ON A FAMILY OF TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS

BY
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1. Introduction

In 1960 Thron and this author [1] studied the continued fraction

$$1 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots, \tag{1.1}$$

whose elements a_n satisfy the conditions

$$\begin{aligned} a_{2n-1} &= c_{2n-1}^2, & \quad | c_{2n-1} \pm ia | &\leqq \rho \\ a_{2n} &= c_{2n}^2, & \quad | c_{2n} \pm i(1 + a)| &\geqq \rho, \end{aligned} \tag{1.2}$$

where a is a complex number and a and ρ satisfy the inequality

$$| a | < \rho < | 1 + a |.$$

It was shown in [1] that the element regions for the a_{2n-1} and the a_{2n} defined by conditions (1.2) form a set of best twin convergence regions for the continued fraction (1.1). It was also shown in [1] that the continued fraction converges uniformly for the real values of the parameter a. However, the problem of proving that the continued fraction converges uniformly for the non-real values of a was at the time unsurmountable. Even to prove ordinary convergence in this case, the authors had to rely on the use of the Stieltjes-Vitali theorem [5, p. 142]. As Perron [3, vol. 1, p. 82] has pointed out, the use of a deep function-theoretic result to obtain a convergence theorem of this nature is aesthetically undesirable. Another disadvantage of this function-theoretic argument is that it gives no information as to how fast the approximants of the continued fraction approach their limiting value.

In this article we are finally able to completely settle the problem for all permissible complex values of a. We are able to prove that the continued fraction (1.1) whose elements satisfy conditions (1.2) converges uniformly, even though the elements a_n be functions of an arbitrary number of variables. In addition, we have obtained in the key inequalities (3.12) and (3.13) usable estimates of the error committed if the continued fraction is replaced by one of its approximants. The methods of proof are elementary in the sense that only fundamental concepts of complex analysis are used.

In the proof we employ a nested circle argument similar to the one given in [1] for real values of a. The success of this method depends on obtaining a sharp enough estimate of B_{2n-1}/B_{2n-2} for complex values of a. This problem,

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dealt with in the next paragraph, turns out to be considerably more difficult than it was in the real case. For a further illustration of the method of proof see Thron [4, pp. 745–749].

For definitions of the fundamental sets and a listing of the basic formulas used throughout this paper refer to [1, pp. 296–297].

2. Three fundamental lemmas

LEMMA 2.1. *Let* $a = |a| e^{i\alpha}$ *be a complex number and* ρ *be a positive real number such that* $|a| < \rho < |1 + a|$. *If*

$$a_{2n-1} \in E_1(ia, \rho) \quad \text{and} \quad z \in M(-ma, m\rho),$$

where $m \geqq 1$, *then*

$$t_{2n-1}(z) \in N(1 + ka, k\rho) \subset N(1 + a, \rho),$$

where $k = 1/m$.

Proof. We shall first dispose of the assertion that

$$N(1 + ka, k\rho) \subset N(1 + a, \rho)$$

by showing that every element of the first set is an element of the second set. Let $w \in N(1 + ka, k\rho)$. Then $w = 1 + ka + k\varepsilon\rho e^{i\phi}$, where $0 \leqq \varepsilon \leqq 1$ and $0 \leqq \phi \leqq 2\pi$. Since,

$$|1 + a - w| = |a(1 - k) - k\varepsilon\rho e^{i\phi}| \leqq |a|(1 - k) + k\rho \leqq \rho,$$

it follows that $w \in N(1 + a, \rho)$.

We shall now prove that

$$t_{2n-1}(z) \in N(1 + ka, k\rho)$$

if $a_{2n-1} \in E_1(ia, \rho)$, $z \in M(-ma, m\rho)$. This is equivalent to showing that

$$|ka - a_{2n-1}/z| \leqq k\rho.$$

If $ka - a_{2n-1}/z$ is considered as a function of a_{2n-1} alone, then $|ka - a_{2n-1}/z|$ assumes its largest value for a point a_{2n-1} on the boundary of $E_1(ia, \rho)$. The boundary of $E_1(ia, \rho)$ consists of all points $a^*_{2n-1} = e^{2i\alpha}(-|a| i + \rho e^{i\theta})^2$, where $\theta_0 \leqq \theta \leqq \pi - \theta_0$ and θ is defined by the equations

$$\sin\theta_0 = |a|/\rho, \qquad \cos\theta_0 = (\rho^2 - |a|^2)^{1/2}/\rho.$$

If we set $r = r(\theta) = 2(\rho\sin\theta - |a|)$, where $0 \leqq r \leqq 2(\rho - |a|)$, then using the above expression for a^*_{2n-1} simple calculations will show that

$$(2.1) \qquad \begin{aligned} a^*_{2n-1} &= e^{2i\alpha}(\rho^2 - |a|^2 + i\rho r e^{i\theta}) \\ |a_{2n-1}| &= \rho^2 - |a|^2 - |a| r. \end{aligned}$$

Now $z \in M(-ma, m\rho)$ implies that $z \in M(-ma, \sigma m\rho)$ for some $\sigma \geqq 1$. Hence

$$
\begin{aligned}
|ka - a_{2n-1}/z| &\leqq \max_{\beta,\sigma,\theta} |ka - a_{2n-1}^*/(-ma + \sigma m\rho e^{i\beta})| \\
&= \max_{\beta,\sigma,\theta} k\,|a - a_{2n-1}^*/(-a + \sigma\rho e^{i\beta})|
\end{aligned}
\tag{2.2}
$$

where $0 \leqq \beta \leqq 2\pi$. The set of all points $1/(-a + \sigma\rho e^{i\beta})$, for a given σ, is the image of the circle $K(-a, \sigma\rho)$ under the transformation $1/z$. It is easily verified that this image set is the circle

$$K(\bar{a}/(\sigma^2\rho^2 - |a|^2), \sigma\rho/(\sigma\rho^2 - |a|^2)),$$

where $\bar{a} = |a|e^{-i\alpha}$ is the conjugate of a. Using this result and inequality (2.2) we obtain

$$
\begin{aligned}
|ak - a_{2n-1}/z| &\leqq k \max_{\beta',\sigma,\theta} |a - a_{2n-1}^*(\bar{a} + \sigma\rho e^{i\beta'})/(\sigma^2\rho^2 - |a|^2)| \\
&= k \max_{\sigma,\theta} \frac{|a|\,|\sigma^2\rho^2 - |a|^2 - e^{-2i\alpha}a_{2n-1}^*| + \sigma\rho\,|a_{2n-1}^*|}{\sigma^2\rho^2 - |a|^2}
\end{aligned}
\tag{2.3}
$$

where $0 \leqq \beta' \leqq 2\pi$.

We now employ formulas (2.1) in inequality (2.3) and arrive at

$$
\begin{aligned}
&|ak - a_{2n-1}/z| \\
&\leqq k \max_{\sigma,\theta,r} \frac{|a|\,|\sigma^2\rho^2 - |a|^2 - \rho^2 + |a|^2 - i\rho r e^{i\theta}| + \sigma\rho(\rho^2 - |a|^2 - |a|r)}{\sigma^2\rho^2 - |a|^2} \\
&\leqq k \max_{\sigma,r} \frac{|a|\,(\rho^2(\sigma^2 - 1) + \rho r) + \sigma\rho(\rho^2 - |a|^2 - |a|r)}{\sigma^2\rho^2 - |a|^2} \\
&\leqq k \max_{\sigma} \frac{\rho(|a|\sigma + \rho)}{\sigma\rho + |a|} \leqq k\rho
\end{aligned}
$$

This completes the proof of the lemma.

LEMMA 2.2. *Let $a = |a|e^{i\alpha}$ be a complex number and let ρ be a positive real number such that $|a| < \rho < |1 + a|$. Let γ be defined by the equations*

$$
\begin{aligned}
\rho\cos\gamma &= -(\rho^2 - |a|^2\sin^2\alpha)^{1/2} \\
\rho\sin\gamma &= -|a|\sin\alpha,
\end{aligned}
\tag{2.4}
$$

and let

$$b = 1 + |a|\cos\alpha + \rho\cos\gamma \tag{2.5}$$

If

$$a_{2n} \in E_2(i(1 + a), \rho) \quad \text{and} \quad z \in N(1 + ka, k\rho)$$

where $0 \leqq k \leqq 1$, then

$$t_{2n}(z) \in M(-ma, m\rho) \subset M(-a, \rho),$$

where

$$m = \frac{1 - k + b}{1 - k + kb} \geqq 1.$$

Since $0 \leqq k \leqq 1$ it is a simple matter to see that $m \geq 1$ if $b > 0$. Hence we first verify the inequality

$$(2.6) \qquad 0 < b = 1 + | a | \cos \alpha + \rho \cos \gamma < 1.$$

The condition $| a | < \rho < | 1 + a |$ insures that

$$-\rho \cos \gamma > 0 \quad \text{and} \quad 1 + | a | \cos \alpha > \tfrac{1}{2} > 0.$$

Furthermore,

$$(1 + | a | \cos \alpha)^2 = | 1 + a |^2 - | a |^2 \sin^2 \alpha > \rho^2 - | a |^2 \sin^2 \alpha = \rho^2 \cos^2 \gamma$$

and

$$\rho^2 \cos^2 \gamma = \rho^2 - | a |^2 + | a |^2 \cos^2 \alpha > | a |^2 \cos^2 \alpha.$$

Therefore, $1 + | a | \cos \alpha > -\rho \cos \gamma$ and $-\rho \cos \gamma > | a | \cos \alpha$, from which inequality (2.6) follows immediately.

We shall now prove that $M(-ma, m\rho) \subset M(-a, \rho)$. If w is an arbitrary element of $M(-ma, m\rho)$, then $w = -ma + \sigma m \rho e^{i\phi}$, where $\sigma \geqq 1$ and $0 \leqq \phi \leqq 2\pi$. Making use of the fact that $m \geqq 1$ we obtain

$$| a + w | = | a - ma + \sigma m \rho e^{i\phi} | \geqq \sigma m \rho - | a |(m - 1)$$

$$\geqq m(\rho - | a |) + | a | \geqq \rho.$$

Therefore, $w \in M(-a, \rho)$ and the assertion

$$M(-ma, m\rho) \subset M(-a, \rho)$$

is true.

Later on in the proof of this lemma we shall need the inequality

$$(2.7) \qquad m | 1 + ka | \geqq k | 1 + ma |.$$

To verify (2.7) it is sufficient to show that

$$m^2 | 1 + ka |^2 - k^2 | 1 + ma |^2 \geqq 0.$$

Recalling that $1 + | a | \cos \alpha > \frac{1}{2}$, we obtain

$$\begin{aligned} m^2 | 1 + ka |^2 - k^2 | 1 + ma |^2 \\ &= m^2(1 + 2k | a | \cos \alpha + k^2 | a |^2) - k^2(1 + 2m | a | \cos \alpha + m^2 | a |^2) \\ &= (m - k)(m + k + 2km | a | \cos \alpha) \geqq (m - k)(m + k - km) \geqq 0. \end{aligned}$$

Equality can occur if $k = m = 1$.

The principal task is to prove that, if

$$a_{2n} \in E_2(i(1 + a), \rho) \quad \text{and} \quad z \in N(1 + ka, k\rho),$$

then

$$t_{2n}(z) \in M(-ma, m\rho).$$

This is equivalent to proving that $| 1 + ma + a_{2n}/z | \geqq m\rho$ if a_{2n} and z satisfy the above conditions. For fixed z and variable a_{2n} the function

$$| 1 + ma + a_{2n}/z |$$

assumes its minimum for some point a_{2n} on the boundary of $E_2(i(1+a), \rho)$, provided $1 + ma + a_{2n}/z \neq 0$. We shall first prove that

$$| 1 + ma + a_{2n}^*/z | \geqq m\rho,$$

where a_{2n}^* is any boundary point of $E_2(i(1+a), \rho)$. Then we shall show that

$$1 + ma + a_{2n}/z \neq 0;$$

so that $| 1 + ma + a_{2n}/z | \geqq m\rho$ for all a_{2n} in $E_2(i(1+a), \rho)$.

If $z \in N(1 + ka, k\rho)$, then $z \in K(1 + ka, \varepsilon k\rho)$ for some ε, where $0 \leqq \varepsilon \leqq 1$. The set $1/K(1 + ka, \varepsilon k\rho)$ is the image of the circle $K(1 + ka, \varepsilon k\rho)$ under the transformation $1/z$. By Lemma 2.1 $N(1 + ka, k\rho) \subset N(1 + a, \rho)$. Hence $| 1 + ka | > k\rho \geqq \varepsilon k\rho$. With this result simple computations will show that the desired image set is the circle

$$K((1 + k\bar{a})/d_\varepsilon, \varepsilon k\rho/d_\varepsilon),$$

where

$$d_\varepsilon = | 1 + ka |^2 - \varepsilon^2 k^2 \rho^2 > 0.$$

It follows that if z is any element of $N(1 + ka, k\rho)$, then $1/z$ can be expressed as

$$1/z = (1 + k\bar{a} + \varepsilon k\rho e^{i\delta})/d_\varepsilon,$$

where $0 \leqq \delta \leqq 2\pi$. Thus setting

$$d = | 1 + ka |^2 - k^2\rho^2$$

we obtain

$$
\begin{aligned}
| 1 + ma + a_{2n}^*/z| &= | 1 + ma + a_{2n}^*(1 + k\bar{a} + \varepsilon k\rho e^{i\delta})/d_\varepsilon | \\
&\geqq | 1 + ma + a_{2n}^*(1 + k\bar{a})/d_\varepsilon | - | a_{2n}^* | \varepsilon k\rho/d_\varepsilon \\
&= | 1 + ma + a_{2n}^*(1 + k\bar{a})/d \\
&\qquad + a_{2n}^*(1 + k\bar{a})(1/d_\varepsilon - 1/d)| - | a_{2n}^* | \varepsilon k\rho/d_\varepsilon \\
&\geqq |1 + ma + a_{2n}^*(1 + k\bar{a})/d | - | a_{2n}^* | k\rho/d \\
&\qquad - | a_{2n}^* | | 1 + k\bar{a} |(1/d - 1/d_\varepsilon) \\
&\qquad + | a_{2n}^* |(k\rho/d - \varepsilon k\rho/d_\varepsilon) \\
&= H + H_\varepsilon \geqq H,
\end{aligned}
\tag{2.8}
$$

where

$$H = | 1 + ma + a_{2n}^*(1 + k\bar{a})/d | - | a_{2n}^* | k\rho/d \tag{2.9}$$

and

$$H_\varepsilon = | a_{2n}^* |[1/(| 1 + k\bar{a} | + k\varepsilon\rho) - 1/(| 1 + k\bar{a} | + k\rho)]. \tag{2.10}$$

Thus we will have shown that $| 1 + ma + a_{2n}^*/z | \geqq m\rho$ if we can prove that $H \geqq m\rho$. For this purpose we employ the fact that a_{2n}^* can be expressed in the form

$$a_{2n}^* = -(1 + a + \rho e^{i\theta})^2$$

where $0 \leqq \theta \leqq 2\pi$. After making this substitution for a_{2n}^* in formula (2.9) it follows that

$$\begin{aligned} H\,d - m\rho\,d &= |\,d(1 + ma) - (1 + a + \rho e^{i\theta})^2(1 + k\bar{a})| \\ &\qquad - \rho[k\,|\,1 + a + \rho e^{i\theta}\,|^2 + m\,d] \\ &= |\,e^{-i\theta}(1 + a + \rho e^{i\theta})^2(1 + k\bar{a}) - de^{-i\theta}(1 + ma)| \\ &\qquad - \rho[k\,|\,1 + a + \rho e^{i\theta}\,|^2 + m\,d] \\ &\geqq \mathcal{R}[e^{-i\theta}(1 + a + \rho e^{i\theta})^2(1 + k\bar{a}) - de^{-i\theta}(1 + ma)] \\ &\qquad - \rho[k\,|\,1 + a + \rho e^{i\theta}\,|^2 + m\,d] \\ &= 2\rho \sin^2 \tfrac{1}{2}(\gamma - \theta)(1 - k)^2(|\,a\,|\cos\alpha - \rho\cos\gamma). \end{aligned}$$

The last expression on the right is obtained from the preceding expression by making repeated use of formulas (2.4). For example these relations allow us to write

$$d = (1 + k\,|\,a\,|\cos\alpha - k\rho\cos\gamma)(1 + k\,|\,a\,|\cos\alpha + k\rho\cos\gamma)$$

$$m\,d = (1 + k\,|\,a\,|\cos\alpha - k\rho\cos\gamma)(2 + |\,a\,|\cos\alpha + \rho\cos\gamma - k).$$

By formula (2.4), $-\rho\cos\gamma = (\rho^2 - |\,a\,|^2 + |\,a\,|^2\cos^2\alpha)^{1/2}$. Therefore,

$$|\,a\,|\cos\alpha - \rho\cos\gamma > |\,a\,|\cos\alpha + |\,a\cos\alpha\,| \geqq 0$$

and it follows from the preceding inequality that $H \geqq m\rho$.

It remains to prove that

$$1 + ma + a_{2n}/z \neq 0$$

for $a_{2n} \in E_2(i(1 + a), \rho)$ and $z \in N(1 + ka, k\rho)$. It is sufficient to show that

$$|(1 + ma)(1 + ka) + a_{2n}\,| > k\rho\,|\,1 + ma\,|.$$

Now

$$\begin{aligned} |(1 + ma)(1 + ka) + a_{2n}\,| &= |(1 + a)^2 + a_{2n} + (1 + ma)(1 + ka) \\ &\qquad - (1 + a)^2\,| \\ &\geqq |(1 + a)^2 + a_{2n}\,| \\ &\qquad - |(1 + ma)(1 + ka) - (1 + a)^2\,|. \end{aligned}$$

Simple calculations will show that

$$|(1 + ma)(1 + ka) - (1 + a)^2\,| = |\,a\,|\,\rho(1 - km).$$

Furthermore, since $\pm i(1 + a) \notin E_2(i(1 + a), \rho)$, it is true that

$$(1 + a)^2 + a_{2n} \neq 0.$$

Thus $|(1 + a)^2 + a_{2n}\,|$ assumes its minimum for a point

$$a_{2n}^* = -(1 + a + \rho e^{i\theta})^2$$

on the boundary of $E_2(i(1+a), \rho)$. Hence,

$$\begin{aligned}|(1+ma)(1+ka)+a_{2n}| &\geqq |(1+a)^2-(1+a+\rho e^{i\theta})^2| \\ &\qquad - |a|\rho(1-km) \\ &\geqq \rho(2|1+a|-\rho)-|a|\rho(1-km) \\ &\geqq \rho(2|1+a|-\rho-|a|)>0.\end{aligned}$$

Since $(1+ma)(1+ka)+a_{2n}\neq 0$, it follows that

$$\begin{aligned}|(1+ma)(1+ka)+a_{2n}| &\geqq \min_{a_{2n}^*}|(1+ma)(1+ka)+a_{2n}^*| \\ &= \min_{a_{2n}^*}|1+ka|\,|1+ma+a_{2n}^*/(1+ka)|.\end{aligned}$$

By taking $\varepsilon = 0$ in inequality (2.8) and formula (2.10) and using the fact that $H \geqq m\rho$ we arrive at the inequality

$$\begin{aligned}|(1+ma)(1+ka)+a_{2n}| &\geqq \min_{a_{2n}^*}|1+ka|[m\rho+|a_{2n}^*|k\rho/ \\ &\qquad [|1+k\bar{a}|^2+k\rho|1+k\bar{a}|]] \\ &\geqq m\rho|1+ka|+k\rho(|1+a|-\rho)^2/ \\ &\qquad (|1+ka|+k\rho).\end{aligned}$$

From inequality (2.7) we have $m\rho|1+ka| \geqq k\rho|1+ma|$. Thus by considering the cases $k=0$ and $0<k\leqq 1$ separately it finally follows from this inequality and the last inequality above that

$$|(1+ma)(1+ka)+a_{2n}| > k\rho|1+ma|.$$

With this our proof is complete.

LEMMA 2.3. *Let $a=|a|e^{i\alpha}$ be a complex number and ρ be a positive real number such that $|a|<\rho|1+a|$. If*

$$a_{2n-1}\in E_1(ia,\rho),\quad a_{2n}\in E_2(i(1+a),\rho),\quad \text{and}\quad s_n = B_{2n-1}/B_{2n-2},$$

then

$$s_n \in N(1+ak_n, \rho k_n)$$

where

$$k_n = (n-1)/(n-1+b) \tag{2.11}$$

and b is given by formula (2.5).

Proof. It follows from the recursion formulas for the B_n that

$$s_1 = 1 \quad\text{and}\quad s_{n+1}=t_{2n+1}(t_{2n}(s_n)).$$

Clearly

$$s_1\in N(1+ak_1,\rho k_1) = N(1,0)\subset N(1+a,\rho).$$

By inequality (2.6), $0<b<1$. Hence

$$0\leqq k_n = (n-1)/(n-1+b)\leqq 1$$

for all $n = 1, 2, \cdots$. It follows from Lemma 2.2 that

$$t_{2n}(N(1 + ak_n, \rho k_n)) \subset M(-am_n, \rho m_n) \subset M(-a, \rho)$$

where

$$m_n = \frac{1 - k_n + b}{1 - k_n + bk_n} \geqq 1.$$

Therefore, by Lemma 2.1

$$t_{2n+1}[M(-am_n \, \rho m_n)] \subset N(1 + ak'_{n+1}, \rho k'_{n+1}) \subset N(1 + a, \rho)$$

where

$$k'_{n+1} = 1/m_n .$$

We shall now show that $k'_{n+1} = k_{n+1}$.

$$\begin{aligned} k'_{n+1} &= \frac{1 - k_n + bk_n}{1 - k_n + b} \\ &= \frac{1 + (b - 1)(n - 1)/(n - 1 + b)}{(1 + b) - (n - 1)/(n - 1 + b)} \\ &= n/(n + b) = k_{n+1}. \end{aligned}$$

Thus, since $s_{n+1} = t_{2n+1}(t_{2n}(s_n))$, it follows that, if

$$s_n \in N(1 + ak_n, \rho k_n),$$

then

$$s_{n+1} \in N(1 + ak_{n+1}, \rho k_{n+1}).$$

By induction $s_n \in N(1 + ak_n, \rho k_n)$ for all n, and the proof is complete.

3. Proof of the main theorem

The approximants $T_n(1) = A_b/B_n$ of the continued fraction (1.1) lie in nested circular disks whose radii R_n are given by

$$R_{2n-1} = \frac{\rho \left| \prod_{m=1}^{2n-1} a_m \right|}{|B_{2n-2}|^2 \, | \rho^2 - |1 + a - s_n|^2 |} \tag{3.1}$$

$$R_{2n} = \frac{\rho \left| \prod_{m=1}^{2n} a_m \right|}{|B_{2n-1}|^2 \, | |a + t_{2n}(s_n)|^2 - \rho^2 |}. \tag{3.2}$$

All of these circular regions are contained in the closed disk $N(1 + a, \rho)$. For the sake of brevity the proofs of these statements and formulas will not be given here since they are, with a few obvious modifications, identical to the ones given in [1, pp. 303–305] for the real case. It should be pointed out that these arguments depend principally on the three lemmas given in the preceding paragraph.

We now set $Q_n = R_{2n}/R_{2n-1}$ and proceed to obtain an upper bound for Q_n.

By Lemma 2.3, $s_n \in N(1 + ak_n, \rho k_n) \subset N(1 + a, \rho)$, where k_n is given by formula (2.11). An application of Lemma 2.2 gives $|a + t_{2n}(s_n)| > \rho$. Therefore, using the relation $s_n(a + t_{2n}(s_n)) = (1 + a)s_n + a_{2n}$, it follows from formulas (3.1) and (3.2) that

$$Q_n = \frac{|a_{2n}|(\rho^2 - |1 + a - s_n|^2)}{|(1 + a)s_n + a_{2n}|^2 - \rho^2|s_n|^2}. \tag{3.3}$$

As a first step towards getting an estimate for Q_n let us consider the function

$$f(a_{2n}, w) = \frac{a_{2n}}{((1 + a)s_n + a_{2n})^2 + w\rho^2 s_n^2},$$

where $a_{2n} \in E_2(i(1 + a), \rho)$ and $|w| \leqq 1$.

It is easily seen that

$$\max_{a_{2n} \in E_2, w \leqq 1} |f(a_{2n}, w)| = \max_{a_{2n} \in E_2, |w|=1} |f(a_{2n}, w)|.$$

If we denote the boundary of $E_2(i(1 + a), \rho)$ by P and make use of the Maximum Modulus Principle we also have that

$$\max_{a_{2n} \in E_2, |w|=1} |f(a_{2n}, w)| = \max_{a_{2n} \in P, |w|=1} \frac{|a_{2n}|}{|((1 + a)s_n + a_{2n})^2 + w\rho^2 s_n^2|}$$
$$= \max_{a_{2n} \in P} \frac{|a_{2n}|}{|(1 + a)s_n + a_{2n}|^2 - \rho^2|s_n|^2}.$$

Thus in order to determine the maximum of Q_n we need to take into consideration only those values of a_{2n} which lie in P.

The boundary of $E_2(i(1 + a), \rho)$ consists of all points of the form $(i(1 + a) + \rho e^{i\theta})^2$, where $0 \leqq \theta \leqq 2\pi$. If we set $1 + a = he^{i\beta}$ and introduce the function

$$r(\theta) = 2(h + \rho \sin(\theta - \beta)),$$

where

$$2(h - \rho) \leqq r \leqq 2(h + \rho),$$

then

$$\begin{aligned}(i(1 + a) + \rho e^{i\theta})^2 &= e^{2i\beta}(ih + \rho e^{i(\theta-\beta)})^2 \\ &= e^{2i\beta}(\rho^2 - h^2 + i\rho r e^{i(\theta-\beta)})\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}|i(1 + a) + \rho e^{i\theta}|^2 &= |ih + \rho e^{i(\theta-\beta)}|^2 \\ &= \rho^2 - h^2 + hr.\end{aligned} \tag{3.5}$$

We shall also need an expression for s_n. It follows from Lemma 2.3 that

$$|1 + a - s_n| \leqq |a| + (\rho - |a|)k_n.$$

Thus we can write

$$s_n = 1 + a + g_n e^{i\delta_n} = he^{i\beta} + g_n e^{i\delta_n}, \tag{3.6}$$

where

(3.7) $$0 \leqq g_n \leqq |a| + (\rho - |a|)k_n \leqq \rho.$$

Employing formulas (3.3) through (3.6) we now have that

$$Q_n \leqq \max_{\theta,r,\delta_n,g_n} \frac{(\rho^2 - h^2 + hr)(\rho^2 - g_n^2)}{|he^{i\beta}(he^{i\beta} + g_n e^{i\delta_n}) + e^{2i\beta}(\rho^2 - h^2 + i\rho re^{i(\theta-\beta)})|^2 - \rho^2|he^{i\beta} + g_n e^{i\delta_n}|^2}$$

$$= \max_{\theta,r,\delta_n,g_n} \frac{(\rho^2 - h^2 + hr)(\rho^2 - g_n^2)}{|hg_n e^{i(\delta_n-\beta)} + \rho^2 + i\rho re^{i(\theta-\beta)}|^2 - \rho^2|h + g_n e^{i(\delta_n-\beta)}|^2}.$$

We set the denominator in the last expression equal to P_n and obtain

$$\begin{aligned} P_n &= (\rho^2 + hg_n \cos(\delta_n - \beta) - \rho r \sin(\theta - \beta))^2 \\ &\quad + (hg_n \sin(\delta_n - \beta) + \rho r \cos(\theta - \beta))^2 \\ &\quad - \rho^2(h^2 + 2hg_n \cos(\delta_n - \beta) + g_n^2) \\ &= (g_n^2 - \rho^2)(h^2 - \rho^2) + \rho^2 r^2 - 2\rho^3 r \sin(\theta - \beta) + 2h\rho r g_n \sin(\delta_n - \theta) \\ &= (g_n^2 - \rho^2)(h^2 - \rho^2) + 2\rho^2 hr + 2h\rho r g_n \sin(\delta_n - \theta). \end{aligned}$$

Clearly,

$$P_n \geqq \min_{r,g_n}[(g_n^2 - \rho^2)(h^2 - \rho^2) + 2\rho hr(\rho - g_n)].$$

It follows that

$$Q_n \leqq \max_{r,g_n} \frac{(\rho^2 - h^2 + hr)(\rho^2 - g_n^2)}{(g_n^2 - \rho^2)(h^2 - \rho^2) + 2\rho hr(\rho - g_n)}$$

$$= \max_{r,g_n} \frac{1}{1 + \left(1 + \dfrac{h^2 - \rho^2}{\rho^2 - h^2 + hr}\right)\left(\dfrac{\rho - g_n}{\rho + g_n}\right)}.$$

The inequalities $|a| < \rho < h$, $2(h - \rho) \leqq r \leqq 2(h + \rho)$, and inequality (3.7) make it almost immediately clear that the last expression above increases with increasing r and increasing g_n. Therefore, after replacing r by $2(h + \rho)$ and g_n by $|a| + (\rho - |a|)k_n$ in this expression we obtain

$$Q_n \leqq 1 - \frac{2h(\rho - |a|)(1 - k_n)}{2\rho(h + \rho) + (\rho - |a|)(h - \rho)(1 - k_n)}.$$

Formula (2.11) gives us $1 - k_n = b/(n - 1 + b)$. After substituting this value for $1 - k_n$ in the inequality above we arrive at

(3.8) $$Q_n \leqq 1 - c/(n - 1 + d)$$

where

(3.9) $$c = \frac{h(\rho - |a|)b}{\rho(h + \rho)}$$

(3.10) $$d = \frac{[2h(\rho - |a|) + (\rho + |a|)(h + \rho)]b}{2\rho(h + \rho)}.$$

Since $0 < b < 1$ by inequality (2.6) and $|a| < \rho < h$, it is easily seen that

$$0 < c < d. \tag{3.11}$$

Since the approximants $T_n(1)$ of the continued fraction (1.1) lie in nested circles of radii R_n we have $|T_n(1) - T_{n+m}(1)| \leqq 2R_n$ for all $m \geqq 0$. Therefore, since $\{R_n\}$ is a monotone non-increasing sequence, convergence of the continued fraction (1.1) will be established if it can be shown that $\lim R_{2n} = 0$. With these statements in mind we proceed to obtain an estimate for R_{2n}.

Employing the fact that $R_{2n+1}/R_{2n} \leqq 1$ for all $n \geq 1$, the following is seen to be true

$$R_{2n} = R_1 \prod_{m=2}^{2n} R_m/R_{m-1} \leqq R_1 \prod_{m=1}^{n} R_{2m}/R_{2m-1} = R_1 \prod_{m=1}^{n} Q_m .$$

Using formula (3.1) it is easily verified that $R_1 \leqq \rho$, and by inequality (3.8), $Q_m \leqq 1 - c/(m - 1 + d)$. Therefore,

$$R_{2n} \leqq \rho \sum_{m=1}^{n} (1 - c/(m - 1 + d)), \tag{3.12}$$

where c and d are given by formulas (3.9) and (3.10). Since $0 < c < d$ by inequality (3.11), we can write

$$1 - \frac{c}{m - 1 + d} = \left(1 + \frac{1}{m - 1 + d}\right)^{-c} - c(1 + c)\int_0^{1/(m-1+d)} \left(\frac{1}{m - 1 + d} + 1 - t\right)^{-c-2} t\, dt.$$

It follows that

$$1 - \frac{c}{m - 1 + d} \leqq \left(1 + \frac{1}{m - 1 + d}\right)^{-c}.$$

If we apply this result to inequality (3.12) we can obtain a more usable estimate for R_{2n} than (3.12).

$$R_{2n} \leqq \rho \prod_{m=1}^{n} \left(1 + \frac{1}{m - 1 + d}\right)^{-c} = \rho \left(\prod_{m=1}^{n} \left(1 + \frac{1}{m - 1 + d}\right)\right)^{-c}$$

or

$$R_{2n} \leqq \rho(1 + n/d)^{-c}. \tag{3.13}$$

Clearly, the sequence $\{\rho(1 + n/d)^{-c}\}$ has the limit zero and furthermore it is independent of the elements a_n of the continued fraction (1.1). Thus we have shown that the continued fraction (1.1) converges uniformly if

$$a_{2n-1} \in E_1(ia, \rho) \quad \text{and} \quad a_{2n} \in E_2(i(1 + a), \rho),$$

where $|a| < \rho < |1 + a|$.

One proof showing that the twin convergence regions

$$E_1(ia, \rho), \qquad E_2(i(1 + a), \rho)$$

must also be best twin convergence regions is given in [1] and a more detailed proof of this fact can be found in [2]. The arguments for bestness are based upon a consideration of certain periodic continued fractions of periods 2 and 4. With this result and the other results of this paragraph, the following theorem is proved:

THEOREM. *Let a be a complex number and ρ be a positive real number such that $|a| < \rho < |1 + a|$. If the elements of the continued fraction (1.1) satisfy the conditions*

$$a_{2n-1} \in E_1(ia, \rho), \qquad a_{2n} \in E_2(i(1 + a), \rho)$$

for all $n \geqq 1$, then the continued fraction converges. Its value lies in the circular disk $N((1 + a), \rho)$. If the elements a_n are functions of an arbitrary number of variables, then the continued fraction converges uniformly provided only that the domains of the variables be chosen such that the a_n satisfy the above conditions for all values of the variables. The twin convergence regions

$$E_1(ia, \rho), \qquad E_2(i(1 + a), \rho)$$

are best twin convergence regions. Finally, if the limit of the sequence of approximants $\{T_n(1)\}$ of the continued fraction (1.1) is L, then $|T_1(1) - L| \leqq 2\rho$ and for $n \geqq 1$

$$|T_{2n+1}(1) - L| \leqq |T_{2n}(1) - L| \leqq 2\rho(1 + n/d)^{-c},$$

where c and d (given by formulas (3.9) and (3.10), respectively) are constants depending only on a and ρ.

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