THE INTEGRAL REPRESENTATION OF FUNCTIONS ON PARTS

BY

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1. Introduction

In his well known paper [7], R. S. Martin considers the following question: "One may ask how great generality in a domain is to be permitted if we are to have for the domain a formula possessing the more significant features of the Poisson-Stieltjes integral formula for the circle or the sphere." This question is then formalized by Martin as follows: "In a given domain is the class of minimal functions [i.e., minimal harmonic functions] sufficiently wide that, with a suitable normalization and a suitable definition of the linear process involved, it contains a basis for the positive harmonic functions of the domain." Martin showed [7, see p. 139] that the answer is affirmative and that the linear process can be realized by an integral, and further that every positive harmonic function in the domain is the limit of convex combinations of minimal functions.

In terms of the Krein-Milman theorem and the work of Choquet [4], we can restate these facts as follows. For any domain, the convex set of normalized positive harmonic functions H_0 is compact in the u.c.c. topology. If E is the closure of the set of extreme points of H_0, then $v \in H_0$ if and only if there is a positive measure μ on E such that

$$v(z) = \int_E K(z, e) \, d\mu(e), \tag{1}$$

where $K(z, e) = e(z)$ for each extreme point (function) e of H_0 and each point z of the domain.

Our purpose here is to extend these ideas from the class of harmonic functions on a domain, to arbitrary linear spaces of bounded real functions on a set. We first prove a theorem which outlines the topological structure involved in an integral representation such as (1). We then turn to a space B of bounded functions on a set Y, and show that Y decomposes into disjoint "parts", and that there is an integral representation of the form (1) on each part. We obtain in this way a general Poisson type of integral representation for linear spaces of bounded function.

2. A general Herglotz theorem

The classical Herglotz theorem which Martin generalized to arbitrary domains states that u is a positive harmonic function in the open unit disc of the complex plane if and only if there is some positive Borel measure μ

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on the unit circle Γ such that

$$u(z) = \int_\Gamma P(z, \theta)\, d\mu(\theta) \tag{2}$$

for all $|z| < 1$, where $P(z, \theta)$ is the Poisson kernel,

$$P(re^{i\varphi},\theta) = (1 - r^2)/[1 + r^2 - 2r \cos(\theta - \varphi)].$$

The functions $P(z, \theta)$ are continuous in θ ($\theta \in \Gamma$) for fixed z ($|z| < 1$), and harmonic in z for fixed θ. In this section we prove a purely topological version of this theorem for an arbitrary compact set in place of Γ, and an arbitrary separating set of continuous functions replacing the kernels $P(z, \theta)$.

Let E be an arbitrary compact Hausdorff space, and let Δ be a separating family of continuous real functions on E. We give Δ the metric (sup-norm) topology it has as a subset of C(E). We will say that E and Δ are *paired* if these conditions hold, and write $(\theta, s) = s(\theta)$ for $s \in \Delta$ and $\theta \in E$ to emphasize that we also regard E as a family of functions on Δ. The ideas of the next two lemmas are essentially known (cf. [6, p. 156]), but for simplicity we assemble the necessary facts in the specific form we need here.

LEMMA 1. *If E and Δ are paired, then evaluation maps E homeomorphically onto a subset of $C(\Delta)$ with the topology of uniform convergence on compact subsets of Δ.*

Proof. Let $\theta_\alpha \to \theta_0$ in the given compact topology of E. We claim that $(\theta_\alpha, s) \to (\theta_0, s)$ uniformly for s in every compact subset of Δ. If A is a compact subset of Δ, then A is equicontinuous [5, p. 233]. Hence there is a neighborhood V of θ_0 such that $|s(\theta) - s(\theta_0)| < \varepsilon$ for all $s \in A$ and all $\theta \in V$. In other words, if $\theta_\alpha \in V$, (θ_α, s) is uniformly close to (θ_0, s) on A.

LEMMA 2. *Let E and Δ be paired, and S be the linear span of Δ in $C(E)$, and U^* the strongly closed unit ball in S^*, with the w^* topology. Then U^*, Δ are paired (and U^*, S are paired) and the w^* topology of U^* is homeomorphic to the topology of uniform convergence on compact subsets of Δ (or S).*

Proof. The set U^* is of course w^* compact, and the functions in S (and therefore Δ) are continuous on U^* in this topology. The sup-norm metric that Δ or S has in $C(E)$ is the same as that in $C(U^*)$. Therefore, the result follows from the preceding lemma by replacing E by U^*.

We now make the additional assumption that Δ, hence S, contains the constant function s_0 which is identically one: $s_0(\theta) \equiv 1$.

Recall that evaluation maps E homeomorphically into a subset E^* of U^*. The space S is isomorphic to all w^* continuous linear functionals on S^*, and this isomorphism is an isometry if the functionals are restricted to the closed convex hull of E^* in U^* [2]. We call this closed convex hull, T_S, the *carrier* of S, and it is shown in [2], or it can be seen from the Hahn-Bänach theorem and the Riesz representation theorem, that

$$T_S = \langle E^* \rangle^- = \{F \in S^* : F(1) = \| F \| = 1\}.$$

With reference to the pairing of U^* and Δ, E^* is also a subset of $C(\Delta)$ with the u.c.c. topology, and this of course coincides with the embedding of E in $C(\Delta)$ from the pairing of E and Δ. Since the w^* topology in U^* coincides with the u.c.c. topology in $C(\Delta)$, the w^* closed convex hull T_S is *linearly* homeomorphic to the u.c.c. closed convex hull of E^* (or E) in $C(\Delta)$. Let H_Δ denote the closed convex hull of E or E^* in $C(\Delta)$. That is, H_Δ is the set of all continuous functions on Δ which are u.c.c. limits of convex combinations v of the form

$$v(s) = \sum \lambda_i(\theta_i, s) = \sum \lambda_i\, s(\theta_i)$$

where $\theta_i \in E$, $\lambda_i > 0$, and $\sum \lambda_i = 1$.

We have proved the following theorem.

THEOREM 3. *If E and Δ are paired and S is the linear span of Δ in $C(E)$, then the carrier T_S of S is linearly homeomorphic to the* u.c.c. *closed convex hull H_Δ of E in $C(\Delta)$. A continuous function $v \in C(\Delta)$ is in H_Δ if and only if there is a unique $F \in S^*$ such that $F(1) = \| F \| = 1$ and $v(s) = F(s)$ for all $s \in \Delta$.*

For $s \in \Delta$ and $\theta \in E$, let us write $K(s, \theta) = s(\theta)$ to emphasize the similarity with (2). For each $F \in S^*$, there is a positive Borel measure μ on E which represents F for S. Hence we have the following abstract form of Herglotz theorem as a corollary of Theorem 3 (cf. [1 Thm 6]).

COROLLARY. *If E and Δ are paired and H_Δ is the u.c.c. closed convex hull of E in $C(\Delta)$, then $v \in H_\Delta$ if and only if there is a positive Borel probability measure μ on E such that for all $s \in \Delta$,*

$$v(s) = \int_E K(s, \theta)\, d\mu(\theta). \tag{3}$$

3. The parts of a set

Let Y be any set, and B a separating linear space of bounded real functions on Y, containing the constants. In [2] we introduced the notion of a part of a compact space X with respect to a linear subspace of $C(X)$, and showed that this was a generalization of the idea of Gleason part for a function algebra. We now extend this idea to the space B of bounded functions on Y.

For points $x, y \in Y$ we write $x \sim y(a)$ if and only if

$$1/a < u(x)/u(y) < a \tag{4}$$

for all positive functions $u \in B$. We write $x \sim y$ if $x \sim y(a)$ for some number a.

It is easy to check that

(i) $x \sim x(a)$ for all $a > 1$
(ii) $x \sim y(a)$ implies $y \sim x(a)$
(iii) $x \sim y(a)$ and $y \sim z(b)$ implies $x \sim z(ab)$.

It follows that $\sim$ is an equivalence relation on Y, and we call the equivalenc classes the *parts* of Y induced by B. For points x, y in the same part we defi e

(5) $R(x, y) = \inf \{a : 1/a < u(x)/u(y) < a \text{ all positive } u \in B\}$.

From (i), (ii), (iii) above it follows that

(i) $R(x, x) = 1$ and $R(x, y) > 1$ if $x \neq y$
(ii) $R(x, y) = R(y, x)$
(iii) $R(x, y)R(y, z) \geq R(x, z)$.

We let

$$d(x, y) = \log R(x, y) \tag{6}$$

for x and y in the same part, and note that d is a metric on each part.

LEMMA 4. *The functions in B are uniformly d-continuous on each part of Y.*

Proof. First let u be a positive function in B, and $x \sim y$. Then

$$-[R(x, y) - 1] \leq \frac{1}{R(x, y)} - 1 \leq \frac{u(x)}{u(y)} - 1 \leq R(x, y) - 1.$$

Hence for $u > 0$, and $x \sim y$, we have

$$\left| \frac{u(x)}{u(y)} - 1 \right| \leq R(x, y) - 1,$$

and consequently

$$| u(x) - u(y) | \leq u(y)[R(x, y) - 1]. \tag{7}$$

Now let u be any function in B and let M be a constant such that $u + M > 0$. Then from (7) we obtain

$$| u(x) - u(y) | \leq [u(x) + M][R(x, y) - 1] \leq [\| u \| + M][R(x, y) - 1].$$

This last says that u is uniformly continuous with respect to d.

Now fix a part Δ of Y, and a point $z_0 \in \Delta$. We consider the normalized positive functions in B, restricted to Δ; let

$$H = \{u \mid \Delta : u \in B, u > 0, u(z_0) = 1\}.$$

A word of caution is necessary here. The set Δ is a part of Y with respect to B as a subspace of $C(Y)$. If we consider the restriction $B \mid \Delta$, then Δ may not be a single part with respect to $B \mid \Delta$. For example, let B be all affine linear functions on a plane set Y consisting of the vertices of a triangle, and a closed segment I in the interior of the triangle. The segment is then one part of Y with respect to B, but there are three parts of the segment (the two end points and the open segment) with respect to $B \mid I$.

LEMMA 5. *The topology of uniform convergence on d-compact subsets of Δ coincides on H and $\bar{H}$ with the pointwise topology. The functions in $\bar{H}$ are d-continuous, and $\bar{H}$ is compact.*

Proof. We show that H is equicontinuous with respect to d at each $x \in \Delta$. If $u \in H$, then $0 < u(x) \leq u(z_0)R(x, z_0) = R(x, z_0)$. For $y \in \Delta$, we obtain from (7) (with x and y interchanged) that

$$| u(x) - u(y) | \leq u(x)[R(x, y) - 1] \leq R(x, z_0)[R(x, y) - 1].$$

Therefore H is d-equicontinuous on Δ, and the same is true for the pointwise closure. Moreover, the u.c.c. (d) topology coincides with the pointwise topology [5, p. 232]. It follows from the Tychonoff theorem that $\bar{H}$ is compact, since H is contained in the compact product $\mathcal{P}\{[0, R(x, z_0)] : x \in \Delta\}$.

Since $\bar{H}$ is convex and compact in a locally convex topology, $\bar{H}$ has extreme points and is the closed convex hull of these extreme points. Let E be the (compact) closure of the extreme points of $\bar{H}$. The set E is of course d-equicontinuous on Δ, and therefore $K(x, e) = e(x)$ is jointly continuous on $\Delta \times E$, where Δ has the d-topology [5, p. 232].

THEOREM 6. *The d-topology on Δ is homeomorphic to the sup-norm topology Δ has as a subset of $C(E)$.*

Proof. The statement is that $d(x_n, x) \to 0$ if and only if $e(x_n) \to e(x)$ uniformly for $e \in E$.

First assume that $| e(x_n) - e(x) | < \varepsilon$ for all $e \in E$ if $n \geq N$. Let $u \in \langle E \rangle$: say $u = \sum \lambda_i e_i$ with $\lambda_i > 0$, $\sum \lambda_i = 1$, and $e_i \in E$. Then if $n \geq N$,

$$| u(x_n) - u(x) | = | \sum \lambda_i e_i(x_n) - \sum \lambda_i e_i(x) |$$
$$\leq \sum \lambda_i | e_i(x_n) - e_i(x) | < \sum \lambda_i \varepsilon = \varepsilon.$$

If $v \in \bar{H} = \langle E \rangle^{-}$, then for each n there is $u_n \in \langle E \rangle$ such that

$$| u_n(x_n) - v(x_n) | < \varepsilon \quad \text{and} \quad | u_n(x) - v(x) | < \varepsilon.$$

It follows that $| v(x_n) - v(x) | < 3\varepsilon$ if $n \geq N$. Thus uniform convergence of x_n on E implies uniform convergence on $\bar{H}$. Since for the fixed point x, $v(x) > 0$ for all v in the compact space $\bar{H}$, it follows that $v(x) \geq c > 0$ for all $v \in \bar{H}$. Hence $v(x_n) \geq c/2$ for all sufficiently large n and all $v \in \bar{H}$, and $v(x_n)/v(x) \to 1$ uniformly for $v \in \bar{H}$. For any positive $w \in B$, $w/w(z_0) \in H$ and

$$\frac{w(x_n)}{w(x)} = \frac{w(x_n)/w(z_0)}{w(x)/w(z_0)} \to 1,$$

and the convergence is independent of w. Hence $R(x_n, x) \to 1$, and $d(x_n, x) \to 0$ if $x_n \to x$ uniformly on E.

Now suppose that $d(x_n, x) \to 0$. For $u \in H$ we have from (7)

$$| u(x_n) - u(x) | \leq u(x)[R(x_n, x) - 1] \leq R(x, z_0)[R(x_n, x) - 1],$$

and $x_n \to x$ uniformly on H, and clearly also on $\bar{H}$, and therefore on $E \subset \bar{H}$.

Now we are in a position to return to the setting of Theorem 3. We have a compact Hausdorff space E, and a separating family $\Delta \subset C(E)$. The function identically one is in Δ, since $e(z_0) = 1$ for all $e \in E$. The set Δ comes with a metric d defined in terms of B. This metric d is topologically equivalent to the sup-norm metric Δ has in $C(E)$, and it is this latter topology which is assumed for Δ in Section 2. The u.c.c. (d) closed convex hull of E in $C(\Delta)$ is $\bar{H}$ by the Krein-Milman Theorem. From Theorem 3, we also know that $\bar{H}$ is linearly homeomorphic to T_S, where S is the linear span of Δ in $C(E)$, and T_S is the set of positive linear functionals of norm one on S. The correspondence is given as follows: $v \in \bar{H}$ if and only if there is $F \in T_S$ such that $v(s) = F(s)$ for all $s \in \Delta \subset S \subset C(E)$. As usual we can extend each $F \in T_S$ to $C(E)$ without changing the norm, and represent it with a measure μ on E. In this way we obtain for each part Δ of Y an integral representation of the type sought by Martin.

THEOREM 7. *If B is a separating linear space of bounded real functions on a set Y, and $1 \in B$, and Δ is a part of Y determined by B, and*

$$H = \{u \mid \Delta : u \in B, u > 0, u(z_0) = 1\}$$

for some $z_0 \in \Delta$, and $\bar{H}$ is the pointwise closure (or u.c.c. *closure) of H, then $v \in \bar{H}$ if and only if there is some positive Borel probability measure μ on the closure E of the set of extreme points of $\bar{H}$ such that*

$$v(s) = \int_E (s, e) \, d\mu(e)$$

for all $s \in \Delta$.

In particular, the development above includes the case considered by Martin, since the defining condition (4) for a part is simply Harnack's inequality for positive harmonic functions in a domain.

4. The parts of a compact space

Suppose now that X is a compact space, and B is a separating linear space of continuous real functions on X, with $1 \in B$. The parts of X with respect to B have been characterized in [2] as the minimal faces of the closed convex hull in B^* of the natural embedding of X. Let us consider the case where $X = \Gamma \cup \Delta$, with Γ the Silov boundary of B in X, and Δ is a single part of X. Each $s \in \Delta$ is represented by a measure μ_s on Γ: for all $u \in B$,

$$u(s) = \int_\Gamma u(\theta) \, d\mu_s(\theta).$$

It follows from recent results of Bishop [3] that these representing measures can be taken to be mutually absolutely continuous. (Bishop shows this for the parts of a function algebra, but the argument works equally well here.) If μ_0 represents the fixed point $z_0 \in \Delta$, then $d\mu_s = g_s\, d\mu_0$ for some bounded Borel measurable function g_s. If each function g_s is *continuous on* Γ, and if $B \mid \Gamma$ is uniformly dense in $C(\Gamma)$, then it follows from the type of argument in [1] that the extreme points of $\bar{H}$ are homeomorphic to Γ, and the functions $g_s(\theta)$ act as the kernel, and that the d-topology coincides with the given topology.

It would be interesting to know in general when the extreme points of $\bar{H}$ are homeomorphic to Γ, and when the d-topology is the given topology on each part.

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