

A DENSITY THEOREM ON SPECTRA OF DISCRETE SUBGROUPS OF SEMI-SIMPLE LIE GROUPS

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This paper is solely devoted to an application of the so called Selberg trace formula, which is a generalized Poisson summation formula for the case of non-commutative groups, to determine, in terms of a certain integral, a relative density of spectra for discrete subgroups which act on a symmetric space. Certain properties of discrete subgroups and the notion of Selberg transform which brings about the trace formula will be indispensable in our discussion.

In obtaining the integral in question we shall select a space of certain positive definite zonal spherical functions over the symmetric space for our domain of definition of Selberg transforms and then use the Plancherel measure on this space for integration.

First we shall describe the problem to be solved and in (II) we prove our main theorem in detail. Then in (III) we shall give an explicit example of the theorem, which is hoped provides some insight into the problem. The elementary notions which might have slipped through our explanation may be easily found in the beginning portions of the papers listed at the end of this note, particularly in [7] and [10].

I. Explanation of the problem

Let X be a globally symmetric Riemannian space of the non-compact type, G the connected component of the identity of the Lie group of all the isometries on X in the compact open topology and U the isotropy subgroup of G at x_0 of X ; then G is a connected semi-simple Lie group which has no compact normal subgroup other than the identity, U is a maximal compact subgroup of G and, in fact, X is identifiable with the quotient space G/U . Suppose that Γ is a discrete subgroup of G , operating on X properly discontinuously with the properties that $\Gamma - \{1\}$ has no fixed points in X and that the homogeneous space $\Gamma \backslash G$ is compact. The G -invariant metric on X is denoted by $ds^2 = \sum_{i,j} g_{ij}(x) dx_i dx_j$, where $g_{ij}(x)$ are the real-valued C^∞ -functions with respect to the local coordinates (x_i) ; the G -invariant measure on X is denoted by dy or $dx = \sqrt{g} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, where $g = \det (g_{ij}(x))$ and the ring of all G -invariant differential operators of finite order on X by L . Then it is well known that the ring L is commutative,¹ since X is a symmetric homo-

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¹ For the commutativity of the ring L , which is necessary for our functional analysis, in particular, for determining the zonal spherical functions, it is sufficient to assume that X is "weakly symmetric" in the sense that we can find an isometry μ of X such that $\mu G \mu^{-1} = G$, $\mu^2 \in G$ and for any pair of points x and y in X , there exists g in G with $gx = \mu y$ and $gy = \mu x$.

geneous space, and actually is, in our case, a polynomial algebra generated by a finite set of fundamental differential operators, $\Delta_1, \Delta_2, \dots$, and Δ_l , where l is the rank of the symmetric space X , i.e., $L \cong \mathbf{C}[\Delta_1, \Delta_2, \dots, \Delta_l]$.

We denote by A the set of all subgroups Γ_α of Γ with a finite index, and put $I = \{\alpha : \Gamma_\alpha \in A\}$, the set of indices of A . Then the index set I is directed by the relation that if $\Gamma_\alpha \subseteq \Gamma_\beta$, then $\alpha \geq \beta$, and similarly is A also directed. We shall further assume one more condition on Γ , that $\bigcap_{\alpha \in I} \Gamma_\alpha = \{1\}$. For $\Gamma_\alpha \in A$ and an l -tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ in \mathbf{C}^l of complex numbers λ_i , the space $\mathfrak{M}(\Gamma_\alpha, \lambda) = \{f : C^\infty\text{-functions on } X, f(\gamma x) = f(x) \text{ for all } \gamma \in \Gamma_\alpha, \text{ and } \Delta_i f = \lambda_i f \text{ for } i = 1, 2, \dots, l\}$ is always of a finite dimension, which is called the multiplicity of λ for Γ_α as shown in [10]. A vector λ in \mathbf{C}^l is said to be a spectrum for Γ_α , if $\mathfrak{M}(\Gamma_\alpha, \lambda) \neq \{0\}$. Then from [9] and [10] we know that the set $\Lambda^{(\alpha)}$ of all spectra for Γ_α , each appearing repeatedly as many times as its multiplicity, is an unbounded, countable discrete set in \mathbf{C}^l .

For a function $h(\lambda)$ defined on \mathbf{C}^l , we shall simply denote $\sum_{j=0}^\infty h(\lambda_j^{(\alpha)})$ with $\lambda_j^{(\alpha)} \in \Lambda^{(\alpha)}$ by $\sum_{\Lambda^{(\alpha)}} h(\lambda)$. Let D_α be a fundamental domain in X of Γ_α and $v(\alpha)$ the volume of D_α measured by the G -invariant measure dx , i.e.,

$$(1) \quad v(\alpha) = \int_{D_\alpha} dx = \int_{\Gamma \backslash X} dx.$$

The main purpose of this note is to answer partially the question stated below, which was originally raised by Prof. M. Kuga. Later, Prof. I. Satake kindly provided its name for us and it now stands as the title of this paper. Here the author would like to mention his sincere gratitude for the continuous encouragement given by the above professors.

The problem is to obtain the following asymptotic density formula upon finding a certain measure $\Omega(\lambda) d\lambda$ on \mathbf{C}^l :

$$(2) \quad \lim_{\alpha \in I} \left\{ \frac{1}{v(\alpha)} \sum_{\Lambda^{(\alpha)}} h(\lambda) \right\} = \int_{\mathbf{C}^l} h(\lambda) \Omega(\lambda) d\lambda,$$

for a ‘‘considerably wide’’ class of functions h defined on \mathbf{C}^l . Here the limit is taken in the sense of directed limit. Note that if $\Gamma_\alpha \subseteq \Gamma_\beta$, that is, $\alpha \geq \beta$, then $\mathfrak{M}(\Gamma_\alpha, \lambda) \supseteq \mathfrak{M}(\Gamma_\beta, \lambda)$, $\Lambda^{(\alpha)} \supseteq \Lambda^{(\beta)}$ and $v(\alpha) \geq v(\beta)$.

II. The main theorem

As used in [9], a C^∞ -class function $k(x, y)$ defined on $X \times X$ is said to be a point-pair invariant if for any x and y in X , $k(gx, gy) = k(x, y)$ for all g in G .

Then the necessary and sufficient condition for an integral operator, defined by

$$(3) \quad \int_X k(x, y) f(y) dy$$

to be G -invariant is that the kernel $k(x, y)$ is a point-pair invariant. Furthermore, if $k(x, y)$ is a point-pair invariant on $X \times X$ with $X = G/U$, then we

can consider the corresponding kernel function on G which is bi-invariant by U , i.e., the kernel is invariant under the actions of the elements of U from both sides, and so we have

$$(4) \quad \int_X k(x, y)f(y) dy = \int_G \varphi(g_2^{-1}g_1)f(g_2) dg_2,$$

where g_1 and g_2 are mapped on x and y respectively under the canonical mapping of G onto G/U , and $\varphi(g_2^{-1}g_1) = k(g_2^{-1}g_1, 1)$.

DEFINITION. A point-pair invariant $k(x, y)$ is said to be of type (a)–(b), if $k(x, y)$ is majorized by another non-negative point-pair invariant $k_1(x, y)$ which satisfies the following:

$$(5) \quad (a) \quad \int_X k_1(x, y) dy < \infty;$$

(b) there exist positive numbers N and δ such that for all x and y in X

$$(6) \quad k_1(x, y) \leq N \int_{\mathcal{K}(y, \delta)} k_1(x, y') dy'$$

where $\mathcal{K}(y, \delta)$ denotes the disc centered at y with radius δ , which is determined by the smallest geodesic distance.

By the convolution $k^{(1)} * k^{(2)}$ of $k^{(1)}$ and $k^{(2)}$, as usual, we mean

$$(7) \quad k^{(1)} * k^{(2)}(x, y) = \int_X k^{(1)}(x, z) k^{(2)}(z, y) dz.$$

DEFINITION. A function $k(x, y)$ on $X \times X$ is said to be admissible if k is expressible as follows:

$$k = k^{(1)} * k^{(2)} + k^{(3)} * k^{(4)} + \dots + k^{(2n-1)} * k^{(2n)},$$

where $k^{(i)} * k^{(i+1)}$ is a convolution of a pair of point-pair invariants of type (a)–(b).

It is an immediate consequence that an admissible function $k(x, y)$, in turn, is a point-pair invariant of type (a)–(b). Hereafter, we shall denote by \mathfrak{A} the space of all admissible functions on $X \times X$. Next we shall consider certain complex-valued functions on X , called the (generalized) spherical functions, which encompass those special functions appearing as the solutions of second order differential equations, which were first used by Cartan as ‘fonctions fondamentales’ in connection with the theory of the irreducible representations of compact Lie groups, and, in fact, are very closely related with the irreducible unitary representations of non-compact semi-simple Lie groups, and, even more so with those of the non-compact symmetric Riemannian spaces.

DEFINITION. A complex-valued continuous function $\omega(x)$ on $X = G/U$

is to be called a zonal spherical function or an elementary spherical function if it satisfies the following conditions:

- (i) $\omega(ux) = \omega(x)$ for all u of U .
- (ii) $\omega(x_0) = 1$.
- (iii) ω is an eigen-function of all G -invariant integral operators i.e., for all point-pair invariants $k(x, y)$, we have $\int_X k(x, y)\omega(y) dy = \lambda_\omega(k)\omega(x)$ with $\lambda_\omega(k) \in \mathbf{C}$.

A function on X with the condition (i) is frequently called a spherical function as in [2] and [8], or a radial function as in [5].

It should be noted that in the above definition of the zonal spherical functions the condition (iii) may be replaced by the following one:

- (iii)' ω is a C^∞ -function on X and an eigen-function of all G -invariant differential operators (of finite order) on X , i.e., we have, for all Δ of L ,

$$\Delta\omega = \lambda(\Delta)\omega \quad \text{with } \lambda(\Delta) \text{ in } \mathbf{C},$$

or, what is the same, for a base, $\Delta_1, \Delta_2, \dots, \Delta_l$ of L over the complex numbers \mathbf{C} , $\Delta_i \omega = \lambda_i \omega$ with λ_i of \mathbf{C} .

Thus a zonal spherical function ω defines a ring-homomorphism,

$$\Delta \rightarrow \lambda(\Delta)$$

of $L = \mathbf{C}[\Delta_1, \Delta_2, \dots, \Delta_l]$ onto the complex numbers \mathbf{C} ; in other words, a zonal spherical function ω determines a point $(\lambda_1, \lambda_2, \dots, \lambda_l)$ in \mathbf{C}^l . On the other hand, we know from [2], [7] and [8] that each ring-homomorphism of L onto \mathbf{C} , in turn, gives rise to a zonal spherical function on X . Therefore the space \mathfrak{S} of all zonal spherical functions on X is bijective to the space Λ of all ring-homomorphisms of L onto \mathbf{C} , which is actually identified with \mathbf{C}^l , i.e., we have $\mathfrak{S} \approx \mathbf{C}^l \approx \Lambda$.

After [9] we shall now define a generalized Fourier transform for the case of non-commutative groups.

DEFINITION. A complex-valued function h defined on \mathfrak{S} is said to be the Selberg transform if it is expressible as follows: for some point-pair invariant $k(x, y)$ on $X \times X$.

$$(8) \quad h(\omega_\lambda) = \int_X k(x_0, y)\omega_\lambda(y, x_0) dy,$$

where $\omega_\lambda(y, x_0) = \omega(y)$ which corresponds to λ under the bijection mentioned above.

We notice immediately that the Selberg transform is nothing but the eigen-value determined in the condition (iii) above: i.e.,

$$(9) \quad \lambda_\omega(k) = \int_X k(x_0, y)\omega_\lambda(y, x_0) dy$$

where $\omega_\lambda = \omega$. The corresponding notion over G then is

$$(10) \quad \hat{\omega}(\varphi) = \int_G \varphi(g)\omega(g^{-1}) dg.$$

We shall henceforth restrict ourselves to the case of \mathfrak{A} , the space of all the admissible kernels, in order for the sum $\sum_{\Lambda(\omega)} h(\lambda)$ to be absolutely convergent. Then from [9] and [10], we have the following Selberg trace formula:

$$(11) \quad \sum_{\Lambda(\alpha)} h(\lambda) = \sum_{\gamma \in \Gamma_\alpha} \int_{D_\alpha} k(x, \gamma x) dx.$$

The equality holds because $k(x, y)$ is an admissible function on $X \times X$ and the fundamental domain D_α is compact. Hereafter we shall denote by \mathfrak{S} the space of all Selberg transforms $h(\lambda) = h(\omega_\lambda)$ of $k(x, y)$ in \mathfrak{A} . For this class \mathfrak{S} of functions we are going to show the density formula in question. In the case of the upper half complex plane, \mathfrak{S} is a considerably wide class of functions.

For a function h of \mathfrak{S} , we shall consider the following sum over the spectra for $\Gamma_\alpha \in A$:

$$(12) \quad \frac{1}{v(\alpha)} \sum_{\Lambda(\alpha)} h(\lambda).$$

By the Selberg trace formula (11), we rewrite the above sum (12) as follows:

$$(13) \quad \begin{aligned} \frac{1}{v(\alpha)} \sum_{\Lambda(\alpha)} h(\lambda) &= \frac{1}{v(\alpha)} \sum_{\gamma \in \Gamma_\alpha} \int_{D_\alpha} k(x, \gamma x) dx \\ &= \frac{1}{v(\alpha)} \int_{D_\alpha} k(x, x) dx + \frac{1}{v(\alpha)} \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} \int_{D_\alpha} k(x, \gamma x) dx. \end{aligned}$$

Then the integrand of the first term of the last part of the above equation becomes constant, since $k(x, x) = k(gx_0, gx_0) = k(x_0, x_0)$ where x_0 is the predetermined fixed point of X and $gx_0 = x$ for some g in G . Therefore it follows that

$$(14) \quad \lim_{\alpha \in I} \left\{ \frac{1}{v(\alpha)} \sum_{\Lambda(\alpha)} h(\lambda) \right\} = k(x_0, x_0),$$

if we prove the following:

PROPOSITION. *For an admissible function $k(x, y)$ on $X \times X$ the following directed limit equation holds:*

$$(15) \quad \lim_{\alpha \in I} \left\{ \frac{1}{v(\alpha)} \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} \int_{D_\alpha} k(x, \gamma x) dx \right\} = 0.$$

Proof. We recall $\mathfrak{K}(x, R)$ to be the circle with radius R centered at x , and let $d(x, y)$ be the distance between x and y , measured by the Riemann

metric ds^2 . Define

$$f(x, \Gamma_\alpha) = \min_{\gamma \in \Gamma_\alpha, \gamma \neq 1} d(x, \gamma x) \quad \text{for } x \text{ of } X$$

It is obvious that $f(x, \Gamma_\alpha)$ is a well defined positive continuous function, since $\Gamma - \{1\}$ does not have fixed points.

LEMMA 1. *If γ is an element of the normalizer of Γ_α , then we have that $f(x, \Gamma_\alpha) = f(\gamma x, \Gamma_\alpha)$.*

In fact, we have $f(x, \Gamma_\alpha) = f(gx, g\Gamma_\alpha g^{-1})$ for any g of G , because $d(x, \gamma x) = d(gx, g\gamma x) = d(gx, g\gamma g^{-1} \cdot gx)$.

Define $\delta(\Gamma_\alpha) = \min_{x \in X} f(x, \Gamma_\alpha)$.

LEMMA 2. *For a given positive number M , there exists a normal subgroup Γ_α of Γ such that $\delta(\Gamma_\alpha) > M$ and the index $[\Gamma : \Gamma_\alpha] < \infty$.*

Proof. Let D_0 be a fundamental domain of Γ containing x_0 with mesh $(D_0) = \eta_0$. Then the set $\{\gamma \in \Gamma : \gamma(D_0) \cap \mathcal{K}(x_0, M + \eta_0) = \emptyset\}$ is a finite set, and denoted by $\{\gamma_0 = 1, \gamma_1, \gamma_2, \dots, \gamma_m\}$. Since $\bigcap_{\alpha \in I} \Gamma_\alpha = \{1\}$, we can easily choose a Γ_α in A , which is normal in Γ and does not contain any of γ_i 's except $\gamma_0 = 1$.² This means that D_0 is thrown outside of $\mathcal{K}(x_0, M + \eta_0)$ by the action of any non-identity element of Γ_α .

On the other hand, by the previous lemma we have $f(x, \Gamma_\alpha) = f(\gamma x, \Gamma_\alpha)$ for any γ of Γ , for Γ_α is a normal subgroup of Γ . Therefore the continuous function $f(x, \Gamma_\alpha)$ certainly takes its minimum value $\delta(\Gamma_\alpha)$ at a point y in D_0 , i.e., $f(y, \Gamma_\alpha) = \delta(\Gamma_\alpha)$.

Now combining these two facts, we see explicitly that for any non-identity $\gamma_\alpha \in \Gamma_\alpha$,

$$d(y, \gamma_\alpha y) > d(x_0, \gamma_\alpha y) - d(x_0, y) > M + \eta_0 - \eta_0 = M,$$

and so we conclude the proof of this lemma by noting

$$\delta(\Gamma_\alpha) = f(y, \Gamma_\alpha) = \min_{\gamma_\alpha \in \Gamma_\alpha, \gamma_\alpha \neq 1} d(y, \gamma_\alpha y) \geq M.$$

LEMMA 3. *For a point-pair invariant $k(x, y)$ on $X \times X$ and a fixed constant R , the integral*

$$(16) \quad \int_{\mathcal{K}(x, R)} k(x, y) dy$$

is independent of x in X .

Proof. For a given x' of X , there exists g in G with $gx = x'$. Then we can find z in X such that $gz = y$. Now observe that

$$\int_{\mathcal{K}(x', R)} k(x', y) dy = \int_{\mathcal{K}(gx, R)} k(gx, gz) d(gz) = \int_{\mathcal{K}(x, R)} k(x, z) dz.$$

Before proceeding to the next lemma, we recall that a point-pair invariant

² This is the only place where we have to use the condition that $\bigcap_{\alpha \in I} \Gamma_\alpha = \{1\}$. However this lemma may be true under a much weaker condition on Γ .

denoted by $k_1(x, y)$ was introduced in way of defining a function of type (a)–(b), namely, as a positive function, $k_1(x, y)$ satisfies the conditions (a) and (b) with the equations (5) and (6).

LEMMA 4. *Let $K_1(x, y)$ be a point-pair invariant of type (a)–(b). Then for a given $\varepsilon > 0$, there exists α in I such that*

$$(17) \quad \sum_{\gamma \in \Gamma_\alpha, \gamma \neq 1} k_1(x, \gamma x) < \varepsilon \quad \text{for all } x \text{ in } X.$$

Proof. Let N and δ be the positive numbers for which $K_1(x, y)$ satisfies the condition (b). Since $\int_x k_1(x, y) dy < \infty$, we can find a sufficiently large number $R (> \delta)$, which depends on x with the property that

$$(18) \quad \int_{x - \mathcal{K}(x, R)} k_1(x, y) dy < \varepsilon/N$$

for a given x in X . However, by virtue of Lemma 3, we can choose such R once for all x of X . If we take a number δ' with $R > \delta' > \delta$, then Lemma 2 provides the existence of a Γ_α in A which is a normal subgroup of Γ and satisfies the inequality $\delta(\Gamma_\alpha) > R + \delta'$.

Now we have a situation where all the discs of the family

$$\{\mathcal{K}(\gamma x, \delta) \text{ for all } \gamma \in \Gamma_\alpha \text{ and } \gamma \neq 1\}$$

are mutually disjoint and scattered outside of $\mathcal{K}(x, R)$. Therefore it is clear that we have

$$\begin{aligned} \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} k_1(x, \gamma x) &\leq N \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} \int_{\mathcal{K}(\gamma x, \delta)} k_1(x, y) dy \\ &\leq N \int_{x - \mathcal{K}(x, R)} k_1(x, \gamma x) dx \leq N \cdot \varepsilon/N = \varepsilon. \end{aligned}$$

In order to complete the proof of our proposition, we first note that if $\alpha \geq \beta$ in I , then $\delta(\Gamma_\alpha) \geq \delta(\Gamma_\beta)$, simply because $\Gamma_\alpha \subseteq \Gamma_\beta$. Hence

$$\sum_{1 \neq \gamma \in \Gamma_\beta} k_1(x, \gamma x) < \varepsilon \text{ implies } \sum_{1 \neq \gamma \in \Gamma_\alpha} k_1(x, \gamma x) < \varepsilon.$$

Finally, from Lemma 4 and from the fact that $k_1(x, y)$ is admissible, it follows that

$$\begin{aligned} \frac{1}{v(\alpha)} \left\{ \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} \int_{D_\alpha} |k(x, \gamma x)| dx \right\} &\leq \frac{1}{v(\alpha)} \left\{ \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} \int_{D_\alpha} k_1(x, \gamma x) dx \right\} \\ &\leq \frac{1}{v(\alpha)} \int_{D_\alpha} \left\{ \sum_{\substack{\gamma \in \Gamma_\alpha \\ \gamma \neq 1}} k_1(x, \gamma x) \right\} dx \\ &\leq \frac{1}{v(\alpha)} \cdot \varepsilon \int_{D_\alpha} dx = \varepsilon. \end{aligned}$$

Thus we have completed the proof of the proposition.

As for a suitable measure for which the integral of the original density formula (2) holds, we shall consider the following notions.

DEFINITION. A zonal spherical function ω is said to be positive definite if it satisfies the inequality:

$$\int_G \int_G \omega(xy^{-1})f(x)\overline{f(y)} dx dy \geq 0$$

for all continuous spherical functions f on G with compact support.

Let Z be the subspace of \mathfrak{S} , consisting of all positive definite zonal spherical functions on X . Then Z is identified with the space of all equivalence classes of irreducible unitary representations T of G for which the trivial representation of the compact subgroup U occurs in its restriction T on U . We shall denote by P a subspace of Z which is bijective to the space of all irreducible unitary representations of the first kind and of the principal series. By making use of another parametrization of the ring $L = \mathbf{C}[\Delta_1, \Delta_2, \dots, \Delta_l]$ of all G -invariant differential operators on X , we shall be able to identify P with a familiar space.

Let $G = U \cdot AN$ be an Iwasawa decomposition of G , A^* the dual of the vector space A , and W the restricted Weyl group of G with respect to A , i.e., $W = N(A)/Z(A)$, operating on A and also on A^* in a natural manner. Then fixing a basis of A and the corresponding dual basis of A^* , we have that $A \approx \mathbf{R}^l \approx A^*$, where \mathbf{R} is the field of all real numbers, and that, by Harish-Chandra, the algebra L is isomorphic to the algebra of all W -invariant polynomial functions on A^* over the complex numbers \mathbf{C} , denoted by

$$\mathbf{C}[X_1, X_2, \dots, X_l]^W.$$

Under this parametrization of L , we can easily establish

$$\mathfrak{S} \approx \text{Hom}(L, \mathbf{C}) \approx \mathbf{C}^l/W$$

by the properties of the functionals of $\mathbf{C}[X_1, X_2, \dots, X_l]^W$.

Using the notations of [8], for a given α of $\text{Hom}(A, \mathbf{C}^*)$, we shall define a Hilbert space \mathfrak{H}^α of all complex-valued measurable functions f on G with the conditions:

- (1) $f(gan) = \alpha(a)f(g)$ for all $g \in G, a \in A, n \in N$,
- (2) $\|f\|^2 = \int_U |f(u)|^2 du$,

and with the inner product, defined by

$$\langle f_1, f_2 \rangle = \int_U \overline{f_1(u)}f_2(u) du.$$

Now we shall define a representation T_σ^α of G into \mathfrak{H}^α by

$$(T_\sigma^\alpha \cdot f)(g') = f(g^{-1}g') \quad \text{for } f \in \mathfrak{H}^\alpha.$$

Then it follows that the representation T_σ^α is of the first kind, since, for f_α of

\mathfrak{H}^α with $f_\alpha(uan) = \alpha(a)$, we have $T_u^\alpha \cdot f_\alpha = f_\alpha$ for all u of U , and that $\omega^\alpha(g)$ defined by

$$\langle f_\alpha, T_g^\alpha \cdot f_\alpha \rangle = \int_U f_\alpha(g^{-1}u) du$$

is a zonal spherical function of G with the property that $\omega^\alpha(g) = \omega^{\alpha'}(g)$ if $\alpha\delta^{1/2} \equiv \alpha'\delta^{1/2} \pmod{W}$, where δ is defined in the volume-element, $d(ana^{-1}) = \delta(a) dn$. However, we know $\text{Hom}(A, \mathbf{C}^*) \cong \mathbf{C}^l$ from the correspondence:

$$(19) \quad \alpha\delta^{1/2} \leftrightarrow s : \text{Hom}(A, \mathbf{C}^*) \cong \mathbf{C}^l$$

where

$$\alpha\delta^{1/2}(a) = \exp \left\{ \sum_{i=1}^l a_i s_i \right\}$$

for $a = (a_1, a_2, \dots, a_l)$ and $s = (s_1, s_2, \dots, s_l)$. Thus we certainly have $\mathfrak{S} \approx \mathbf{C}^l/W$ and $\omega^\alpha(g) \leftrightarrow \omega_s(g)$ by (19).

The necessary and sufficient condition for the representation T_g^α to be unitary is that $|\alpha|^2 \delta = 1$, because

$$\| T_g^\alpha \cdot f \|^2 = \int_U |f(u')|^2 \cdot |\alpha(a)|^2 \cdot \delta(a) du'$$

But $|\alpha|^2 \delta = 1$ if and only if $|\exp \{ \sum_{i=1}^l a_i s_i \}| = 1$. Therefore we can identify P with $(i\mathbf{R})^l/W \cong \mathbf{R}^l/W$.

In summary, we have the following relation:

$$\begin{aligned} L &\approx \mathbf{C}[X_1, X_2, \dots, X_l] \approx \mathbf{C}[X_1, X_2, \dots, X_l]^W \\ \Lambda &\approx \mathfrak{S} \approx \text{Hom}(L, \mathbf{C}) \approx \mathbf{C}^l \approx \mathbf{C}^l/W \approx \text{Hom}(A, \mathbf{C}^*)/W \\ &\cup \\ &Z \\ &\cup \\ P &\approx A^*/W \approx (i\mathbf{R})^l/W \approx \text{the principal series.} \end{aligned}$$

By a point-pair invariant $k(x, y)$ on $X \times X$ with compact support we obviously mean that for fixed x' or y' in X , either $k(x', y)$ or $k(x, y')$ has compact support.

Godement showed in [1] the unique existence of the Plancherel measure on the space Z of all positive definite zonal spherical functions, by which the Plancherel formula for the point-pair invariants with compact support holds. In [2], Harish-Chandra calculated the explicit form of the Plancherel measure on \mathbf{R}^l/W , namely, for s in \mathbf{R}^l/W , and λ in $\Lambda \approx \mathfrak{S}$,

$$(20) \quad \Omega'(\lambda) d\lambda = |\mathbf{c}(s)|^{-2} ds$$

where $\mathbf{c}(s)$ is a function, which occurs in the leading term of the asymptotic expansion of the corresponding positive definite zonal spherical function ω_s and ds is the Euclidean measure on \mathbf{R}^l/W . With this measure, the inverse

of the Selberg transform holds for a certain class,³ I_0 of point-pair invariants on $X \times X$. Let J be the largest class of functions for which the inverse of the Selberg transform holds with respect to the explicit Plancherel measure (20). Thus for the functions $k(x, y)$ in $\mathfrak{A} \cap J$, we shall have the inverse transform:

$$\begin{aligned} k(x, y) &= \frac{1}{m} \int_{\mathbf{R}^l/W} \hat{h}(s) \omega_s(x, y) |\mathbf{c}(s)|^{-2} ds \\ &= \int_P h(\lambda) \omega_\lambda(x, y) \Omega(\lambda) d\lambda, \end{aligned}$$

where m is the order of the Weyl group W , $\Omega(\lambda) d\lambda = (1/m)\Omega'(\lambda) d\lambda$ and

$$\hat{h}(s) = h(\lambda) = \int_X k(x_0, y) \omega_\lambda(y, x_0) dy.$$

Using this inverse form, the equation (14), resulted from the proposition, and the fact that $\omega_\lambda(x_0, x_0) = \omega_\lambda(x_0) = 1$, we obtain the required integral formula:

$$\begin{aligned} \lim_{\alpha \in I} \left\{ \frac{1}{v(\alpha)} \sum_{\Lambda(\alpha)} h(\lambda) \right\} &= \int_P h(\lambda) \omega_\lambda(x_0, x_0) \Omega(\lambda) d\lambda \\ (21) \qquad \qquad \qquad &= \int_P h(\lambda) \Omega(\lambda) d\lambda. \end{aligned}$$

In summary, we have established a density theorem on spectra of the following kind:

THEOREM. *Let G, Γ and $\Lambda^{(\alpha)}$ be defined as in (I), and \mathfrak{S}' the space of all the Selberg transforms of $\mathfrak{A} \cap J$. Then for a given function $\hat{h}(s)$ of \mathfrak{S}' defined on \mathbf{C}^l/W , it follows that*

$$\lim_{\alpha \in I} \left\{ \frac{1}{v(\alpha)} \sum_{\Lambda(\alpha)} \hat{h}(\lambda) \right\} = m^{-1} \int_{\mathbf{R}^l/W} \hat{h}(s) \cdot |\mathbf{c}(s)|^{-2} ds.$$

III. Example

Let $X = \{Z = x + iy : y > 0\}$ be the upper complex half plane. For $G = SL(2, R)/\{\pm E\}$ and $U = SO(2, R)/\{\pm E\}$, which is the isotropic subgroup of i of X , X is identified with the homogeneous space G/U , since G operates transitively on X from the left by

$$g(z) = (az + b)/(cz + d)$$

for z in X and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in G . Then the G -invariant Riemannian metric on X is $ds^2 = (dx^2 + dy^2)/y^2$, hence the corresponding G -invariant measure is

³ See the footnote on p. 68 of [9]. As shown by Harish-Chandra in [2], the inverse transform of the functions in the class $I_0 (= I_0(A)$ in his notation) is established by using the explicit measure (20). For the further information, see p. 780 of [5].

$dz = dx dy/y^2$ and the G -invariant Laplacian operator on X is

$$\Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2),$$

which, in fact, generates the ring L of all G -invariant differential operators on X .

The geodesic distance $\zeta_0(z, z')$ between z and z' on X is

$$(22) \quad \zeta_0(z, z') = \log \left[\left(\frac{|z - z'|^2}{yy'} \right) + 2 + \sqrt{\left(\frac{|z - z'|^2}{yy'} \right)^2 + 4 \frac{|z - z'|^2}{yy'}} \right] / 2$$

which is a point-pair invariant on $X \times X$. Then as shown in [1] and [4], for $\zeta = \zeta_0(i, z)$,

$$(23) \quad \omega_s(z) = \frac{1}{2\pi} \int_0^{2\pi} (\cosh \zeta + \sinh \zeta \cdot \cos \theta)^{s-1/2} d\theta$$

is a zonal spherical function. Incidentally, the equation (23) justifies $\mathfrak{S} \approx \mathbf{C}/W \approx \mathbf{C}/\{\pm 1\} \approx \mathbf{C}$. Since the eigen-values $\{\lambda\}$ of the Laplacian operator Δ , having the positive definite zonal spherical functions as the eigen-functions, are the non-positive real numbers, the equation, $s^2 - \frac{1}{4} = \lambda$, which comes from

$$\Delta \omega_s = y^2(\partial^2 \omega_s / \partial x^2 + \partial^2 \omega_s / \partial y^2) = (s^2 - \frac{1}{4}) \omega_s = \lambda \cdot \omega_s$$

gives us the correspondence

$$\{\lambda\} = (-\infty, 0] \leftrightarrow \{\omega_s\} \approx [-\frac{1}{2}, \frac{1}{2}] \mathbf{u} i\mathbf{R},$$

and

$$(-\infty, -\frac{1}{4}] \leftrightarrow i\mathbf{R}/W;$$

in other words, the space Z of the positive definite zonal spherical functions on X is identified with $[-\frac{1}{2}, \frac{1}{2}] \mathbf{u} i\mathbf{R}$ and the subspace P of Z which corresponds to the principal series is bijective to $i\mathbf{R}/W$.

If we restrict ourselves to the correspondence

$$(-\infty, -\frac{1}{4}] \leftrightarrow i\mathbf{R}/W,$$

then we actually have the equation $\lambda = -r^2 - \frac{1}{4}$ where $r = |s|$. Any point-pair invariant $k(z, z')$ on $X \times X$ is to be of the form $k(t)$, where $t = |z - z'|^2 / yy'$ with $\text{Im } z = y$ and $\text{Im } z' = y'$. Let $h(\lambda)$ be the Selberg transform of $k(t)$ and put $\hat{h}(r) = h(-r^2 - \frac{1}{4})$. Then as shown in [9], we have the following relation: for $\eta = \log(y'/y)$,

$$(24) \quad k(t) = \frac{-1}{\pi} \int_t^\infty dg(\eta) / \sqrt{w - t},$$

where $g(\eta) = (1/2\pi) \int_{-\infty}^\infty \exp(-i r \eta) \cdot h(r) dr$ and $w = e^\eta + e^{-\eta} - 2$. Now

using the fact that $\hat{h}(-r) = \hat{h}(r)$ and substituting $t = 0$ in (24), we obtain the density formula

$$(25) \quad \lim_{\alpha \downarrow I} \left\{ \frac{1}{v(\alpha)} \sum_{\Lambda(\alpha)} h(\lambda) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{h}(r) \cdot r \cdot \tanh(\pi r) dr$$

$$= \frac{-1}{4\pi} \int_{-1/4}^{-\infty} h(\lambda) \cdot \tanh\left(\pi \cdot \sqrt{-\lambda - \frac{1}{4}}\right) d\lambda.$$

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