SOME PROPERTIES OF A SINGULAR DIFFERENTIAL OPERATOR

BY

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1. Introduction

Let X denote the B-space of all bounded and uniformly continuous functions from the real line E_1 into the set C of all complex numbers. Suppose p is a function from E_1 into E_1 which has uniformly bounded difference quotients, and let $F(p)$ denote the class of all functions from E_1 into C which are differentiable at every non-zero of p. For each x in $F(p)$, let x^* be defined on E_1 by $x^*(s) = x'(s)$ if $p(s) \neq 0$, $x^*(s) = 0$ if $p(s) = 0$. Let D denote the linear subspace of X consisting of all x in X such that x is in $F(p)$ and px^* is in X, and let A denote the linear transformation from D into X defined by $Ax = px^*$. Let D^2 denote the linear subspace of X consisting of all x in D for which Ax is in D, and let A^2 denote the linear transformation from D^2 into X defined by $A^2x = A(Ax)$. Various properties are developed for the transformations $A, A^2, A + Q$, and $A^2 + PA + Q$, where P and Q denote bounded linear transformations from X into X, and the results have applications to partial and ordinary differential equations. The results all carry over if X is taken to be the B-space of all bounded and uniformly continuous complex-valued functions defined on an interval $[a, b]$ if $p(a) = p(b) = 0$, $[a, \infty)$ if $p(a) = 0$, or $(-\infty, b]$ if $p(b) = 0$.

Most of the main results require that p be bounded and are obtained by first establishing the fact that A is the infinitesimal generator of a strongly continuous group $[T(t), -\infty < t < \infty]$ of bounded operators in X and giving a simple formula for $T(t)$. This yields the fact that A^2 is the infinitesimal generator of a strongly continuous semi-group $[V(t), 0 \leq t < \infty]$ of bounded operators in X and a formula for $V(t)$. The theory of semi-groups of operators is discussed by Dunford and Schwartz in [2] and more completely by Hille and Phillips in [4]. In this paper, as in [2], the term "strongly continuous" means belonging to the class $(C, 0)$ of [4]. In some of the applications to differential equations, advanced calculus methods are used to sharpen the results beyond what the semi-group theory alone would yield.

Glazman [3], Stone [5], and Weyl [6] have treated similar singular differential operators. They considered the differential operators as operators in a Hilbert space of Lebesgue square-integrable functions and allowed singularites to occur only at the end points of the domain of the functions in the space. In this paper, singularities (even intervals of them) are allowed to occur within the interval on which the functions considered are defined, but the results are not as complete.

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2. An ordinary differential equation

Let K denote an upper bound for the difference quotients of p. E_2 denotes the real number plane, and for any real or complex-valued function defined on a subset of E_2, the subscripts 1 and 2 denote partial derivatives. Let $G(p)$ denote the class of all functions from E_2 into C such that $g_1(s, t)$ exists if $p(s) \neq 0$, and for each g in $G(p)$, let g_1^* be defined on E_2 by $g_1^*(s, t) = g_1(s, t)$ if $p(s) \neq 0$, $g_1^*(s, t) = 0$ if $p(s) = 0$. Observe that p satisfies the following conditions.

$(S, 1)$ If J is a bounded open interval containing no zeroes of p but having a zero of p for one of its end points, then p^{-1} is not integrable on J.

$(S, 2)$ If J is an unbounded interval containing no zeroes of p, then p^{-1} is not integrable on J.

It is perhaps of intersest to compare these conditions with [5, Th. 10.11, p. 458] and [3, eq. (4), p. 10].

2.1 Lemma. *If s is in E_1, then there is only one function f from E_1 into E_1 such that $f(0) = s$ and $f' = p(f)$. If f is such a function then $p(f)$ has no zeroes unless $p(s) = 0$, and*

$$\int_s^{f(t)} p^{-1} = t \tag{2.1.1}$$

for all t if $p(s) \neq 0$.

Proof. Suppose s is in E_1. There is an open interval Q containing 0 such that $\{f(0) = s, f' = p(f)\}$ has a unique solution on Q, [1, Th. 2.3, p. 10]. Let J denote the maximal such open interval, and let f denote the function from J into E_1 which satisfies $f(0) = s$, $f' = p(f)$. If $p(s) = 0$, then $J = E_1$ and $f(t) = s$ for all t. Suppose $p(s) \neq 0$. Then for sufficiently small t,

$$0 \neq p(f(t)) = f'(t) \quad \text{and} \quad \int_s^{f(t)} p^{-1} = t.$$

Condition $(S, 1)$ now shows that $p(f)$ has no zeroes on J so that (2.1.1) holds for all t in J. If J is bounded on the right, then it follows from (2.1.1) and $(S, 2)$ that $f(t)$ remains bounded as t approaches b, the right endpoint of J. Therefore, if a is a negative number in J, then the closure of $f\{(a, b)\}$ is contained in an open interval on which p is bounded, so that J is not maximal by [1, Th. 4.1, p. 15]. Similarly, J is not bounded to the left.

2.2. Definition. Let y denote the function from E_2 into E_1 which satisfies $y_2 = p(y)$, $y(s, 0) = s$.

2.3. Lemma. *y is continuous, and*

$$|y(s, t) - y(s_0, t)| \leq |s - s_0| \exp(K|t|) \quad \textit{for all} \quad s, s_0, t. \tag{2.3.1}$$

If p is bounded, then

$$| y(s, t) - y(s, t_0)| \leq \| p \| \cdot | t - t_0 | \qquad \textit{for all} \quad s, t, t_0 . \tag{2.3.2}$$

Proof. (2.3.1) follows from [1, Th. 2.1, p. 8]. The continuity of y follows from (2.3.1) and the fact that $y(s_0, \cdot)$ is continuous for all s_0. (2.3.2) is trivial.

2.4. LEMMA. *If $p(s) \neq 0$, then $p(y(s, \cdot))$ has no zeroes, and*

$$\int_s^{y(s,t)} p^{-1} = t \qquad \textit{for all } t. \tag{2.4.1}$$

Proof. This follows from Lemma 2.1 and the definition of y.

2.5. LEMMA. *y is in $G(p)$, and*

$$y_2(s, t) = p(s) y_1^*(s, t) \qquad \textit{for all} \quad (s, t) \tag{2.5.1}$$

If $p(s) = 0$ and $p'(s)$ exists, then

$$y_1(s, t) = \exp(tp'(s)) \qquad \textit{for all} \quad t.$$

Proof. If $p(s) = 0$, then $y(s, t) = s$ for all t. Suppose $p(s_0) \neq 0$; let J denote the maximal open interval containing s_0 but no zeroes of p, let α be defined on J by

$$\alpha(s) = \int_{s_0}^{s} p^{-1},$$

and let β denote the inverse of α. By $(S, 1)$ and $(S, 2)$, β has domain E_1. α has a continuous nonvanishing derivative on J, so that β has a continuous derivative on E_1. Let M denote the strip consisting of all (s, t) with s in J. By (2.4.1),

$$\alpha(y(s, t)) - \alpha(s) = t \quad \text{and} \quad y(s, t) = \beta(\alpha(s) + t)$$

for all (s, t) in M. Therefore, $y_1(s, t)$ exists for all (s, t) in M. (2.51) now follows from (2.4.1).

If $p(u) = 0$, and $p'(u)$ exists, then

$$(y(s, t) - u)/(s - u) = \exp\left[tp'(u) + \int_0^t \varepsilon(y(s, \cdot))\right]$$

for all t and all $s \neq u$, where

$$\varepsilon(v) = p(v)/(v - u) - p'(u)$$

for $v \neq u$, $\varepsilon(u) = 0$.

2.6. LEMMA. *$y(y(s, u), t) = y(s, u + t)$ for all s, u, t.*

Proof. Let

$$g(t) = y(y(s, u), t), \qquad h(t) = y(s, u + t).$$

Then

$$g'(t) = y_2(y(s, u)t) = p(g(t)),$$

$$h'(t) = y_2(s, u + t) = p(h(t)),$$

and

$$g(0) = y(y(s, u), 0) = y(s, u) = h(0).$$

2.7. *Remark.* If J is an interval $[a, b]$ with $p(a) = p(b) = 0$, an interval $[a, \infty)$ with $p(a) = 0$, or an interval $(-\infty, b]$ with $p(b) = 0$, then Lemma 2.4 shows that $y(s, t)$ is in J for all t if s is in J. In the theorems to follow, it may be noted that this fact makes the theorem carry over for functions defined on J rather than on E_1 or on $J \times E_1$ rather than E_2.

3. A partial differential equation

Here the results of Section 2 are applied to a partial differential equation. This is the only one of the main results which does not depend on the semigroup theory and the only one which does not require that p be bounded.

3.1. THEOREM. *If x is in $F(p)$ and px^* is continuous, then there is only one function g in $G(p)$ such that g_2 is continuous and*

$$g_2(s, t) = p(s)g_1^*(s, t)$$

for all (s, t), $g(s, 0) = x(s)$ for all s.

Proof. The function g defined on E_2 by $g(s, t) = x(y(s, t))$ is as required.

Suppose ξ and η are two such functions, and let $\phi = \xi - \eta$. Then ϕ is in $G(p)$, ϕ_2 is continuous,

$$\phi_2(s, t) = p(s)\phi_1^*(s, t)$$

for all (s, t), and $\phi(s, 0) = 0$ for all s.

If $p(s) = 0$, then $\phi(s, t) = 0$ for all t. Suppose $p(s_0) \neq 0$, and take J, α, β, and M as in the proof of Lemma 2.5. Let

$$\theta(u, t) = \phi(\beta(\alpha(u) - t), t)$$

for all (u, t) in M. Since ϕ_1 and ϕ_2 exist and are continuous on M, then θ_1 and θ_2 exist and are continuous on M. Also,

$$\phi(s, t) = \theta(y(s, t), t)$$

for all (s, t) in M. Some applications of the chain rule for partial derivatives yields the fact that ϕ is identically zero on M.

3.2. *Remark.* The function $g(s, t) = x(y(s, t))$ is bounded if x is bounded and continuous if x is continuous. Moreover, $g(s, t)$ exists if $x'(s)$ and $p'(s)$ exist, even if $p(s) = 0$.

4. Semi-groups

For the rest of the paper, p is assumed to be bounded. For each real t, let $T(t)$ denote the transformation with domain X defined by $T(t)x = x(y(\cdot, t))$. By (2.3.1), $T(t)x$ is in X for each x in X and t in E_1. Clearly, $T(t)$ is a bounded operator in X and $\| T(t) \| \leq 1$ for each real t.

4.1. THEOREM. *$[T(t), -\infty < t < \infty]$ is a strongly continuous group of operators with infinitesimal generator A.*

Proof. By Lemma 2.6, $[T(t)]$ is a group, and by (2.3.2), $[T(t)]$ is strongly continuous. For each $h > 0$, let $A_h = (T(h) - I)/h$.

If x is in D and $p(s) = 0$, then

$$[A_h x](s) = [Ax](s) = 0.$$

If x is in D, $p(s) \neq 0$, and $h > 0$, then

$$[A_h x](s) - [Ax](s) = \operatorname{Re} \{[Ax](y(s, h_1))\} + i \operatorname{Im} \{[Ax](y(s, h_2))\} - [Ax](s)$$

for some h_1, h_2 in $(0, h)$ by the mean value theorem for derivatives.

Therefore, by (2.3.2),

$$\lim \| A_h x - Ax \| = 0$$

for all x in D.

Conversely, if x is in X, z is in X, and

$$\lim \| A_h x - z \| = 0,$$

then $z(s) = 0$ if $p(s) = 0$, and for $p(s) \neq 0$,

$$\lim [x(y(s, h)) - x(s)]/[y(s, h) - s]$$

exists and is equal to $z(s)/y_2(s, 0)$, so that $z = px^*$, x is in D, and $z = Ax$.

4.2. THEOREM. *If* $\operatorname{Re}(\lambda) \neq 0$, *then λ is in the resolvent set of A,*

$$\| R(\lambda, A) \| \leq | \operatorname{Re}(\lambda) |^{-1},$$

and $R(\lambda, A)$ is given by

$$[R(\lambda, A)x](s) = \int_0^\infty e^{-\lambda t} x(y(s, t)) \, dt$$

for $\operatorname{Re}(\lambda) > 0$, *and*

$$[R(\lambda, A)x](s) = -\int_{-\infty}^0 e^{-\lambda t} x(y(s, t)) \, dt$$

for $\operatorname{Re}(\lambda) < 0$.

Proof. $[T(t), 0 \leq t < \infty]$ is a strongly continuous semi-group with infinitesimal generator A, $[T(-t), 0 \leq t < \infty]$ is a strongly continuous semi-group with infinitesimal generator $-A$, and $-\lambda$ in $\rho(-A)$ implies that λ is in

$\rho(A)$ and $R(\lambda, A) = -R(-\lambda, -A)$. Thus [2, Th. 11, p. 822] implies everything but the inequality, which follows from the formulas.

4.3. THEOREM. *If Q is a bounded operator in X, then $A + Q$ is the infinitesimal generator of a strongly continuous group $[T(t, Q), -\infty < t < \infty]$ of operators in X such that*

$$\| T(t, Q)\| \leq \exp(|t|\cdot\| Q \|)$$

for all t, the resolvent set of $A + Q$ includes all λ such that $|\operatorname{Re}(\lambda)| > \| Q \|$, *and*

$$\| R(\lambda, A + Q)\| \leq (|\operatorname{Re}(\lambda)| - \| Q \|)^{-1}$$

for $|\operatorname{Re}(\lambda)| > \| Q \|$.

Proof. If $|\operatorname{Re}(\lambda)| > \| Q \|$, then

$$\lambda I - A - Q = (I - QR(\lambda, A))(\lambda I - A)$$

so that $\lambda I - A - Q$ is invertible and

$$R(\lambda, A + Q) = R(\lambda, A)(I - QR(\lambda, A))^{-1},$$

$$\| R(\lambda, A)\| \leq |\operatorname{Re}(\lambda)|^{-1}(1 - \| Q \|\cdot|\operatorname{Re}(\lambda)|^{-1})^{-1}$$
$$= (|\operatorname{Re}(\lambda)| - \| Q \|)^{-1}.$$

Since A is closed and D is dense by [2, Lemma 8, p. 620], the rest follows from [2, Cor. 17, p. 628].

4.4. *Remark.* [2, Th. 19, p. 631] gives a construction for $T(t, Q)$. Also, see (E_5) through (E_{10}) [4, p. 354]. Aside from the formula given for $R(\lambda, A + Q)$ in the above proof, one has the series

$$R(\lambda, A) \textstyle\sum_0^\infty [QR(\lambda, A)]^n$$

for the formula of [2, Th. 11, p. 622].

4.5. THEOREM. *A^2 is the infinitesimal generator of a strongly continuous semigroup of operators (which we shall denote by $[V(t), 0 \leq t < \infty]$) such that $\| V(t)\| \leq 1$ for $t \geq 0$, and*

$$V(r)x = (4\pi r)^{-1/2}\int_{-\infty}^{\infty} \exp(-t^2/4r)T(t)x\,dt \tag{4.5.1}$$

for $r > 0$, x in X. The resolvent set of A^2 includes all λ such that $\operatorname{Re}(\lambda) > 0$, $\| R(\lambda, A^2)\| \leq \operatorname{Re}(\lambda)^{-1}$ *for* $\operatorname{Re}(\lambda) > 0$, *and*

$$R(\lambda, A^2)x = (4\lambda)^{-1/2}\int_{-\infty}^{\infty} \exp(-|t|\sqrt{\lambda})T(t)x\,dt \tag{4.5.2}$$

for $\lambda > 0$, x in X.

Proof. D^2 is dense in X by [4, Th. 10.3.4, p. 308] and A^2 is closed by

[2, Th. 7, p. 602]. If $\lambda > 0$, then $\sqrt{\lambda}$ and $-\sqrt{\lambda}$ are in $\rho(A)$ and

$$\lambda I - A^2 = -(\sqrt{\lambda} I - A)(-\sqrt{\lambda} I - A)$$

so that $\lambda I - A^2$ is invertible and

$$R(\lambda, A^2) = -R(-\sqrt{\lambda}, A)R(\sqrt{\lambda}, A),$$
$$R(\lambda, A^2) \leq \lambda^{-1}.$$

Therefore, by [2, Cor. 14, p. 626], A^2 generates a strongly continuous semi-group $[V(t),\ 0 \leq t < \infty]$ such that $\| V(t) \| \leq 1$ for $t \geq 0$. By [2, Th. 11, p. 622] we have that λ is in $\rho(A^2)$ and $\| R(\lambda, A^2) \| \leq \operatorname{Re}(\lambda)^{-1}$ if $\operatorname{Re}(\lambda) > 0$.

The integral operator U of (4.5.2) is simply

$$U = (4\lambda)^{-1/2}[R(\sqrt{\lambda}, A) - R(-\sqrt{\lambda}, A)]$$

(see Theorem 4.2), so that $\lambda U - A^2 U = I$, $U = R(\lambda, A^2)$. Consulting a table of Laplace transforms, one obtains

$$(\lambda)^{-1/2} \exp(-| t | \sqrt{\lambda}) = \int_0^\infty \exp(-\lambda r) \exp(-t^2/4r)(\pi r)^{-1/2}\, dr.$$

Substituting and interchanging the order of integration, one obtains

$$R(\lambda, A^2)x = \int_0^\infty \exp(-\lambda r) \left\{ \int_{-\infty}^\infty \exp(-t^2/4r)(4\pi r)^{-1/2} T(t)x\, dt \right\} dr.$$

(4.5.1) now follows from [2, Cor. 16, p. 627].

4.6. Theorem. *If P and Q are bounded operators in X, then*

$$A^2 + PA + Q$$

is the infinitesimal generator of a strongly continuous semi-group of operators in X (which we shall denote by $[V(t, P, Q),\ 0 \leq t < \infty]$).

Proof. If $r > 0$ and x is in D, then

$$AV(r)x = (4\pi r)^{-1/2} \int_{-\infty}^\infty (t/2r) \exp(-t^2/4r) T(t)x\, dt,$$

$$\| AV(r)X \| \leqq \| X \| (4\pi r)^{-1/2} \int_{-\infty}^\infty | t/2r | \exp(-t^2/4r)\, dt$$
$$= 2 \| x \| (4\pi r)^{-1/2}.$$

The conclusion follows from [2, Th. 19, p. 631].

4.7. Theorem. *If P and Q are bounded operators in X, $\omega > \| P \|^2$, and $\gamma = \| P \| \omega^{-1/2}$, then*

$$\| V(t, P, Q) \| \leq (1 - \gamma)^{-1} \exp(t[\omega + \| Q \| (1 - \gamma)^{-1}])$$

for $t \geq 0$, and if $P = 0$, we may take $\omega = 0$, $\gamma = 0$.

Proof. For each $r > 0$, $PAV(r)$ has a unique extension to a bounded operator (which we shall denote by $W(r)$) defined on all of X. For $t > 0$, let $\chi(t) = \| V(t)\|$, $\psi(t) = \| W(t)\|$.

Then $\chi(t) \leq 1$ and $\psi(t) \leq \| P \|(\pi t)^{-1/2}$ for $t > 0$. Let $\psi^{(1)} = \psi$, $\psi^{(n+1)} = \psi^{(n)} * \psi$, $\chi^{(0)} = \chi$, $\chi^{(n)} = \chi * \psi^{(n)}$, where $(*)$ denotes convolution, see [2, Def. 23, pp. 633, 634].

By [2, Th. 19, p. 631], $V(t, P, 0)$ is given by

$$V(t, P, 0) = \textstyle\sum_0^\infty S_n(t),$$

where by [2, eq. (iii), p. 636], $\| S_n(t)\| \leq \chi^{(n)}(t)$.

If $\omega > \| P \|^2$, $\gamma = \| P \| \omega^{-1/2}$, then

$$\int_0^\infty e^{-\omega t}\psi(t)\, dt \leqq \| P \| \ (\pi)^{-1/2} \int_0^\infty e^{-\omega t} t^{-1/2}\, dt$$

$$= \| P \| \ \omega^{-1/2} = \gamma < 1.$$

$\chi^{(0)}(t) \leq 1 \leq \gamma^0 e^{\omega t}$. If $\chi^{(n)}(t) \leq \gamma^n e^{\omega t}$, then

$$\chi^{(n+1)}(t) \leqq \gamma^n e^{\omega t} \int_0^t e^{-\omega s}\psi(s)\, ds \leqq \gamma^{n+1} e^{\omega t}.$$

Therefore $\| S_n(t)\| \leq \gamma^n e^{\omega t}$ for each n and each $t \geq 0$, so that

$$\| V(t, P, 0)\| \leq (1 - \gamma)^{-1} e^{\omega t}.$$

If $\operatorname{Re}(\lambda) > \omega$, then λ is in $\rho(A^2 + PA)$, and

$$\| R(\lambda, A^2 + PA)^n \| \leq (1 - \gamma)^{-1}(\operatorname{Re}(\lambda) - \omega)^{-n} \quad \text{for} \quad n = 1, 2, 3, \cdots$$

by [2, Lemma 12, p. 623].

If $\operatorname{Re}(\lambda) > \omega + \| Q \| (1 - \gamma)^{-1}$, then

$$\lambda I - A^2 - PA - Q = (I - QR(\lambda, A^2 + PA))(\lambda I - A^2 - PA)$$

so that $\lambda I - A^2 - PA - Q$ is invertible and

$$R(\lambda, A^2 + PA + Q) = R(\lambda, A^2 + PA)(I - QR(\lambda, A^2 + PA))^{-1}.$$

$$\|(I - QR(\lambda, A^2 + PA))^{-n} \| \leq (1 - \| Q \|(\operatorname{Re}(\lambda) - \omega)^{-1}(1 - \gamma)^{-1})^{-n}$$

so that

$$\| R(\lambda, A^2 + PA + Q)^n \| \leq (1 - \gamma)^{-1}[\operatorname{Re}(\lambda) - \omega - \| Q \|(1 - \gamma)^{-1}]^{-n}.$$

The inequality for $\| V(t, P, Q)\|$ is now established. If $P = 0$, then an argument similar to the one given for Theorem 4.3 may be applied to show that we may take $\omega = 0$, $\gamma = 0$.

4.8. Theorem. *If P and Q are bounded operators in X, $\omega > \| P \|^2$, $\gamma = \| P \|\omega^{-1/2}$, and* $\operatorname{Re}(\lambda) > \omega + \| Q \|(1 - \gamma)^{-1}$, *then λ is in $\rho(A^2 + PA + Q)$ and*

$$\| R(\lambda, A^2 + PA + Q)^n \| \leq (1-\gamma)^{-1}[\mathrm{Re}\,(\lambda) - \omega - \| Q \|(1-\gamma)^{-1}]^{-n}$$

for $n = 1, 2, 3, \cdots$. *If* $P = 0$, *we may take* $\omega = 0, \gamma = 0$.

Proof. This follows immediately from Theorem 4.7.

4.9. THEOREM. *If P and Q are bounded operators in X, then*

$$\begin{aligned} R(\lambda, A^2 + PA) &= R(\lambda, A^2)(I - PAR(\lambda, A))^{-1} \\ &= R(\lambda, A^2)\textstyle\sum_0^\infty [PAR(\lambda, A^2)]^n \end{aligned}$$

if $\mathrm{Re}\,(\lambda) > \| P \|^2$;

$$\begin{aligned} R(\lambda, A^2 + PA + Q) &= R(\lambda, A^2)[I - (PA + Q)R(\lambda, A^2)]^{-1} \\ &= R(\lambda, A^2)\textstyle\sum_0^\infty [(PA + Q)R(\lambda, A^2)]^n \end{aligned}$$

if $\mathrm{Re}\,(\lambda) > \| Q \| + \| P \| \,\mathrm{Re}\,(\lambda)^{1/2}$; *and*

$$\begin{aligned} R(\lambda, A^2 + PA + Q) &= R(\lambda, A^2 + PA)[I - QR(\lambda, A^2 + PA)]^{-1} \\ &= R(\lambda, A^2 + PA)\textstyle\sum_0^\infty [QR(\lambda, A^2 + PA)]^n \end{aligned}$$

if $\omega > \| P \|^2$, $\gamma = \| P \|\omega^{-1/2}$, $\mathrm{Re}\,(\lambda) > \omega + \| Q \|(1-\gamma)^{-1}$, *and if* $P = 0$, *we can take* $\omega = 0$, $\gamma = 0$.

Proof. The range of $R(\lambda, A^2)$ is D^2, so that the domain of $AR(\lambda, A^2)$ is all of X. If $\mathrm{Re}\,(\lambda) > \| P \|^2$, then

$$\| PAR(\lambda, A^2)x \| = \left\| \int_0^\infty e^{-\lambda t} W(t)x \, dt \right\| \leqq \| P \| \cdot \| x \| \,\mathrm{Re}\,(\lambda)^{-1/2}.$$

This establishes the first and the second formula. The inequalities of Theorem 4.8 establish the third formula.

4.10. *Remark.* The perturbed semi-groups can be constructed as in [2, Th. 19, p. 631] and then used to get the perturbed resolvents as in [2, Th. 11, p. 622]. Also, the perturbed resolvents may be constructed as in Theorem 4.9 and then used to get the perturbed semi-group by means of one of the formulas (E_5) thru (E_{10}) [4, p. 354].

5. Applications

Here the results of Section 4 are applied to a partial differential equation, some ordinary differential equations, and an abstract Cauchy problem [4, pp. 617–622].

5.1. THEOREM. *If q is in X and x is in D, then there is only one function g in $G(p)$ such that g and g_2 are continuous and*

$$g_2(s, t) = p(s)g_1^*(s, t) + q(s)g(s, t)$$

for all (s, t), $g(s, 0) = x(s)$ *for all* s.

Proof. Let Q denote the operator defined on X by $Qx = qx$, and let

$$[T(t, Q), -\infty < t < \infty]$$

denote the group generated by $A + Q$. The function g defined on E_2 by $g(s, t) = [T(t, Q)X](s)$ is as required. The uniqueness claim may be established by an argument similar to the one given for Theorem 3.1.

5.2. THEOREM. *If z is in X, q is in X, and* Re (q) *is bounded away from zero, then there is only one bounded function x in $F(p)$ such that $px^* + qx = z$.*

Proof. Let d denote g.l.b. $|\operatorname{Re}(q)|$. If λ is a real number having the same sign as Re (q) and $\lambda > \| q \|^2/2\, d$, then $\| q - \lambda \| < |\lambda|$. Let Q denote the operator defined on X by $Qx = qx - \lambda x$ for some such λ. Then

$$x = -R(-\lambda, A + Q)z$$

is as required.

Suppose ξ and η are two bounded functions in $F(p)$ satisfying

$$p\xi^* + q\xi = z \quad \text{and} \quad p\eta^* + q\eta = z,$$

and let $\phi = \xi - \eta$. Then ϕ is a bounded function in $X(p)$ and $p\phi^* + q\phi = 0$.

If $p(x) = 0$, then $\phi(s) = 0$. Suppose $p(s_0) \neq 0$, and let J denote the maximal open interval containing s_0 but no zeroes of p. Then

$$\phi(s) = \phi(s_0) \exp\left(-\int_{s_0}^{s} q/p\right)$$

for all s in J. Conditions $(S, 1)$ and $(S, 2)$ now show that $\phi(s_0) = 0$.

5.3. THEOREM. *If q and z are in X, and q has a negative real part which is* bounded away from zero, then there is only one function x *in D^2 such that $p(px^*)^* + qx = z$.*

Proof. Let d denote g.l.b. $|\operatorname{Re}(q)|$. If $\lambda > \| q \|^2/2\, d$, then $\| q + \lambda \| < \lambda$. Let Q denote the operator defined on X by $Qx = qx + \lambda x$ for some such λ. Then $x = -R(\lambda, A^2 + Q)z$ is the only function in D^2 having the required property.

5.4. THEOREM. *If u and v are functions in X, and there exist numbers, $\omega > \| u \|^2$, $r > 0$, $\lambda > \omega + r(1 - \gamma)^{-1}$, where $\gamma = \| u \| \omega^{-1/2}$ such that the values of v all lie in the circular disk with center $-\lambda$, radius r, then there is only one function x in D^2 such that $p(px^*)^* + upx^* + vx = z$.*

Proof. Let P and Q denote the operators defined on X by $Px = ux$, and $Qx = vx + \lambda x$, respectively. Then $\lambda > \omega + \| Q \| (1 - \gamma)^{-1}$, so that $x = -R(\lambda, A^2 + PA + Q)z$ is the only function in D^2 having the required property.

5.5. THEOREM. *If P and Q are bounded operators in X and x is in D^2, then there is only one function from* $[0, \infty)$ *into D^2 such that f is strongly con-*

tinuously differentiable as a function from $[0, \infty)$ *into* X, $f(0) = x$, *and*

$$f'(t) = (A^2 + PA + Q)f(t) \qquad \text{for all } t \geq 0.$$

Proof. $f(t) = V(t, P, Q)x$ is the only such function; see the corollary of [4, Th. 23.8.1, p. 622].

5.6. *Remark.* The results of Section 4 give properties and constructions of the solutions to the equations considered in this section.

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