ON THE KERNEL OF CONSTANT-SUM SIMPLE GAMES WITH HOMOGENEOUS WEIGHTS[1]

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Introduction

The kernel of a characteristic function game was defined by M. Davis and M. Maschler in [1]. It is proved in [1] that the kernel is a subset of the bargaining set $M_1^{(i)}$; so each outcome in the kernel is "stable", in the sense defined in [2]. Indeed, unlike the classical solution of von Neumann and Morgenstern, in which a set of outcomes is considered as one solution, each particular outcome of the bargaining set possesses its own stability. Thus, the accumulated theoretical and experimental evidence that justifies the outcomes of the bargaining set theory, justifies a fortiori each single outcome of the kernel. On the other hand there is an example (see [1, Section 6]) which indicates that the outcomes of the kernel should not be considered as preferred to other outcomes of $M_1^{(i)}$, since restricting the outcomes to the kernel may lead to the omission of "reasonable" outcomes.

It would be very interesting to study the question of the exact location of the kernel in the bargaining set $M_1^{(i)}$. Apparently, the kernel represents either a specific extreme type of negotiation, or, perhaps, it can be interpreted as a final stage of the negotiations, when the players are determined in forming specific coalition structures. We refer the reader to [1, Section 6] and to [5, Section 14] for heuristic information on this subject.

In addition to being a subset of $M_1^{(i)}$, the kernel has many interesting mathematical properties (see [1] and [5]); for example the kernel is highly sensitive to many possible symmetries that a game may possess [5]. This makes the kernel a good "indicator" of certain symmetries that may exist in a game. This paper and [6] show also that the kernel is sensitive to additive structures of the game.

At present, however, the kernel should mainly be regarded as a tool for investigating the bargaining sets. As such it proves quite useful (see [5]). The present paper is a contribution in this spirit. We investigate here the geometrical structure of the kernels of certain classes of simple games. We conjecture that similar results are true for a wider class of games, and, moreover, that they hold also for the bargaining set itself.

We now describe the contents of the paper. Section 1 supplies the necessary definitions and Section 2 the proofs of the lemmas. The proofs of the main results—(a) the main simple vector is an extreme point of the convex hull of

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the kernel (Theorem 3.5) and (b) the kernel of an n-person constant sum simple game with exactly n minimal winning coalitions is star-shaped, (Theorem 4.4)—are given in sections 3 and 4 respectively. We conjecture that Theorem 4.4 can be generalized to cover all simple games.

Strong use is made of Isbell's results on weighted majority constant-sum homogeneous games in [3] and [4].

We are indebted to Professor R. J. Aumann for a stimulating discussion concerning Theorem 3.5.

1. Definitions

A *simple game* is a pair (N, W), where $N = \{1, 2, \cdots, n\}$ is a set with n members, and W is a set of subsets of N. The members of N are called *players*; subsets of N are termed *coalitions*. The elements of W are termed *winning coalitions*. The set of the *minimal winning coalitions* is denoted by W^m.

A simple game (N, W) is *constant-sum* if a coalition $S \in W \Leftrightarrow N - S \notin W$. A constant-sum simple game has homogeneous weights if there exist n positive weights $w_1, \cdots, w_n$ and a number q, such that

$$S \in W \Leftrightarrow \sum_S w_i \geqq q \quad \text{and} \quad S \in W^m \Leftrightarrow \sum_S w_i = q.$$

q is called a *quota*.

Constant-sum games with homogeneous weights were defined by Von Neumann and Morgenstern in [7].

Let (N, W) be a constant-sum game with homogeneous weights. We shall always assume that there are no dummy players, so that each player is in at lease one minimal winning set. It is known (see [3]) that there exist minimum integer homogeneous weights for (N, W), which are unique. We shall use the notation $[w_1, \cdots, w_n]$ for (N, W), where $w_1, \cdots, w_n$ are the minimum integer weights. We shall also assume that the weights are given in non-decreasing order, i.e. that $w_1 \leqq \cdots \leqq w_n$.

A quota that corresponds to $w_1, \cdots, w_n$ is $\frac{1}{2}(1 + \sum_{i=1}^n w_i)$.

Let $G = (N, W)$ be a constant-sum game with homogeneous weights. The *characteristic function* of G is the function v defined on the subsets of N by $v(S) = 1$ if $S \in W$, and $v(S) = 0$ if $S \notin W$.

An *imputation* is an n-tuple of real numbers that satisfies $x_i \geqq 0$, $i = 1, \cdots, n$, and $\sum_{i=1}^n x_i = 1$.

Let x be an imputation and i and j different players. For $S \subset N$ we denote $e(S, x) = e(S) = v(S) - \sum_S x_k$. Also we denote

$$W_{ij} = \{Q : Q \in W, i \in Q \text{ and } j \notin Q\} \quad \text{and}$$

$$s_{ij}(x) = s_{ij} = \max \{e(T) : T \in W_{ij}\}.$$

i outweighs j, written $i \gg j$, if $x_j > 0$ and $s_{ij} > s_{ji}$. The imputation x is *balanced* if there exists no pair of players k and l such that $k \gg l$. The *kernel* of the game G is the set of all balanced imputations.

2. Some properties of the s_{ij} functions

Let $[w_1, \cdots, w_n]$ be an n-person constant-sum game with homogeneous weights. Two players are symmetric if and only if they have equal weights. We denote by $T_1, \cdots, T_m$ the different classes of symmetric players, arranged in increasing order of weights.

LEMMA 2.1. *Let x be an imputation in the kernel; if $i \in T_h$, $j \in T_k$ and $k \geqq h$, then $x_j \geqq x_i$.*

Proof. Suppose $x_i > x_j$. Let $s_{ij} = e(S)$. If $S_1 = (S - \{i\}) \cup \{j\}$ then $S_1 \in W_{ji}$ and $s_{ji} \geqq e(S_1) > e(S)$; so j outweighs i, which is impossible. Hence we conclude that $x_j \geqq x_i$.

A proof of a more general result is given in [5].

Let i and j be two different players; we denote

$$W_{ij}^m = \{Q : Q \in W^m, i \in Q \quad \text{and} \quad j \notin Q\}.$$

LEMMA 2.2. *If i and j are two different players then W_{ij}^m is not empty.*

A proof of this result can be found in [4, p. 438]. See also [5].

LEMMA 2.3. *Let x be an imputation in the kernel; if $i \in T_h$, $j \in T_{h-1}$ and either $x_j = 0$ or s_{ji} is attained by a coalition $S \in W_{ji}^m$, then there exist players $p_1, \cdots, p_t$ such that $w_{p_u} < w_i$, $u = 1, \cdots, t$, $w_i = \sum_{u=1}^t w_{p_u}$, and $x_i \geqq \sum_{u=1}^t x_{p_u}$.*

Proof. Let $Q = S$ if $x_j > 0$, and an arbitrary set in W_{ji}^m otherwise. Let $R = (Q - \{j\}) \cup \{i\}$. $R \in W_{ij}$. Since $Q - \{j\} \notin W$, R contains a set $R_1 \in W_{ij}^m$. Let $p_1, \cdots, p_t$ be the players of $Q - R_1$. $w_i = \sum_{u=1}^t w_{p_u}$ and $w_{p_u} < w_i$, $u = 1, \cdots, t$. If $x_j = 0$ then by Lemma 2.1, $x_{p_u} = 0$, $u = 1, \cdots, t$, and $x_i \geqq 0 = \sum_{u=1}^t x_{p_u}$. If $x_j > 0$ then we have $e(Q) = s_{ji} = s_{ij} \geqq e(R_1)$, and the inequality $x_i \geqq \sum_{u=1}^t x_{p_u}$ follows.

Let $S \in W^m$. We denote $d(S) = \min \{h : T_h \cap S \neq \emptyset\}$. A coalition $P \in W^m$ *is obtained from S by substitution* if $P = (S - \{i\}) \cup Q$, where $i \in T_{d(S)} \cap S$ and $Q \subset \bigcup \{T_g : g < d(S)\}$. We remark that if S, P, $R \in W^m$, P is obtained from S by substitution, and R is obtained from P by substitution, then R is obtained from S by substitution. These definitions and remarks will be useful in what follows.

LEMMA 2.4. *Let x be an imputation; if i and j are different players, then either there is an $S \in W_{ij}^m$ such that $s_{ij} = e(S)$, or there is an $R \in W_{ij}$, $R = \{i\} \cup R_0$, $R_0 \in W^m$ and $w_t > w_i$ for all $t \in R_0$, such that $s_{ij} = e(R)$.*

The proof, which is straightforward, is omitted.

LEMMA 2.5. *Let x be an imputation in the kernel, and let $T_{a+1}, \cdots, T_m$, $a \geqq 0$, be the classes of the players that are assigned positive payments in x. If $i \in T_h$ and $j \in T_k$, $a + 1 \leqq h < k$, then both s_{ij} and s_{ji} are attained by minimal winning coalitions.*

Proof. If $m - 1 \geqq a + 1$, $i \in T_{m-1}$ and $j \in T_m$ then, by Lemma 2.4, either s_{ij} is attained by a coalition in W_{ij}^m, or there is an $R \in W_{ij}$, $R = \{i\} \cup R_0$, $R_0 \in W^m$, $R_0 \subset T_m$, such that $s_{ij} = e(R)$. If the second possibility holds then, since the players of T_m are symmetric, we have that

$$s_{ji} \geqq e(R_0) > e(R) = s_{ij},$$

which is impossible; so s_{ij} is attained by a coalition in W_{ij}^m. The proof that s_{ji} is attained by a minimal winning coalition follows directly from Lemma 2.4.

We continue the proof by induction on h. We assume that if $h_1 > h \geqq a + 1$, $k > h_1$, $i \in T_{h_1}$ and $j \in T_k$, then both s_{ij} and s_{ji} are attained in W^m; we shall show that this is true also for $i \in T_h$ and $j \in T_k$, where $k > h$. This will be done by induction on k. So we assume now that if $h + 1 \leqq k_1 < k$, $i \in T_h$ and $j \in T_{k_1}$, then s_{ij} and s_{ji} are attained by minimal winning coalitions. We shall prove now that if $i \in T_h$ and $j \in T_k$, then s_{ij} is attained in W_{ij}^m. Suppose there is no $R \in W_{ij}^m$ such that $s_{ij} = e(R)$; by Lemma 2.4 there is an S such that $s_{ij} = e(S)$, $S = \{i\} \cup S_0$ and $S_0 \in W^m$. From the existence of such an S we shall derive a contradiction. Denote $e(S) = \varepsilon$ and $e(S_0) = \delta$. $\delta = \varepsilon + x_i$. Our first step will be to show

(2.5.1) If $Q \in W^m$, $e(Q) \geqq \delta$ and $Q \cap (T_1 \cup T_2 \cup \cdots \cup T_k) \neq \emptyset$, then $Q \supset T_{a+1} \cup T_{a+2} \cup \cdots \cup T_k$.

To prove (2.5.1) we shall show firstly that $Q \cap (T_h \cup T_k) \neq \emptyset$. Assume, per absurdum, that $Q \cap (T_h \cup T_k) = \emptyset$. Let $r \in Q \cap T_{d(Q)}$; if $d(Q) \leqq h - 1$ then $(Q - \{r\}) \cup \{i\}$ contains a set $P \in W_{ij}^m$ such that $e(P) \geqq \varepsilon$, contradicting our assumption that s_{ij} is not attained by a minimal winning coalition. If $d(Q) \geqq h + 1$ let $p \in T_{d(Q)-1}$; by our induction hypotheses s_{pr} is attained by a coalition in W_{pr}^m. It follows from Lemma 2.3 that there exist players $p_1, \cdots, p_t$ such that $w_r = \sum_{u=1}^{t} w_{p_u}$, $x_r \geqq \sum_{u=1}^{t} x_{p_u}$ and $w_{p_u} < w_r$, $u = 1, \cdots, t$. Let $Q_1 = (Q - \{r\}) \cup \{p_1, \cdots, p_t\}$. $e(Q_1) \geqq e(Q)$ and $Q_1 \cap T_k = \emptyset$, so $Q_1 \cap T_h \neq \emptyset$ is impossible. Also, by what we have already shown $d(Q_1) \geqq h + 1$. $d(Q) > d(Q_1)$. We can continue and construct a sequence of sets Q_g, $g = 1, 2, \cdots$, having the same properties as Q, such that $d(Q_{g+1}) < d(Q_g)$, which is absurd. So we conclude that $Q \cap (T_h \cup T_k) \neq \emptyset$. We proceed now to show that $Q \supset T_h \cup T_k$. If $Q \cap T_h \neq \emptyset$ and $T_k - Q \neq \emptyset$, then we can, by interchanging symmetric players, obtain a set Q_1 that satisfies $e(Q_1) = e(Q)$, $i \in Q_1$ and $j \notin Q_1$. Since $e(Q_1) \geqq \delta > \varepsilon = s_{ij}$, this is impossible. Hence if $Q \cap T_h \neq \emptyset$ then $Q \supset T_k$. In a similar way we can show that if $T_k \cap Q \neq \emptyset$ then $Q \supset T_h$. Since $Q \cap (T_k \cup T_j) \neq \emptyset$, it follows that $Q \supset T_h \cup T_k$. We are able now to complete the proof of (2.5.1). Suppose there is a $p \in (T_{a+1} \cup \cdots \cup T_k) - Q$. Since $j \in Q$ and $x_p > 0$ we have that $s_{pj} = s_{jp} \geqq e(Q)$. Let $s_{pj} = e(U)$. Since $s_{ij} = \varepsilon < e(U)$, $i \notin U$. If $w_p > w_i$ then, by our first induction hypothesis, we can choose $U \in W^m$. But, by what we have already shown, U must contain T_k, which is absurd. So $w_p < w_i$. Let $U_1 = (U - \{p\}) \cup \{i\}$. $U_1 \in W_{ij}$ and

$$e(U_1) = e(U) + x_p - x_i \geqq \delta + x_p - x_i > \delta - x_i = \varepsilon = s_{ij},$$

again a contradiction. Hence $Q \supset T_{a+1} \cup \cdots \cup T_k$ and the proof of (2.5.1) is complete. Our second step will be to show

(2.5.2) There is no coalition $Q \in W^m$ that satisfies

$$e(Q) \geqq \delta \quad \text{and} \quad Q \supset T_{a+1} \cup \cdots \cup T_k .$$

Let

$$D = \{Q : Q \in W^m, e(Q) \geqq \delta \quad \text{and} \quad Q \supset T_{a+1} \cup \cdots \cup T_k\}.$$

We shall show that if $Q \in D$ then $Q \supset T_{a+1} \cup \cdots \cup T_m$. This will prove that D is empty, since if $Q \supset T_{a+1} \cup \cdots \cup T_m$ then $e(Q) = 0$, while $\delta > 0$. Assume that each $P \in D$ contains $T_{a+1} \cup \cdots \cup T_{k+g-1}$, and let $Q \in D$. If $T_{k+g} - Q \neq \emptyset$, let $c \in T_{k+g} - Q$. $s_{jc} \geqq e(Q)$, so there is a coalition $U \in W^m_{cj}$ such that $s_{cj} = e(U) \geqq e(Q)$. Since $j \notin U$, by (2.5.1), $U \cap (T_1 \cup \cdots \cup T_k) = \emptyset$. Let $r \in U \cap T_{d(U)}$ and $p \in T_{d(U)-1}$. Since s_{pr} is attained by a minimal winning coalition there exist, according to Lemma 2.3, players $p_1, \cdots, p_t$, $w_{p_u} < w_r$, $u = 1, \cdots, t$ such that

$$U_1 = (U - \{r\}) \cup \{p_1, \cdots, p_t\} \in W^m \quad \text{and} \quad e(U_1) \geqq e(U).$$

We can continue, using this method of substitution, and obtain eventually a coalition $U_0 \in W^m$ such that

$$e(U_0) \geqq \delta \quad \text{and} \quad U_0 \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset.$$

By (2.5.1), $U_0 \supset T_{a+1} \cup \cdots \cup T_k$. Now, if $d(U) < k + g$ then

$$U_0 \not\supset T_{a+1} \cup \cdots \cup T_{k+g-1},$$

contradicting our assumption that each $P \in D$ contains this set. If $d(U) = k + g$ then we have $U_0 \supset T_{a+1} \cup \cdots \cup T_{k+g-1}$; using Isbell's inequality[2] [3, p. 185] $w_r \leqq w(T_1) + \cdots + w(T_{k+g-1} - \{p\}) + 1$, we have

$$w((T_1 \cup \cdots \cup T_a) - U_0) \geqq w_p - 1 \geqq w_i .$$

Hence $(U_0 - \{i\}) \cup T_1 \cup \cdots \cup T_a$ contains a coalition $R \in W^m$. Since $i \notin R$ and $e(R) \geqq \delta$, this contradicts (2.5.1). So we can proceed to show by induction that each $Q \in D$ contains $T_{a+1} \cup \cdots \cup T_m$, and complete the proof of (2.5.2).

We conclude from (2.5.1) and (2.5.2) that there is no $Q \in W^m$ that satisfies $e(Q) \geqq \delta$ and $Q \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset$. But, starting with S_0 and using our

[2] If F is a coalition then $w(F) = \sum_{b \in F} w_b$.

first induction hypothesis and Lemma 2.3, we can apply the method of substitution and obtain an $S_1 \in W^m$, such that

$$e(S_1) \geqq \delta \quad \text{and} \quad S_1 \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset.$$

So we reached the desired contradiction, and thus proved that s_{ij} is attained by a minimal winning coalition. The proof that s_{ji} is attained in W^m_{ji} is similar. The proof of Lemma 2.5 is now complete.

LEMMA 2.6. *Let x be a balanced imputation and let*

$$E = \{S : S \in W^m, e(S) \geqq e(Q) \text{ for all } Q \subset N\};$$

then $\bigcap \{S : S \in E\} = \emptyset$.

Proof. Suppose $M = \bigcap \{S : S \in E\} \neq \emptyset$. If $i \in M$ and $j \notin M$ then $s_{ij} > s_{ji}$. Since x is balanced, $x_t = 0$ for $t \notin M$. It follows that $\max \{e(Q) : Q \in W^m\} = 0$, which is impossible since $[w_1, \cdots, w_n]$ has an empty core.

A detailed proof of a more general result is given in [5].

LEMMA 2.7. *Let x be a balanced imputation, and let $T_{a+1}, \cdots, T_m$ be the classes of the players that are assigned positive payments in x. If $i \in T_h$, $h \leqq a$, and $j \in T_k$, $k \geqq a + 1$, then $s_{ij} = s_{ji}$, and s_{ji} is attained by a minimal winning coalition.*

Proof. Let E be as in Lemma 2.6. There is an $S \in E$ such that $j \notin S$. We have that $s_{ij} \geqq e(S \cup \{i\}) = e(S) \geqq s_{ji}$; since $x_j > 0$, $s_{ji} \geqq s_{ij}$, and therefore $s_{ij} = s_{ji}$. Also, if there is no $R \in W^m_{ji}$, such that $s_{ji} = e(R)$, then, by Lemma 2.4 there is a $P = \{j\} \cup P_0$, $P_0 \in W^m$, such that $s_{ji} = e(P)$. Since $e(P_0) \leqq e(S)$ and $x_j > 0$, $e(P) < e(S) = s_{ij}$, contradicting the equality $s_{ij} = s_{ji}$; so s_{ji} is attained in W^m.

LEMMA 2.8. *Let x be a balanced imputation. If i and j are different players* then $s_{ij} = s_{ji}$.

Proof. If $x_i > 0$ and $x_j > 0$ then $s_{ji} = s_{ji}$ since x is balanced. If $x_i = 0$, let E be as in Lemma 2.6. There is an $S \in E$ such that $j \notin S$.

$$s_{ij} \geqq e(S \cup \{i\}) = e(S) \geqq s_{ji}.$$

If $x_j > 0$ then by Lemma 2.7, $s_{ij} = s_{ji}$; if $x_j = 0$, the same argument shows that $s_{ji} \geqq s_{ij}$, and therefore $s_{ij} = s_{ji}$.

LEMMA 2.9. *Let x be a balanced imputation and let $T_1, \cdots, T_a$ be the classes of the players that are assigned a zero payment in x. If $i \in T_h$, $j \in T_k$, $h < k \leqq a$, then s_{ji} is attained by a minimal winning coalition.*

Proof. We assume that if $h < h_1 < k \leqq a$, $i \in T_{h_1}$ and $j \in T_k$ then s_{ji} is attained in W^m, and we shall show that the same is true if $i \in T_h$ and $j \in T_k$, $k > h$. This will be done by induction on k. So we assume now that if

$h < k < k_1 \leqq a$, $i \in T_h$ and $j \in T_{k_1}$, then s_{ji} is attained in W^m, and shall prove that if $i \in T_h$ and $j \in T_k$ then s_{ji} is attained by a minimal winning coalition. If s_{ji} is not attained in W^m, then there is a coalition S, $S = \{j\} \cup S_0$, $S_0 \in W^m$, such that $s_{ji} = e(S)$. We shall show that the existence of a coalition S with the above properties leads to a contradiction. Let $e(S) = \delta$. Our first step is to prove

(2.9.1) If $Q \in W^m$, $e(Q) \geqq \delta$ and $Q \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset$ then $Q \supset T_h$.

Suppose that $T_h - Q \neq \emptyset$. By interchanging symmetric players, if necessary, we can obtain a $Q_1 \in W^m$ such that $i \notin Q_1$ and $e(Q_1) = e(Q)$. $j \notin Q_1$ since s_{ji} is not attained in W^m. Let $u \in Q_1 \cap T_b$ where $b \leqq k$. $(Q_1 - \{u\}) \cup \{j\}$ contains a coalition $Q_2 \in W^m_{ji}$ that satisfies $e(Q_2) = e(Q_1)$, contradicting our assumption that s_{ji} is not attained by a minimal winning coalition. Hence we conclude that $Q \supset T_h$.

The next result that we need is

(2.9.2) There is no $Q \in W^m$ such that $Q \supset T_h$ and $e(Q) \geqq \delta$.

Let $D = \{Q : Q \in W^m, e(Q) \geqq \delta \text{ and } Q \supset T_h\}$. We shall show that if $Q \in D$ then

$$Q \supset T_{k+1} \cup \cdots \cup T_m .$$

This will prove that D is empty, since if $Q \supset T_{k+1} \cup \cdots \cup T_m$ then $e(Q) = 0$, while we know (see the proof of Lemmas 2.6 and 2.8) that

$$\delta = \max \{e(P) : P \in W^m\} > 0.$$

Let

$$b = \min \{g : g \geqq k + 1, \exists\, Q \in D \text{ such that } T_g - Q \neq \emptyset\}$$

and let $P \in D$ such that $T_b - P \neq \emptyset$. If $c \in T_b - P$ then, by Lemma 2.7 or our second induction hypothesis, there is a coalition U in W^m_{ci} such that $s_{ci} = e(U) \geqq e(P)$. Since $i \notin U$, $U \cap (T_1 \cup \cdots \cup T_k) = \emptyset$. Let $r \in U \cap T_{d(U)}$ and $p \in T_{d(U)-1}$; then either $x_p = 0$, or, by Lemma 2.5, s_{pr} is attained by a minimal winning coalition. So, according to Lemma 2.3, there exist players $p_1, \cdots, p_t$, $w_{p_u} < w_r$, $u = 1, \cdots, t$, such that

$$U_1 = (U - \{r\}) \cup \{p_1, \cdots, p_t\} \in W^m \quad \text{and} \quad e(U_1) \geqq e(U).$$

We can continue, using this method of substitution, and obtain eventually a coalition $U_0 \in W^m$ such that

$$e(U_0) \geqq \delta \quad \text{and} \quad U_0 \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset.$$

By (2.9.1), $U_0 \supset T_h$. Now, if $d(U) < b$ then $U_0 \not\supset T_{k+1} \cup \cdots \cup T_{b-1}$, contradicting the definition of b. If $d(U) = b$ then $U_0 \supset T_{k+1} \cup \cdots \cup T_{b-1}$; using Isbell's inequality we have

$$w((T_1 \cup \cdots \cup T_k) - U_0) \geqq w_p - 1 \geqq w_i .$$

Hence $(U_0 \cup T_1 \cup \cdots \cup T_k) - \{i\}$ contains a coalition $R \in W^m$. Since $i \notin R$ and $e(R) \geqq \delta$, this contradicts (2.9.1). This completes the proof of (2.9.2).

It follows from (2.9.1) and (2.9.2) that there is no $Q \in W^m$ such that $e(Q) \geqq \delta$ and $Q \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset$. But, starting with S_0 and using Lemmas 2.5 and 2.3, we can obtain by substitution a coalition $S_1 \in W^m$, such that $e(S_1) \geqq \delta$ and $S_1 \cap (T_1 \cup \cdots \cup T_k) \neq \emptyset$. So we reached the desired contradiction and the proof is complete.

LEMMA 2.10. *Let x be a balanced imputation and let $T_1, \cdots, T_a$ be the classes of the players that are assigned a zero payment in x. If for each $i \in T_k$, $k > 1$, s_{1i} is attained in W^m, then for all $1 < h \leqq \min(a, k-1)$ and for each $j \in T_h$, s_{ji} is attained by a minimal winning coalition.*

Proof. Let $s_{1i} = e(S)$, $S \in W^m$. We know (see the proof of Lemmas 2.7 and 2.8) that $e(S) = \max\{e(Q) : Q \in W\}$. If $j \in T_h$, $1 < h \leqq \min(a, k-1)$, and $j \notin S$, then $S_1 = (S - \{1\}) \cup \{j\}$ contains a coalition $S_2 \in W^m_{ji}$ such that $e(S_2) = e(S)$. $s_{ji} = e(S_2)$.

3. Extremeness of the main simple vector

Let $[w_1, \cdots, w_n]$ be an n-person constant-sum game with homogeneous weights. The *normalized main simple vector* is the vector

$$x^0 = (w_1, \cdots, w_n)/\textstyle\sum_{i=1}^n w_i .$$

In this section we shall prove that x^0 is an extreme point of the convex hull of the kernel.

LEMMA 3.1. *The normalized main simple vector x^0 is balanced.*

Proof. Since $w_1, \cdots, w_n$ are homogeneous weights,

$$e(S) = \max\{e(Q) : Q \in W\}$$

for each $S \in W^m$. By Lemma 2.2, for each pair of distinct players i and j there is a minimal winning coalition that contains i and not j; so $s_{ij} = s_{ji}$, and x^0 is balanced.

A proof of Lemma 3.1 is also given in [5].

LEMMA 3.2. *If x is a balanced imputation then $x_i/w_i \geqq x_1/w_1$ for $i = 1, \cdots, n$.*

Proof. Let $T_1, \cdots, T_m$ be the classes of symmetric players. We shall prove by induction on k that if $i \in T_k$ then $x_i/w_i \geqq x_1/w_1$. If $x_1 = 0$ the inequalities follow since the x_i are non-negative. If $x_1 > 0$, assume that the inequalities hold for $i \in T_h$, $h \leqq k-1$, and let $r \in T_{k-1}$ and $j \in T_k$. By Lemma 2.5, s_{rj} is attained by a minimal winning coalition, and therefore, by Lemma 2.3, there exist players $p_1, \cdots, p_t$ such that $w_{p_u} < w_j$, $u = 1, \cdots, t$, $w_j = \sum_{u=1}^t w_{p_u}$ and $x_j \geqq \sum_{u=1}^t x_{p_u}$. By our assumption $x_{p_u} \geqq (w_{p_u}/w_1)x_1$, and so we have

$$x_j \geqq \sum_{u=1}^{t} \frac{w_{p_u}}{w_1} x_1 = \frac{w_j}{w_1} x_1 .$$

COROLLARY 3.3. *The maximum amount that player* 1 *gets in the kernel is* $w_1/\sum_{i=1}^{n} w_i$, *and it is achieved only in the normalized main simple vector.*

Remark 3.4. The more general inequalities

$$x_1/w_1 \leqq x_2/w_2 \leqq \cdots \leqq x_n/w_n$$

are not always satisfied by balanced imputations; e.g. $(0, 0, 0, 0, \frac{1}{2}, \frac{1}{2})$ is in the kernel of [1, 1, 1, 1, 2, 3].

THEOREM 3.5. *The normalized main simple vector is an extreme point of the convex hull of the kernel.*

Proof. Lemma 3.1. and Corollary 3.3.

4. The kernel of partition games

Let $G = [w_1, \cdots, w_n]$ be a constant-sum game with homogeneous weights; G is a *partition game* if it has exactly n minimal winning sets. Partition games are described and discussed by Isbell in [3] and [4]. The name is due to Isbell [4, p. 433]. Let $T_1, \cdots, T_m$ be the classes of symmetric players in G. If G is a partition game then (see [3], [4]), T_m consists of exactly one player, T_{m-1} has at least two players, and T_1 contains also two or more players. The minimal winning sets of G, which we shall denote by $S_1, \cdots, S_m$, are given by

$$S_m = \bigcup_{i=1}^{[(m+1)/2]} T_{2i-1} ,$$

and for $j = 1, \cdots, m-1$,

$$S_j = \{j\} \cup \bigcup_{i=1}^{[(m-k+1)/2]} T_{k+2i-1} ,$$

where T_k is the class that contains j.

Let G be a partition game. We divide W^m into two sets:

$$A_1 = \{S_j : j < m, j \in T_i, i \equiv 1 \ (2)\} \quad \text{and} \quad A_2 = W^m - A_1 .$$

It follows from the above description of the members of W^m that

(i) $\bigcap \{S : S \in A_1\} \neq \emptyset$ and $\bigcap \{S : S \in A_2\} \neq \emptyset$.
(ii) If $S \in A_j$ and P is obtained from S by substitution, then $P \in A_j$.
(iii) Player 1 is in exactly two minimal winning sets: $S_1 \in A_1$ and $S_m \in A_2$.

LEMMA 4.1. *If x is a balanced imputation then*

$$e(S_m) = e(S_1) = \max \{e(S) : S \in W\}.$$

Proof. Let $E = \{S : S \in W^m, e(S) \geqq e(Q) \text{ for all } Q \in W\}$. We know (see

Lemma 2.6) that $\bigcap \{S : S \in E\} = \emptyset$. Hence E intersects both A_1 and A_2. From Lemmas 2.5 and 2.3, (ii) and (iii), it follows that both S_1 and S_m are in E.

COROLLARY 4.2. *If x is balanced and i and j are non-symmetric players, then s_{ij} is attained by a minimal winning coalition.*

Proof. It follows from Lemma 4.1, (iii) and Lemma 2.2 that s_{1k} is attained by a minimal winning coalition for all $k \neq 1$. Lemmas 2.5, 2.7, 2.9 and 2.10 complete the proof.

LEMMA 4.3. *The kernel of a partition game is star-shaped.*

Proof. Let $G = [w_1, \cdots, w_n]$ be a partition game, and let

$$x^0 = (w_1, \cdots, w_n)/\sum_{i=1}^n w_i$$

be the normalized main simple vector of G. We shall show that if x is balanced then the whole segment $[x^0x]$ is contained in the kernel. Let i and j be different players, and let $\tilde{x} = tx^0 + (1 - t)x$, where $0 < t < 1$. If i and j are symmetric then, since $x_i^0 = x_j^0$ and $x_i = x_j$,

$$tx_j^0 + (1 - t)x_i = tx_j^0 + (1 - t)x_j \quad \text{and} \quad s_{ij}(\tilde{x}) = s_{ji}(\tilde{x}).$$

If i and j are not symmetric let $s_{ij}(x) = e(P, x)$ and $s_{ji} = e(Q, x)$. By Corollary 4.2 we may assume that both P and Q are minimal winning sets. Since $e(S, x^0) = \max \{e(Q, x^0) : Q \in W\}$ for all $S \in W^m$, $s_{ij}(x^0) = e(P, x^0)$ and $s_{ji}(x^0) = e(Q, x^0)$. Hence $s_{ij}(\tilde{x}) = e(P, \tilde{x})$ and $s_{ji}(\tilde{x}) = e(Q, \tilde{x})$. By Lemma 2.8 $s_{ij}(x) = s_{ji}(x)$, so $s_{ij}(\tilde{x}) = s_{ji}(\tilde{x})$, and the proof is complete.

THEOREM 4.4. *Let G be an n-person constant-sum simple game with exactly n minimal winning sets; then the kernel of G is star-shaped.*

Proof. G is either the seven-player projective game or a partition game (Isbell, [4]). The kernel of the seven-player projective game is known to be star-shaped (though not convex, [5]).

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