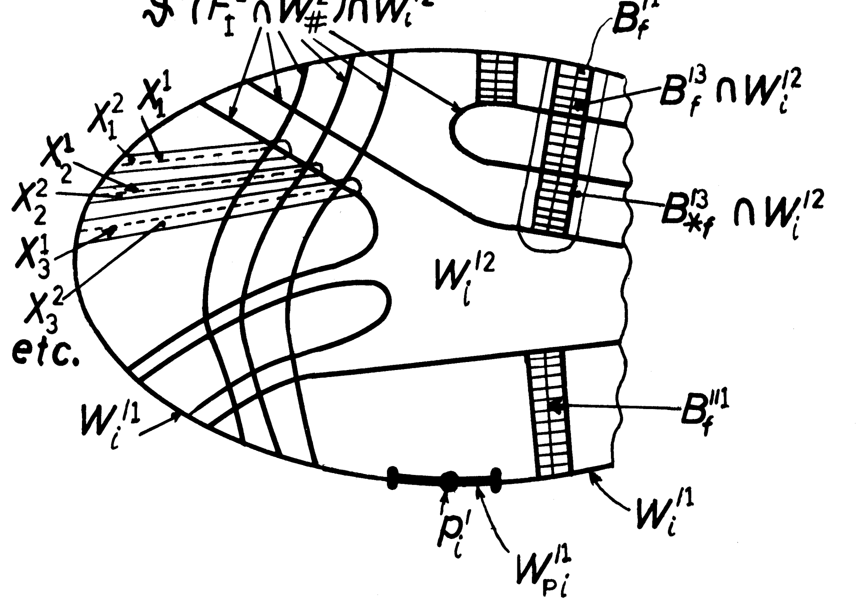

FIGURE 10

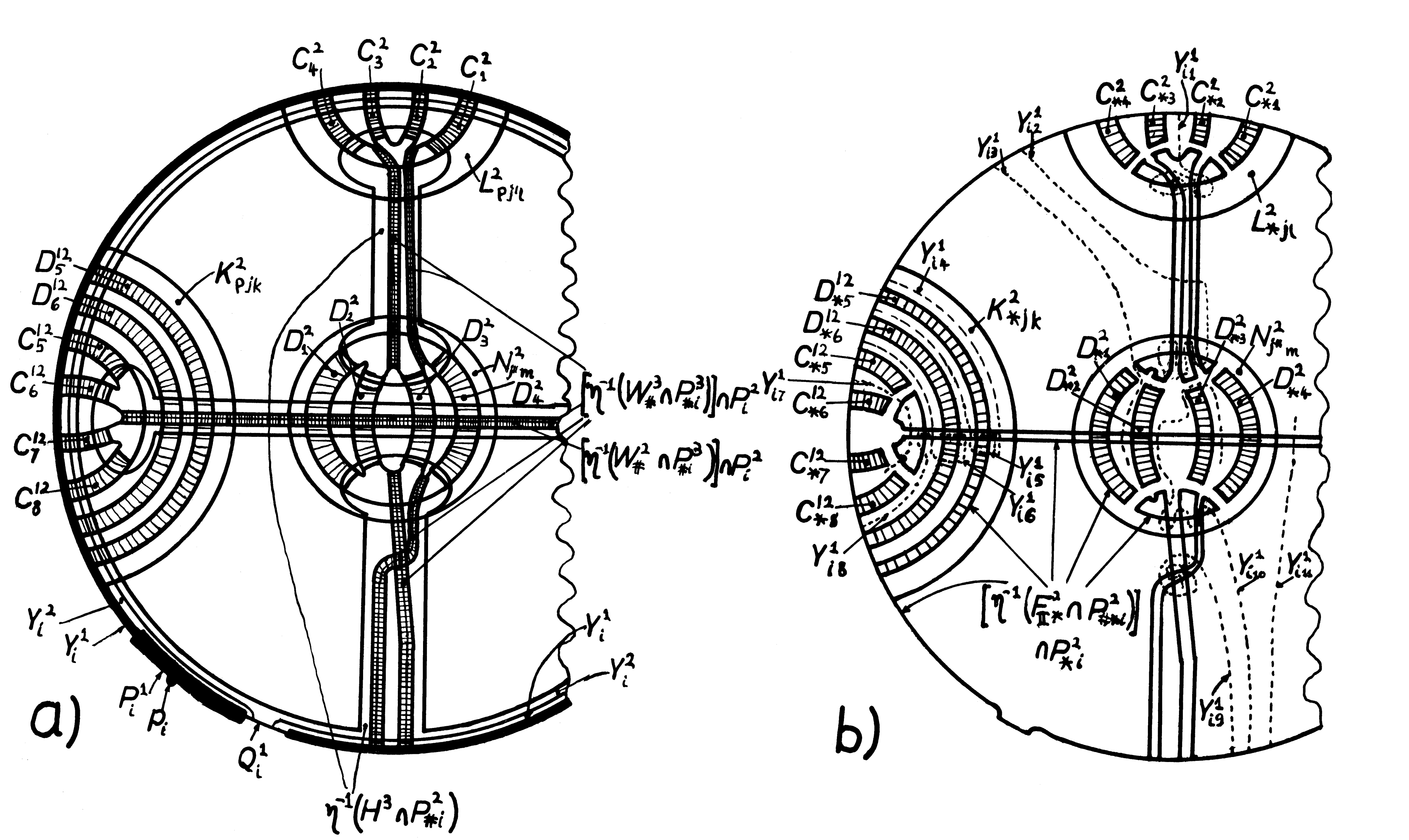

FIGURE 11

Compare with Figure 6. (a) depicts P^2_i. (b) depicts P^2_{*i}.

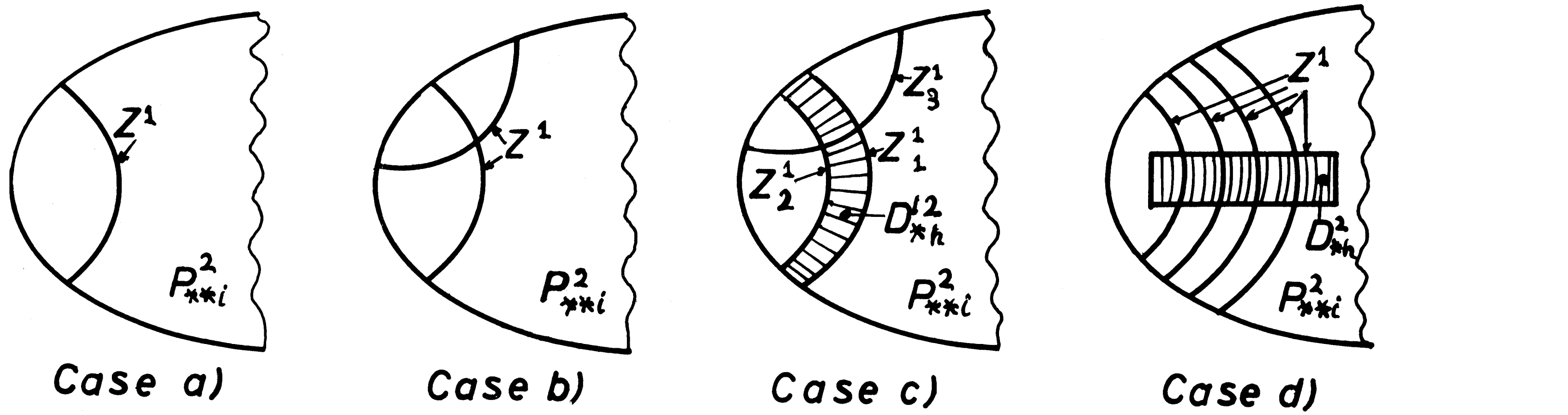

FIGURE 12