THE REPRESENTATION LATTICE OF A LOCALLY COMPACT GROUP

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We begin by defining the term "representation lattice" used in the title. Let G be a separable locally compact group. By a representation of G we shall always mean a strongly continuous homomorphism of G into a group of unitary operators acting on a (not necessarily separable) Hilbert space. Two representations L and M of G are said to be *disjoint*, denoted $L \, \delta \, M$, *if* no subrepresentation of L is (unitary) equivalent to any subrepresentation of M. We say L *covers* M, denoted $L \, \} \, M$, if no subrepresentation of M is disjoint from L. We say L is *quasi-equivalent* to M, denoted $L \sim M$, if L covers M and M covers L. (For all of these concepts, see [10] and [11].) The collection Q of all quasi-equivalence classes of representations of G forms a complete distributive lattice with respect to the ordering given by the covering relation. The lattice Q ($= Q(G)$) will be called the *representation lattice of* G The collection Q_σ of all quasi-equivalence classes of separable representations of G forms a σ-complete sublattice of $Q(G)$, which we call the *separable representation lattice of* G.

The properties of quasi-equivalence and the covering relation (cf. [10], [11] and Proposition 1 of [6]) are reminiscent of a projection lattice. Our first theorem proves that this is not a mere impression. $Q(G)$ is lattice isomorphic to the lattice of all projections in an abelian von Neumann algebra. The appropriate von Neumann algebra is just the center of the big group algebra $\mathfrak{A}(G)$, introduced in [7].

There has been a rather extensive study of Q_σ, using the tool of direct integral decomposition theory for representations. (Recall that a group G is said to be type I if it admits only type I representations. A representation is type I if its range generates a type I von Neumann algebra.) In the case where G is type I, G. W. Mackey has characterized $Q_\sigma(G)$ as being lattice isomorphic to the lattice of all standard σ-finite measure classes on the dual $\hat{G}$ of G. (Cf, [12].) ($\hat{G}$ denotes the set of unitary equivalence classes of separable irreducible representations of G.) The measures on $\hat{G}$ arise from the central decomposition of separable multiplicity free representations, as a direct integral of irreducible representations. If G is not type I, the lattice $Q_{\sigma-I}$ of all quasi-equivalence classes of separable type I representations of G is lattice isomorphic to some lattice $\mathfrak{L}_I$ of standard σ-finite measure classes on the dual $\hat{G}$. However $\mathfrak{L}_I$ is a proper sublattice of the lattice of all standard σ-finite measure classes on $\hat{G}$. The characterization of $\mathfrak{L}_I$, when G is not type I, has remained an open problem.

In [6], the author presented a generalization of the Mackey decomposition

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theory designed to cover the case where G is not type I. In this theory the dual $\hat{G}$ is replaced by the quasi-dual $\tilde{G}$, which is the set of all quasi-equivalence classes of primary representations. (A representation is said to be *primary* if the von Neumann algebra generated by its range is a factor.) The two dual objects $\hat{G}$ and $\tilde{G}$ coincide in the type I case. Cf. [4]. It is shown in [6, Theorem 2] that the central decomposition of any separable representation may be taken over the space $\tilde{G}$, relative to some measure μ on G. In this way decomposition theory once again leads to a description of the separable representation lattice Q_σ of G, in terms of a measure lattice. Thus Q_σ is lattice isomorphic to a lattice $\mathcal{CM}(\tilde{G})$ of standard σ-finite measure classes on $\tilde{G}$. $\mathcal{CM}(\tilde{G})$ is called the canonical measure lattice on $\tilde{G}$. However when G is not type I, it is known that $\mathcal{CM}(\tilde{G})$ is a proper sublattice of the lattice of all standard σ-finite measure classes. Thus the unsolved problem of identifying $\mathcal{L}_I$ in the Mackey theory [12] emerges once again as the problem of identifying the sublattice $\mathcal{CM}(\tilde{G})$. Various attempts have been made to ameliorate this situation. Thus Ed Effros [5] has designed a decomposition theory emphasizing a different kind of equivalence between representations (called weak equivalence) specifically to circumvent this obstacle. A. Guichardet [9] has obtained a number of results identifying some of the measures in $\mathcal{CM}(\tilde{G})$.

When the representations act on nonseparable Hilbert space, many of the techniques of direct integral decomposition theory are no longer available. Nevertheless a combination of the big group algebra theory ([7] and [8]) and the structure theory for abelian von Neumann algebras does enable us to characterize the representation lattice Q in terms of a measure lattice. Indeed (Theorem 2) there exists a locally compact measure space (Z, ν), called the *central space of* G, such that Q is lattice isomorphic to the lattice $\mathcal{L}(Z, \nu)$ of *all* measure classes on Z, absolutely continuous with respect to ν. Further the sublattice Q_σ of Q is isomorphic to the sublattice $\mathcal{L}_\sigma(Z, \nu)$ of the σ-finite classes $\mathcal{L}(Z, \nu)$. The space (Z, ν) is called the central space of G because the center of the big group algebra $\mathcal{A}(G)$, (cf. [7]) is isomorphic to $L_C^\infty(Z, \nu)$, the algebra of ν-essentially bounded complex-valued measurable functions on Z.

This characterization helps one to understand the enigma of the canonical measure lattice $\mathcal{CM}(\tilde{G})$. Thus (Theorem 3) there exists a lattice isomorphism φ of $\mathcal{CM}(\tilde{G})$ onto the lattice $\mathcal{L}_\sigma(Z, \nu)$ such that $L_C^\infty(G, \mu) \simeq L_C^\infty(Z, \varphi(\mu))$ for all μ in $\mathcal{CM}(\tilde{G})$. The atoms of ν correspond to the points of $\tilde{G}$. Thus a measure in $\mathcal{L}_\sigma(Z, \nu)$ which is concentrated on an atom of ν corresponds to the measure in $\mathcal{CM}(\tilde{G})$ concentrated at the corresponding point of $\tilde{G}$.

In terms of the Mackey theory [12], this observation takes the following form. There exists a measure ν_I on the central space (Z, ν) of G such that $\nu_I \ll \nu$ and the lattice $\mathcal{L}_I$ of standard measure classes on $\hat{G}$ which arise from the central decomposition of multiplicity free representations is lattice isomorphic to the lattice of those elements of $\mathcal{L}_\sigma(Z, \nu)$ which are absolutely continuous with respect to ν_I.

1. Preliminary considerations of separability

PROPOSITION 1. *Every primary representation of a separable locally compact group is quasi-equivalent to a separable primary representation.*

Proof. Let L be a primary representation of G. Then L contains a cyclic subrepresentation L'. Since G is separable, L' must be separable. Since L is primary, we conclude that L and L' are quasi-equivalent.

COROLLARY. *The quasi-dual $\tilde{G}$ of a separable locally compact group G is the same whether one uses the set of quasi-equivalence classes of all primary representationsof G, or the set of quasi-equivalence classes of separable primary representations of G.*

We next generalize Theorem 8.3 of [7] to cover the case of nonseparable representations. Recall that a *normal* $*$-representation of a von Neumann algebra $\mathcal{A}$ is a $*$-algebra homomorphism of $\mathcal{A}$ onto a von Neumann algebra $\mathcal{B}$ which, when restricted to the positive cone of $\mathcal{A}$, preserves least upper bounds.

PROPOSITION 2. *Every strongly continuous unitary representation of a separable locally compact group G has a unique extension to a normal $*$-representation of its big group algebra $\mathcal{A}(G)$. (Cf. [7].) Further the restriction of any normal $*$-representation of $\mathcal{A}(G)$, to G, gives a strongly continuous unitary representation of G.*

Proof. Suppose L is a strongly continuous unitary representation of G. Following the usual argument, one concludes that L is a direct sum of cyclic representations, $L = \sum_{\alpha \epsilon \Lambda} L^{\alpha}$. Since G is separable, we conclude that each cyclic representation L^{α} is separable. We may then use Theorem 8.3 of [7] to conclude that each L^{α} has a unique extension to a normal $*$-representation $L^{\alpha\prime}$ of the big group algebra $\mathcal{A}(G)$. We may then form the direct sum $L' = \sum_{\alpha \epsilon \Lambda} L^{\alpha\prime}$. Clearly L' is a normal $*$-representation of $\mathcal{A}(G)$ whose restriction to G is L. L' is unique since G generates the von Neumann algebra $\mathcal{A}(G)$. (Theorem 7.2 of [7].)

We next let M denote a normal $*$-representation of $\mathcal{A}(G)$. By Theorem 8.4 of [7] it follows that M is σ-weak continuous. Let M' denote its restriction to G. Since the elements of G are unitary elements in $\mathcal{A}(G)$, (Theorem 2.3 of [7]) and since M is a $*$-representation, it follows that M' is a unitary representation of G. Further the σ-weak topology of $\mathcal{A}(G)$ induces the given topology on G. (Corollary 5.8 of [7].) Thus M' is a σ-weakly and hence strongly continuous unitary representation of G.

2. The representation lattice is a projection lattice

Proposition 2 shows that the study of normal $*$-representations of von Neumann algebras actually subsumes the theory of unitary representations of separable locally compact groups. The main result of this section will be proved in the more general context of normal $*$-representations of von Neumann algebras. We assume the global concepts of unitary representations ([10] and [11]) such as unitary equivalence, quasi-equivalence, subrepresentation, primary representation etc., have been extended to normal $*$-representations. For any representation L, the notation $\mathfrak{A}(L)$ will always denote the von Neumann algebra generated by the range of L. Often an adjective referring to the von Neumann algebra $\mathfrak{A}(L)$ will also be used to refer to the normal $*$-representation L. Once again $Q = Q(\mathfrak{A})$ will denote the lattice of all quasi-equivalence classes of normal $*$-representations of a von Neumann algebra $\mathfrak{A}$. The quasi-dual, denoted $\tilde{\mathfrak{A}}$, is the set of quasi-equivalence classes of primary normal $*$-representations of $\mathfrak{A}$.

PROPOSITION 3. *Every normal $*$-representation φ of a von Neumann algebra $\mathfrak{A}$ is a quasi-equivalent to an induction, $\mathfrak{A} \rightarrow \mathfrak{A}_E$, for some central projection E of $\mathfrak{A}$.*

Proof. The map φ is a normal homomorphism of $\mathfrak{A}$ onto a von Neumann algebra $\varphi(\mathfrak{A})$. According to the structure theorem for normal homomorphisms (Theorem 3, page 58 of [3]), φ is the composite of three maps,

$$\varphi = \varphi_3 \circ \varphi_2 \circ \varphi_1 ,$$

where φ_1 is an exemplification ("ampliation" in French) of $\mathfrak{A}$ onto a von Neumann algebra $\mathfrak{B}$, φ_2 is an induction $\mathfrak{B} \rightarrow \mathfrak{B}_F$, where F is a projection in the commutator of $\mathfrak{B}$ and φ_3 is a spatial isomorphism of $\mathfrak{B}_F$ onto $\varphi(\mathfrak{A})$. Let F' denote the central support of F. Then $F' \in \mathfrak{B}$ and the induction φ_2 may be expressed as the composite of two inductions,

$$\varphi_2 = \varphi_2' \circ \varphi_2''$$

where $\varphi_2' : \mathfrak{B}_{F'} \rightarrow \mathfrak{B}_F$ and $\varphi_2'' : \mathfrak{B} \rightarrow \mathfrak{B}_{F'}$. Since F' is the central support of F, Proposition 2, page 19 of [3] implies that φ_2' is an isomorphism. Since an exemplification is an isomorphism we have that φ_1 is an isomorphism and $E = \varphi_1^{-1}(F')$ is a central projection of $\mathfrak{A}$. Let φ' denote the induction $\mathfrak{A} \rightarrow \mathfrak{A}_E$.

We next verify that φ and φ' have the same kernel. Indeed

$$\begin{aligned} T \in \operatorname{Ker} \varphi \;&\Leftrightarrow\; \varphi_1(T)F' = 0 \;\Leftrightarrow\; \varphi_1(T)\varphi_1(E) = 0 \\ &\Leftrightarrow\; \varphi_1(TE) = 0 \;\Leftrightarrow\; TE = 0 \;\Leftrightarrow\; T \in \operatorname{Ker} \varphi'. \end{aligned}$$

Thus there exists an isomorphism θ of $\mathfrak{A}_E$ onto $\varphi(\mathfrak{A})$ such that $\theta(\varphi'(T)) = \varphi(T)$ for all T in $\mathfrak{A}$. It follows from Lemma 4 of [6] that φ and φ' are quasi-equivalent.

THEOREM 1. *Let $\mathfrak{A}$ denote a von Neumann algebra. Then the representation lattice Q of $\mathfrak{A}$ is isomorphic to the lattice P of central projections in $\mathfrak{A}$.*

Proof. For each projection E in P, let $\Phi(E)$ denote the quasi-equivalence class containing the normal $*$-representation $\mathfrak{A} \rightarrow \mathfrak{A}_E$. We proceed to verify that Φ is a lattice isomorphism of P onto Q.

Clearly if E and F are in P and $F \geq E$, then the representation $\mathcal{A} \to \mathcal{A}_E$ is a subrepresentation of $\mathcal{A} \to \mathcal{A}_F$. Thus $\Phi(F) \} \Phi(E)$ and Φ is order preserving. Conversely if $\Phi(F) \} \Phi(E)$, then we must have $F \geq E$. Indeed if not, then $E = E_1 - E_2$ where $E_1 \leq F$ and E_2 is a non-zero central projection such that $E_2 F = 0$. Thus the representation $\mathcal{A} \to \mathcal{A}_{E_2}$ is a nontrivial subrepresentation of $\mathcal{A} \to \mathcal{A}_E$ which is disjoint from $\mathcal{A} \to \mathcal{A}_F$. However this contradicts the fact that $\Phi(F) \} \Phi(E)$. Thus the inverse map of Φ is also order preserving.

We next verify that Φ is one to one. Suppose E and F are distinct elements of P. Let L denote the representation $\mathcal{A} \to \mathcal{A}_E$, and let M denote the representation $\mathcal{A} \to \mathcal{A}_F$. Since $E \neq F$, either $E - EF \neq 0$ or $F - EF \neq 0$. Without loss of generality, we suppose the former. Then $L' : \mathcal{A} \to \mathcal{A}_{(E-EF)}$ is a nontrivial subrepresentation of L. Since $L'_F = 0$, we have that $L'_T = 0$ for any T for which $M_T \neq 0$. Thus no subrepresentation of L' can be equivalent to a subrepresentation of M. Thus L' is disjoint from M and hence L and M are not quasi-equivalent. Hence $\Phi(E) \neq \Phi(F)$.

The proof that Φ maps P onto Q has been separated out in Proposition 3.

Corollary 1. *If $\mathcal{A}$ is a von Neumann algebra, then the quasi-dual $\tilde{\mathcal{A}}$ is in one-to-one correspondence with the set of minimal central projections in $\mathcal{A}$.*

Proof. The primary classes in Q are just the minimal classes relative to the partial ordering in Q.

Corollary 2. *Let G denote a separable locally compact group. Then the representation lattice Q of G is isomorphic to the lattice of all central projections in its big group algebra $\mathcal{A}(G)$.*

Proof. This result is obtained by applying the previous theorem to the big group algebra $\mathcal{A}(G)$, and then using Proposition 2 to obtain a statement about the representation theory of G.

3. The representation lattice is a measure lattice

Theorem 2. *To each separable locally compact group G we may associate a locally compact measure space (Z, ν), called the central space of G, such that Z is the support of ν and which has the following properties.*

1. *The representation lattice Q of G is isomorphic to the lattice $\mathcal{L}(Z, \nu)$ of all measure classes on Z, absolutely continuous with respect to ν.*
2. *The separable representation lattice Q_σ of G is isomorphic to the lattice $\mathcal{L}_\sigma(Z, \nu)$ of all σ-finite measure classes on Z, absolutely continuous with respect to ν.*

Proof. We refer the reader to [1] and [2] for the terminology to be used in referring to (not necessarily σ-finite) measures on a locally compact space.

To obtain the central space (Z, ν), we apply the structure theory for abelian von Neumann algebras to the center $\mathcal{Z}(G)$ of the big group algebra $\mathcal{A}(G)$. Thus according to Theorem 1, page 117 of [3], there exists a locally compact space Z, a positive measure ν on Z whose support is Z, and an isometric isomorphism between the normed $*$-algebra $\mathcal{Z}(G)$ and the normed $*$-algebra $L_C^\infty(Z, \nu)$. Here $L_C^\infty(Z, \nu)$ denotes the algebra of ν-measurable, essentially bounded, complex-valued functions on Z, where one identifies two functions which are equal locally almost everywhere. (Cf. pp. 207–208 of [1].)

Since (Corollary 2 to Theorem 1) Q is isomorphic to the lattice of projections P in $\mathcal{Z}(G)$, it is sufficient to show that there is a one-to-one order preserving map Φ of P onto $\mathcal{L}(Z, \nu)$. It is an easy exercise to show that the projections in $L_C^\infty(Z, \nu)$ are just the functions of the form χ_X where X is a ν-measurable subset of Z and χ_X denotes its characteristic function. For each projection E in $\mathcal{Z}(G)$, let $\chi(E)$ denote the corresponding function in $L_C^\infty(Z, \nu)$. For each E in P, let $\Phi(E)$ denote the equivalence class of the measure $\chi(E) \cdot \nu$. We next note that if E_1 and E_2 are distinct projections in $\mathcal{Z}(G)$, then the corresponding measures $\mu_1 = \chi(E_1) \cdot \nu$ and $\mu_2 = \chi(E_2) \cdot \nu$ are not equivalent. Indeed $\chi(E_1)$ (respectively $\chi(E_2)$) contains a characteristic function χ_X (respectively χ_Y) where X and Y are ν-measurable sets and the functions χ_X and χ_Y are not equivalent. Thus the set $(X - Y) \cup (Y - X)$ is not locally ν-negligible. Thus one of the sets $(X - Y)$ and $(Y - X)$ is not locally ν-negligible. If $(X - Y)$ is not locally ν-negligible, then $(X - Y)$ is a locally μ_2-negligible set which is not a locally μ_1-negligible set. Thus μ_1 and μ_2 are not equivalent. This result follows similarly if $(Y - X)$ is not locally ν-negligible. Thus Φ is a one-to-one mapping of P into $\mathcal{L}(Z, \nu)$ which is clearly an order isomorphism. It remains to show that Φ maps P onto $\mathcal{L}(Z, \nu)$.

Let $\bar{\mu}$ denote an element of $\mathcal{L}(Z, \nu)$ and let μ denote a measure in the class $\bar{\mu}$. By the Lebesgue-Nikodym theorem (Theorem 2, page 47 of [2]), there exists a finite nonnegative locally ν-integrable function g on Z such that $\mu = g \cdot \nu$. Let X denote the measurable set $\{x : x \in Z \text{ and } g(x) \neq 0\}$. Then χ_X is also a finite nonnegative locally ν-integrable function and $\chi_X \cdot \nu$ is equivalent to $g \cdot \nu$. But χ_X is a projection in $L_C^\infty(Z, \nu)$. If we denote this projection by the letter E, we have $\Phi(E) = \bar{\mu}$.

To prove part 2 of the theorem, we shall apply some well known facts about the structure theory of abelian von Neumann algebras, summarized in exercises 2 and 3, page 119 of [3]. A quasi-equivalence class of representations of G will be called separable if it contains at least one separable representation. Similarly a measure class will be called σ-finite if it contains at least one σ-finite measure.

Suppose E is contained in P, and that X is the corresponding measurable subset of Z such that χ_X is the associated projection in $L_C^\infty(Z, \nu)$. Then $\mathcal{Z}(G)_E$ is of countable type if and only if X is, up to a set locally ν-negligible, the union of a countable number of integrable sets. (Recall that a von Neumann algebra $\mathfrak{A}$ is said to be of countable type if every family of projections of $\mathfrak{A}$, nonzero and two-by-two orthogonal, is countable.) Thus the measure $\chi_X \cdot \nu$ is equivalent to a σ-finite measure on Z. Thus for each projection E in P, $\mathcal{Z}(G)_E$ is of countable type if and only if $\Phi(E)$ is a σ-finite measure class.

Note that since the group G is separable, Corollary 5.8 and Theorem 7.2 of [7] imply that the big group algebra $\mathfrak{A}(G)$ is countably generated. Thus in particular, for every E in P, $\mathfrak{A}(G)_E$ is countably generated.

Let $\bar{L}$ denote a separable quasi-equivalence class in Q and let E denote the corresponding element of P. Then the restriction M of the induction $\mathfrak{A}(G) \to \mathfrak{A}(G)_E$ to the group G, is contained in the quasi-equivalence class $\bar{L}$. Further $\bar{L}$ contains a separable representation L. Thus $\mathfrak{A}(M) = \mathfrak{A}(G)_E$, the von Neumann algebra generated by the range of M, is isomorphic to the von Neumann algebra $\mathfrak{A}(L)$ generated by the range of L. Since $\mathfrak{A}(G)_E$ is isomorphic to a von Neumann algebra acting on a separable Hilbert space, $\mathfrak{A}(G)_E$ is of countable type. Thus its center, $\mathfrak{Z}(G)_E$ is also of countable type. Hence $\Phi(E)$ is a σ-finite measure class.

Conversely suppose $\Phi(E)$ is a σ-finite measure class. Then $\mathfrak{Z}(G)_E$ is of countable type. Since $\mathfrak{A}(G)_E$ is generated by a countable family of elements, exercise 3c, page 119 of [3] implies that $\mathfrak{A}(G)_E$ is isomorphic to a von Neumann algebra, say $\mathfrak{B}$, acting on a separable Hilbert space. Let φ denote the isomorphism of $\mathfrak{A}(G)_E$ onto $\mathfrak{B}$. Then the restriction L of the composite map

$$\mathfrak{A}(G) \to \mathfrak{A}(G)_E \xrightarrow{\varphi} \mathfrak{B}$$

to G, gives a separable strongly continuous unitary representation of G for which $\mathfrak{A}(L) = \mathfrak{B}$. (Cf. Theorem 8.3 of [7]). Thus the representation of G determined by the induction $\mathfrak{A}(G) \to \mathfrak{A}(G)_E$ is quasi-equivalent to the separable representation L. Thus the quasi-equivalence class corresponding to the projection E, and hence to the σ-finite measure class $\Phi(E)$, is separable.

4. The canonical measure lattice $\mathfrak{CM}(\tilde{G})$

In [6], the author developed a decomposition theory which associated, with each separable unitary representation L of G, a measure class $\mathfrak{C}(L)$ on the quasi-dual $\tilde{G}$ of quasi-equivalence classes of primary representations. This result sets up a lattice isomorphism between the separable representation lattice Q_σ and a lattice $\mathfrak{CM}(\tilde{G})$ of σ-finite measure classes on $\tilde{G}$. $\mathfrak{CM}(\tilde{G})$ is called the canonical measure lattice on $\tilde{G}$. When G is not type I, $\mathfrak{CM}(\tilde{G})$ is a proper sublattice of the lattice of all standard σ-finite measure classes on $\tilde{G}$ and the identification of this sublattice has remained an open problem.

THEOREM 3. *Let G denote a separable locally compact group, (Z, ν) its central space, and $\mathfrak{CM}(\tilde{G})$ its canonical measure lattice. Then there exists a lattice isomorphism φ of the lattice $\mathfrak{L}_\sigma(Z, \nu)$ of all σ-finite measure classes on Z which are absolutely continuous with respect to ν, onto the canonical measure lattice $\mathfrak{CM}(\tilde{G})$. The map φ has the additional property that*

$$L_C^\infty(Z, \mu) \simeq L_C^\infty(\tilde{G}, \varphi(\mu)),$$

for every μ in $\mathfrak{L}_\sigma(Z, \nu)$.

Proof. Note that both $\mathfrak{CM}(\tilde{G})$ and $\mathfrak{L}_\sigma(Z, \nu)$ are lattice isomorphic to Q_σ. (Cf. Theorems 3 and 4 of [6] and Theorem 2 above.) Thus there exists a lattice isomorphism φ of $\mathfrak{L}_\sigma(Z, \nu)$ onto $\mathcal{CM}(\tilde{G})$. Let $\mu \in \mathfrak{L}_\sigma(Z, \nu)$ and let $\tilde{L}$ denote the corresponding element of Q_σ. (Cf. Theorem 2.) Then for any L in the class $\tilde{L}$, $L_C^\infty(\tilde{G}, \varphi(\mu))$ is isomorphic to the algebra of diagonalizable operators in the central decomposition of L. That is to say, $L_C^\infty(\tilde{G}, \varphi(\mu))$ is isomorphic to the center of the von Neumann algebra $\mathfrak{A}(L)$ generated by the range of L. On the other hand the measure class μ contains a measure of the form $\chi_X \cdot \nu$ where χ_X is the characteristic function of a measurable set X and χ_X is a projection E_μ in $L_C^\infty(Z, \nu)$. Thus $L_C^\infty(Z, \mu) \simeq L_C^\infty(Z, \chi_X \cdot \nu)$ is isomorphic to $\mathfrak{Z}(G)_{E_\mu}$, which is the center of $\mathfrak{A}(G)_{E_\mu}$. But $\mathfrak{A}(G)_{E_\mu}$ is the algebra $\mathfrak{A}(M)$ generated by the range of the representation M of G, determined by the induction $\mathfrak{A}(G) \to \mathfrak{A}(G)_{E_\mu}$. By the definition of φ, L and M are quasi-equivalent. Hence by Lemma 4 of [6], the center of $\mathfrak{A}(L)$ is isomorphic to the center of $\mathfrak{A}(M)$. Thus $L_C^\infty(Z, \mu) \simeq L_C^\infty(\tilde{G}, \varphi(\mu))$.

Remark. A nontrivial measure class μ, which is minimal in the lattice $\mathfrak{L}(Z, \nu)$ will be called a *measure atom of* ν. It is an easy exercise to show that each measure atom of ν is the class of a finite measure of compact support. Note that the points of $\tilde{G}$ correspond to the measure atoms of ν. Thus a measure atom μ in $\mathfrak{L}_\sigma(Z, \nu)$ corresponds to the measure in $\mathcal{CM}(\tilde{G})$ concentrated at the corresponding point of $\tilde{G}$.

This same procedure may be used to describe the lattice $\mathfrak{L}_I$ of those standard σ-finite measure classes on the dual $\hat{G}$ which arise from the central decomposition of separable multiplicity free representations. (Cf. [12].) It should be pointed out that J. Dixmier [4] has shown that $\hat{G}$ is Borel isomorphic to the Borel subset $\tilde{G}_I$ of $\tilde{G}$, consisting of the type I classes in $\tilde{G}$. With respect to this isomorphism, $\mathfrak{L}_I$ corresponds to the sublattice of $\mathcal{CM}(\tilde{G})$ consisting of the measure classes which arise from the central decomposition of type I representations.

COROLLARY. *Let G denote a separable locally compact group, $\hat{G}$ its dual and (Z, ν) its central space. Then there exists a measure ν_I on Z, absolutely continuous with respect to ν, such that the lattice $\mathfrak{L}_I$ of standard measure classes on $\hat{G}$ which arise from the central decomposition of separable multiplicity free representations of G, is lattice isomorphic to the lattice $\mathfrak{L}_\sigma(Z, \nu_I)$ of all σ-finite measure classes on Z which are absolutely continuous with respect to ν_I.*

Proof. Since the lattice of unitary equivalence classes of separable multiplicity free representations is lattice isomorphic to the lattice of quasi-equivalence classes of separable type I representations, we simply identify the sublattice of $\mathfrak{L}_\sigma(Z, \nu)$ which, by the isomorphism of Theorem 2, corresponds to the type I representations. By Corollary 1, page 121 of [3], there exists a maximal central projection E_I of $\mathfrak{A}(G)$ such that $\mathfrak{A}(G)_{E_I}$ is type I. Since $\mathfrak{Z}(G)$ is isomorphic to $L_C^\infty(Z, \nu)$, E_I corresponds to a non-negative function χ of $L_C^\infty(Z, \nu)$. Let $\nu_I = \chi \cdot \nu \cdot$ Then since the type I separable representations all correspond to inductions of the form $\mathfrak{A}(G) \to \mathfrak{A}(G)_E$, where E is a central projection for which $E \leq E_I$, it follows that the type I elements of Q_σ correspond to those measures in $\mathcal{L}_\sigma(Z, \nu)$ which are absolutely continuous with respect to ν_I.

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