GENERALIZATION OF A FORMULA OF HAYMAN AND ITS APPLICATION TO THE STUDY OF RIEMANN'S ZETA FUNCTION

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1. Introduction. In [5] Hayman considers functions $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, analytic inside $|z| < R$ ($\leq \infty$) and satisfying some additional conditions and obtains for their coefficients α_n an asymptotic estimate which generalizes Stirling's formula $1/n! \sim (e/n)^n(2\pi n)^{-1/2}$, to which it reduces in the case $f(z) = e^z$.

In the first part of the present paper, we obtain an asymptotic series for the coefficients α_n of an appropriate class of functions $f(z)$; this is the analog of the well-known asymptotic series for the factorials, to which it reduces in the case $f(z) = e^z$.

In the second part, we use the results of the first part, in order to prove that a certain condition, necessary for the validity of the Riemann hypothesis, is in fact satisfied.

Part I

2. Notations and main results. Let $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ be a function analytic inside the circle $|z| < R$ ($\leq \infty$), real on the real axis and such that $\lim_{x \to R} f(x) = \infty$. Set

$$a_1(z) = z \frac{d(\log f(z))}{dz} = z(f'(z)/f(z))$$

and define inductively for $\nu > 1$,

$$a_\nu(z) = z \frac{da_{\nu-1}(z)}{dz}.$$

Let A be the class of real-valued functions $a(x)$ such that, for $\nu \geq 3$, $a_\nu(x) \leq a(x)$ for sufficiently large x (which might depend on ν). We assume furthermore, that there exists a function $\delta(x)$, satisfying the following conditions for some $a \in A$:

(i) $\lim_{x \to R} \delta_2(x) a_2(x) = \infty$;
(ii) $\lim_{x \to R} \delta^3(x) a(x) = 0$;

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also, setting $\lambda(x;\delta) = \max_{|\theta|\geq\delta} f(x)^{-1}\,|f(xe^{i\theta})|$, we require that

(iii) $\lim_{x\to R} \lambda(x;\delta) = 0$.

The functions $f(z)$ for which such functions $\delta(x)$ can be defined will be said to belong to the class F.

Denote by $r = r(n)$ the unique (for proof of uniqueness, see [5]) root of

$$a_1(r) = n, \tag{1}$$

which approaches R, when $n \to \infty$, and by $\Delta = \Delta(a)$, the class of functions $\delta = \delta(r)$, satisfying (i), (ii), (iii) for a given $a(x) \in A$.

For fixed natural integer m and $a(x) \in A$, set

$$b(x) = \max\{a^{2m+1}a_2^{-3m-5/2},\ a_2^{-3m-3/2}a_3^{2m}a_4,\ a_2^{-3m-5/2}a_3^{2m+2},\ a_2^{-3m-7/2}a^{2m+2}\};$$

next, selecting also $\delta(x) \in \Delta(a)$, define

$$\varphi_m(x;\delta,a) = a_2(x)^{1/2}\max\{(\delta a_2)^{-1}\exp(-\tfrac{1}{2}\delta^2 a_2),\ \lambda(x,\delta),\ b(x)\}. \tag{2}$$

Finally, denoting the greatest integer function by square brackets and the multinomial coefficient

$$\frac{N!}{\nu_1!\,\nu_2!\cdots\nu_k!}\ (\nu_1+\nu_2+\cdots+\nu_k = N) \quad \text{by} \quad \binom{N}{\nu}_k,$$

set

$$A_N(x) = \frac{(-1)^{N/2}}{N!}\sum_{k=1}^{k_1}\frac{1}{k!}\sum\binom{N}{\nu}_k a_{\nu_1}(x)\cdots a_{\nu_k}(x),\ N \text{ even},\ k_1 = [N/3]; \tag{3}$$

here, and in what follows, a summation without limits is understood to range over all sets of integers $\nu_j \geq 3$, satisfying $\nu_1 + \cdots + \nu_k = N$. By (3), $A_2 = 0$ and we agree to set $A_0 = 1$. Any 0-term occurring in the paper is understood for $n \to \infty$, or, equivalently, for $r \to R$. With these notations we now state the main result.

THEOREM. *If $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \in F$ and $r = r(n)$ is the root defined by (1), then, for any selection of the natural integer m, of $a(r) \in A$ and $\delta(r) \in \Delta(a)$, the coefficients α_n of $f(z)$ admit the following asymptotic expansion:*

$$\begin{aligned}\alpha_n &= f(r)\cdot r^{-n}(2\pi^2 a_2(r))^{-1/2}\\ &\quad\cdot\{\textstyle\sum_{\nu=0}^{3m}(2a_2^{-1}(r))^{\nu}\Gamma(\nu+\tfrac{1}{2})A_{2\nu}(r) + O(\varphi_m(r;\delta,a))\}\\ &= f(r)\cdot r^{-n}(2\pi a_2(r))^{-1/2}\\ &\quad\cdot\{1+\pi^{-1/2}\textstyle\sum_{\nu=2}^{3m}(2a_2^{-1}(r))^{\nu}\Gamma(\nu+\tfrac{1}{2})A_{2\nu}(r) + O(\varphi_m(r;\delta,a))\}.\end{aligned} \tag{4}$$

3. Comments and a corollary. In (4), the error term depends upon the choice of $a(x)$ and $\delta(x)$. In many cases it is possible to choose these functions so as to minimize the order of this term. For instance, if we set $\tilde{a}(x) = \text{g.l.bd.}_{a\in A}\, a(x)$, it might happen that $\tilde{a}(x) \in A$. This, e.g., is always the case if $f(z) \epsilon F$ is of order less than one, because then $a_\nu(x) \leq a_1(x)$ for $x \to R$. In general, we shall say that $f(z) \epsilon F_1$, if $f(z) \epsilon F$ and $\tilde{a}(x) \epsilon A$. Clearly, $F_1 \subset F$ and, if $f(z) \epsilon F_1$ one will select $a(x) = \tilde{a}(x)$. Concerning $\delta(x)$, the choice is less well defined. However, one observes that in (2) $b(x)$ is independent of $\delta(x)$. If for some $\delta(x) \epsilon \Delta(\tilde{a})$ and x sufficiently large,

$$\max \{b(x), \lambda(x; \delta), (\delta a_2)^{-1} \exp(-\tfrac{1}{2}\delta^2 a_2)\} = b(x),$$

then $\varphi_m(x; \delta, \tilde{a}) = a_2^{1/2}(x) b(x)$ and this result cannot be improved by any different choice of $\delta \epsilon \Delta(\tilde{a})$. If, however, for all $\delta \epsilon \Delta(\tilde{a})$ and $x \to R$,

$$b(x) \leq \max \{\lambda(x; \delta), (\delta a_2)^{-1} \exp(-\tfrac{1}{2}\delta^2 a_2)\},$$

then one selects the largest possible δ, because, as δ increases, $\lambda(x; \delta)$ and $(\delta a_2)^{-1} \exp(-\frac{1}{2}\delta^2 a_2)$ both can only decrease. In order to do that, we may set $\delta_0(x) = \text{l.u.bd.}_{\delta \epsilon \Delta(\tilde{a})} \delta(x)$; but then in general $\delta_0(x) \notin \Delta(\tilde{a})$. However, there exists a sequence $\{\delta_K(x)\}$, $\delta_K \epsilon \Delta(\tilde{a})$, such that $\lim_{K \to \infty} \delta_K(x) = \delta_0(x)$, and we may choose a specific $\delta_K(x)$, with conveniently large K. We shall denote by $\varphi_m(x)$ the function $\varphi_m(x; \delta, a)$, corresponding to $a(x) = \tilde{a}(x)$ and the specific choice of $\delta(x)$ described. We may formalize the result just obtained in the

COROLLARY. *If $f(z) \epsilon F_1$, then the error term in* (4) *may be replaced by* $O(\varphi_m(r))$.

For completeness we also recall that (see [5]) in (4), $r^{-n}f(r)$ may be replaced by $M_n = \min_x (1/x) \sup_{|z|=x} |f(z)|$.

Finally, it should be mentioned that the usefulness of the theorem is limited mainly by the difficulty one has to solve (1) with sufficient accuracy. Indeed, suppose that we obtain $r(n)$ in the form $r(n) = r_1(n) + \varepsilon(n)$, where $\varepsilon(n) = o(r_1(n))$ is an error term of which we have only limited information.

Then, when we substitute this in (4), it may happen that the uncertainty introduced by $\varepsilon(n)$ is so large, that it wipes out any advantage obtained from carrying along the asymptotic expansion. In these cases, (4) does not represent any effective improvement over the simpler result of Hayman as far as the computation of the coefficients α_n is concerned. However, even in some of these situations, the present theorem may lead to results not directly obtainable by Hayman's Theorem. One such example will be presented in Part II.

4. Proofs.

LEMMA 1. $I_{2\nu} = \int_{-\infty}^{\infty} e^{-x^2} x^{2\nu}\, dx = \Gamma(\nu + \frac{1}{2})$

Proof. Set $x^2 = y$; then $I_{2\nu} = \int_0^\infty e^{-y} y^{\nu - 1/2}\, dy = \Gamma(\nu + \frac{1}{2})$.

LEMMA 2. *Let* $c = (\delta^2 a_2/2)^{1/2}$; *then* $\eta_{2\nu} = \int_c^\infty e^{-x^2} x^{2\nu}\, dx < c^{2\nu-1} e^{-c^2}$.

Proof. Integrating by parts and using property (i) of δ,

$$\eta_{2\nu} = \tfrac{1}{2}c^{2\nu-1}e^{-c^2} + \tfrac{1}{2}(2\nu - 1)\eta_{2\nu-2} < \tfrac{1}{2}c^{2\nu-1}e^{-c^2} + \tfrac{1}{2}c^{-2}(2\nu - 1)\eta_{2\nu}$$

and the result follows, because, by (i), $c^2 = \frac{1}{2}\delta^2 a_2 > 2\nu - 1$ for fixed ν and sufficiently large x.

LEMMA 3.

$$\int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2(x)} \theta^{2\nu}\, d\theta = (2a_2^{-1}(x))^{\nu+1/2}\Gamma(\nu + \tfrac{1}{2}) - 2\gamma_\nu(2a_2^{-1}(x))\delta^{2\nu-1}e^{-(1/2)\delta^2 a_2(x)}$$

$$(0 < \gamma_\nu < 1).$$

Proof. Follows from (i), Lemma 1 and Lemma 2.

Proof of the theorem. Let $f(z) \in F$ and set $z = xe^{i\theta}$, so that

$$\log f(z) = \log f(x) + \sum_{\nu=1}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu(x);$$

hence,

$$f(z) = f(x) \exp\left\{\sum_{\nu=1}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu(x)\right\}.$$

By Cauchy's theorem,

$$\alpha_n x^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(xe^{i\theta})e^{-i\theta n}\, d\theta. \tag{5}$$

We split the integral (5) into two parts: $\int_{-\pi}^{\pi} = \int_{-\delta}^{\delta} + \int_{|\theta|>\delta}$. By the assumption $f(z) \in F$ we can select a function $a(x) \in A$, and then $\delta(x) \in \Delta(a)$, such that, using (iii), the last integral is $O\{\lambda(x;\delta)\cdot f(x)\}$ and it remains to compute

$$\int_{-\delta}^{\delta} f(x) \exp\left\{\sum_{\nu=1}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu(x)\right\} e^{i\theta n}\, d\theta.$$

In this last integral we take for x the root $r = r(n)$ of (1); then the terms $i\theta a_1(r) - i\theta n$ of the exponent cancel and the integral becomes

$$f(r) \int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2(r)} \exp\left\{\sum_{\nu=3}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu(r)\right\} d\theta = f(r)I(r,\delta). \tag{6}$$

We select a fixed integer m and write, for simplicity a_ν rather than $a_\nu(r)$, when ever there is no danger of confusion; then the exponential in (6) may be ex panded as follows:

$$\exp\left\{\sum_{\nu=3}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu\right\} = \sum_{k=0}^{2m} \frac{1}{k!}\left(\sum_{\nu=3}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu\right)^k + \sum_{k=2m+1}^{\infty} \frac{1}{k!}\left(\sum_{\nu=3}^{\infty} \frac{(i\theta)^\nu}{\nu!} a_\nu\right)^k$$
$$= S_1 + S_2.$$

Here

$$S_2 = \sum_{k=2m+1}^{\infty} \frac{1}{k!} \sum_{N=3k}^{\infty} (i\theta)^N \sum \frac{a_{\nu_1} \cdots a_{\nu_k}}{\nu_1! \cdots \nu_k!}$$

$$= \sum_{N=3(2m+1)}^{\infty} (i\theta)^N \sum_{k=2m+1}^{k_1} \frac{1}{k!} \sum \frac{a_{\nu_1} \cdots a_{\nu_k}}{\nu_1! \cdots \nu_k!}$$

$$= \sum_{N=6m+3}^{\infty} \frac{(i\theta)^N}{N!} \sum_{k=2m+1}^{k_1} \frac{1}{k!} \sum \binom{N}{\nu}_k a_{\nu_1} \cdots a_{\nu_k},$$

the symbols k_1, $\binom{N}{\nu}_k$ and $\sum$ (without indication of limits) having the same meaning as in **2.** We set $S_2 = S_2' + S_2''$, S_2' being the sum of the even powers and S_2'' the sum of the odd powers of θ. Clearly, $\int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2} S_2'' \, d\theta = 0$ and it is sufficient to consider only S_2'.

$$S_2' = \sum_{n=3m+2}^{\infty} (-1)^n \frac{\theta^{2n}}{(2n)!} \sum_{k=2m+1}^{[2n/3]} \frac{1}{k!} \sum \binom{2n}{\nu}_k a_{\nu_1} \cdots a_{\nu_k}$$

$$= (-1)^m \frac{\theta^{6m+4}}{(6m+4)!} \left\{ \frac{2m+1}{(2m+1)!} \frac{(6m+4)!}{(3!)^{2m} 4!} a_3^{2m} a_4 \right\}$$

$$+ (-1)^{m+1} \frac{\theta^{6m+6}}{(6m+6)!} \left\{ \frac{1}{(2m+1)!} \left(\frac{(6m+6)!\,(2m+1)}{(3!)^{2m} 6!} a_3^{2m} a_6 \right. \right.$$

$$+ \frac{(6m+6)!\,(2m+1)(2m)}{(3!)^{2m-1} 4!\, 5!} a_3^{2m-1} a_4\, a_5$$

$$\left. \left. + \frac{(6m+6)!}{(3!)^{2m-2}(4!)^3} \binom{2m+1}{3} a_3^{2m-2} a_4^3 \right) + \frac{1}{(2m+2)!} \frac{(6m+6)!}{(3!)^{2m+2}} a_3^{2m+2} \right\}$$

$$+ (-1)^{m+2} \frac{\theta^{6m+8}}{(6m+8)!} O(a^{2m+2})$$

use having been made of (ii), $\delta^3 a \to 0$ (which implies $\delta \to 0$), for $r \to R$. Hence, using (ii) once more,

$$S_2' = (-1)^m \left\{ \frac{\theta^{6m+4}}{(2m)!\,(3!)^{2m} 4!} a_3^{2m} a_4 - \frac{\theta^{6m+6}}{(2m+2)!\,(3!)^{2m+2}} a_3^{2m+2} \right\} \tag{7}$$
$$+ O(\theta^{6m+6} a^{2m+1} + \theta^{6m+8} a^{2m+2}).$$

As for S_1, it may be rewritten as follows:

$$S_1 = 1 + \sum_{k=1}^{2m} \frac{1}{k!} \sum_{N=3k}^{\infty} (i\theta)^N \sum \frac{a_{\nu_1} \cdots a_{\nu_k}}{\nu_1! \cdots \nu_k!}$$

$$= 1 + \sum_{N=3}^{\infty} (i\theta)^N \sum_{k=1}^{k_2} \frac{1}{k!} \frac{a_{\nu_1} \cdots a_{\nu_k}}{\nu_1! \cdots \nu_k!}$$

where $k_2 = \min([N/3], 2m)$. Hence, $S_1 = 1 + S_3 + S_4$, with

$$S_3 = \sum_{N=3}^{3 \cdot 2m+1} \frac{(i\theta)^N}{N!} \sum_{k=1}^{k_1} \frac{1}{k!} \sum \binom{N}{\nu}_k a_{\nu_1} \cdots a_{\nu_k}$$

and

$$S_4 = \sum_{N=3\cdot 2m+2}^{\infty} \frac{(i\theta)^N}{N!} \sum_{k=1}^{2m} \frac{1}{k!} \sum \binom{N}{\nu}_k a_{\nu_1} \cdots a_{\nu_k}.$$

In S_3 we separate the even and odd powers of θ, so that $S_3 = S_3' + S_3''$, with

$$S_3' = \sum_{\nu=2}^{3m} A_{2\nu}(r)\theta^{2\nu}, \qquad S_3'' = \sum_{\nu=1}^{3m} B_{2\nu+1}(r)\theta^{2\nu+1},$$

where the $A_{2\nu}(r)$'s are given by (3) and the $B_{2\nu+1}(r)$'s have similar expressions. We do not need to know the $B_{2\nu+1}$'s explicitly, because $\int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2} S_3''\, d\theta = 0$. For later use we record that

$$A_{2\nu}(r) = O(a^{k_1}(r)), \qquad k_1 = [2\nu/3]. \tag{8}$$

One handles S_4 in the same way as S_2 and obtains $S_4 = S_4' + S_4''$, where S_4'' is the sum of the odd powers of θ in S_4, so that $\int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2} S_4''\, d\theta = 0$, while

$$S_4' = (-1)^{m-1} \frac{\theta^{6m+2}}{(6m+2)!} g(r),$$

$$g(r) = \binom{2m}{1} \frac{a_3^{2m-1} a_5}{(3!)^{2m-1} 5!} + \binom{2m}{2} \frac{a_3^{2m-2} a_4^2}{(3!)^{2m-2}(4!)^2} + O(a^{2m-1}) + O(\delta^2 a^m) \tag{9}$$
$$= O(a^{2m}).$$

Substituting in the integral of (6),

$$I(r,\delta) = \int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2}(1 + S_2' + S_3' + S_4')\, d\theta$$

and, by Lemma 3,

$$I(r,\delta) = (2\pi a_2^{-1})^{1/2} + \sum_{\nu=2}^{3m} (2a_2^{-1})^{\nu+1/2}\Gamma(\nu + \tfrac{1}{2}) A_{2\nu}(r) + R,$$

with an error term

$$R = \int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2}(S_2' + S_4')\, d\theta - 4\gamma_0(a_2\,\delta)^{-1} e^{-(1/2)\delta^2 a_2}$$
$$- 2\sum_{\nu=2}^{3m} A_{2\nu}(r)(2a_2^{-1})\delta^{2\nu-1}\gamma_\nu\, e^{-(1/2)\delta^2 a_2} \qquad (0 < \gamma_i < 1).$$

From Lemma 3, (7), (9) and the definition of $b(x)$ (see (2)) follows

$$\int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2} S_2'\, d\theta = O\{a_2^{-(3m+2)+1/2} a_3^{2m} a_4 + a_2^{-(3m+3)+1/2} a_3^{2m+2}$$
$$+ a_2^{-(3m+3)+1/2} a^{2m+1} + a_2^{-(3m+4)+1/2} a^{2m+2}\} = O(b(x)),$$

and

$$\int_{-\delta}^{\delta} e^{-(1/2)\theta^2 a_2} S_4'\, d\theta = O\{a^{2m}(2a_2^{-1})^{3m+3/2}\} = O(b(x)),$$

respectively. Also, by (ii) and (8), $A_{2\nu}\,\delta^{2\nu} = O(a^{2\nu/3}\delta^{2\nu}) = o(1)$; hence, the last two summands of R are $O((a_2\,\delta)^{-1}e^{-(1/2)\delta^2 a_2})$. Consequently, if

$$\psi_m(r;\,\delta,\,a) = \max\{b(r),\,(a_2\,\delta)^{-1}e^{-(1/2)\delta^2 a_2}\},$$

then $R = O(\psi_m(r;\,\delta,\,a))$ and (5) becomes

$$\begin{aligned}\alpha_n\,r^n &= (2\pi)^{-1}f(r)(I(r,\,\delta) + O(\lambda(r;\,\delta)))\\ &= (2\pi)^{-1}f(r)(2\pi a_2^{-1})^{1/2}\{1 + \pi^{-1/2}\textstyle\sum_{\nu=2}^{3m}(2a_2^{-1})^{\nu}\Gamma(\nu + \tfrac{1}{2})A_{2\nu}(r)\\ &\qquad + a_2^{1/2}\cdot O(\psi_m(r;\,\delta,\,a) + \lambda(r,\,\delta))\}.\end{aligned}$$

Hence, by (2),

$$\alpha_n\,r^n = (2\pi a_2)^{-1/2}f(r)\{1 + \pi^{-1/2}\textstyle\sum_{\nu=2}^{3m}(2a_2^{-1})^{\nu}\Gamma(\nu + \tfrac{1}{2})A_{2\nu}(r) + O(\varphi_m(r;\delta,a))\}$$

and this finishes the proof of the theorem.

5. An illustration. Let $f(z) = e^z$, so that $R = \infty$. Then, $\alpha_n = 1/n!$, $a_\nu(z) = z$ so that we may take $a(z) = \tilde{a}(z) = z$ and $e^z \in F_1$; also, (1) becomes $r = n$. Conditions (i) and (ii) require a choice of $\delta(x)$ such that $x\delta^2 \to \infty$, $x\delta^3 \to 0$ for $x \to \infty$. Therefore,

$$\delta_0(x) = \text{l.u.bd.}_{\delta\in\Delta(x)}\,\delta(x) = \text{l.u.bd.}\,x^{-1/3}/\chi(x)$$

where $\chi(x)$ may be any function that increases monotonically to infinity slower than $x^{1/6}$. Without loss of generality, we may normalize $\chi(x)$ by setting $\chi(1) = 1$. Then $\delta_0(x) = x^{-1/3}$ and $\delta_0^3(x)a(x) = 1$ so that $\delta_0(x) \notin \Delta(x)$. We may, however, select the sequence $\delta_K(x) = (x^{1/3}\log_K x)^{-1}$ where $\log_K x$ stands for the K times iterated logarithm. With this choice we obtain for any natural integer m, $b(x) = x^{-m-1/2}$, $\lambda(x) < \exp\{-\tfrac{1}{2}x^{1/3}\log_K^{-2} x\}$ and $\varphi_m(x) = a_2^{1/2}(x)b(x) = x^{-m}$. The coefficients $A_{2q}(r) = A_{2q}(n)$ become

$$\frac{(-1)^q}{(2q)!}\sum_{k=1}^{k_1}\frac{1}{k!}\sum\binom{2q}{\nu}_k n^k$$

so that

$$(2a_2^{-1})^q A_{2q}(n) = \frac{(-2)^q}{(2q)!}\sum_{k=1}^{[2q/3]}\frac{1}{k!}\sum\binom{2q}{\nu}_k n^{-(q-k)}$$

and

$$\begin{aligned}\pi^{-1/2}\sum_{q=2}^{3m}(2a_2^{-1})^q\Gamma(q + \tfrac{1}{2})A_{2q}(n) &= \sum_{q=2}^{3m}\frac{(-1)^q}{2^q q!}\sum_{k=1}^{[2q/3]}\frac{1}{k!}\sum\binom{2q}{\nu}_k n^{-(q-k)}\\ &= \sum_{j=1}^{3m-1}\frac{c_j}{n^j}\end{aligned}$$

with

$$c_j = \frac{(-1)^j}{2^j}\sum_{k=1}^{2j}\frac{(-1)^k}{2^k k!\,(j+k)!}\sum\binom{2j+2k}{\nu}_k. \tag{10}$$

Hence, (4) becomes

$$1/n! = (e/n)^n(2\pi n)^{-1/2}\{1 + \textstyle\sum_{j=1}^{m-1} c_j/n^j + O(n^{-m})\}.$$

It is, of course, classical, that $1/n! = (e/n)^n(2\pi n)^{-1/2}G(n)$, where

$$G(n) = \exp\left\{-\sum_{\nu=1}^{m} \frac{B_{2\nu}}{2\nu(2\nu-1)} n^{1-2\nu} + O(n^{-2m-1})\right\}.$$

Hence, it follows from the uniqueness of the asymptotic expansion (see [3, p. 13]), that the coefficients c_j, defined by (10), have as (formal) generating function

$$\exp\left\{-\sum_{\nu=1}^{\infty} \frac{B_{2\nu}}{2\nu(2\nu-1)} n^{1-2\nu}\right\}.$$

This result may be compared with a similar one in [4], where the coefficients of the formal power series expansion of $G(n)$ are represented by an explicit formula different from, but somewhat similar to (10).

Part II

6. Introduction. As already mentioned in **3**, a severe limitation in the use of our theorem is the fact that (1) is, in general, a transcendental equation and the lack of accuracy inherent in its solution wipes out any advantage that (4) may have over the simpler result of Hayman. Here we shall see an example of precisely this nature, where (1) cannot be solved with the accuracy needed to give more than the principal term in (4), for the coefficients α_n, but where a significant result can nevertheless be obtained.

7. Statement of the problem and main result. We consider the function

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{1}{2}s)\zeta(s),$$

where $\zeta(s)$ is the Riemann Zeta function. Setting $s = \frac{1}{2} + it$, $\xi(\frac{1}{2} + it) = \Xi(t) = \sum_{n=0}^{\infty} c_n t^n$ is an entire function of order one of t. The Riemann hypothesis is equivalent to the assertion that all roots of $\Xi(t) = 0$ are real. If this is the case, then, observing that c_0 is real, one of the consequences is (see [2]) that

$$\mu_n = nc_n^2 - (n+1)c_{n-1}c_{n+1} > 0.$$

Actually, this condition is satisfied rather trivially; indeed, $\Xi(t)$ is an even function and the coefficients of t^2 alternate in sign, so that (see [5])

$$\Xi(t) = \sum_{m=0}^{\infty} (-1)^m \alpha_m t^{2m}, \qquad \alpha_m > 0,$$

and one immediately checks that

$$\mu_{2m} = 2m\alpha_m^2 > 0, \qquad \mu_{2m+1} = (2m+2)\alpha_m\alpha_{m+1} > 0.$$

One may, however, consider $\Xi(t)$ as a function of t^2; still better, using Hayman's notation, set $z = -t^2$, so that $\Xi(t) = f(-t^2) = f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, an entire function of order $\frac{1}{2}$. If t_0 is a real zero of $\Xi(t)$, then $z_0 = -t_0^2$ is a negative zero of $f(z)$ and the Riemann hypothesis is equivalent to the statement that all zeros of $f(z)$ are negative. This means in particular that they have to be real, which, as mentioned, implies that $D_n = n\alpha_n^2 - (n+1)\alpha_{n-1}\alpha_{n+1}$ must be positive for all $n \geq 1$. Although the proof that

$$D_n > 0 \tag{11}$$

would not settle anything, the verification of (11) is of interest (if (11) fails for some integer $n > 0$, then the Riemann hypothesis cannot be true) and the problem of determining the sign of D_n has been raised by Pólya already in 1927. (See [7, p. 16].)

It is known [5] that for $f(z)$ defined as here above,

$$\log f(z) = \tfrac{1}{2}z^{1/2}\log(z^{1/2}/2\pi e) + \tfrac{7}{8}\log z + \tfrac{1}{4}\log \pi/2 + o(1)$$

holds for $|z| \to \infty$, uniformly in $|\arg z| < \pi - \delta(0 < \delta < \pi)$ and that (notations as in Part I) if $M_n = \inf_{r>1} r^{-n}f(r)$, then $\alpha_n \sim (\pi n)^{-1/2}M_n$ $(n \to \infty)$. This asymptotic relation is, however, insufficient for the proof of (11). In what follows, we propose to show, using our theorem from Part I, that for $n \to \infty$,

$$D_n = \alpha_n^2(1 + O(\log^{-1} n)). \tag{12}$$

One may observe that (12) will be obtained, although, on account of the difficulties already mentioned, (4) does not lead even to an asymptotic formula for α_n (only for $\log \alpha_n$). From (12) it follows that (11) holds at least for sufficiently large values of n, so that this necessary condition for the validity of the Riemann hypothesis is indeed satisfied. It may be added that (11) may be verified directly for any specific value of n and that the present method permits, in principle, to replace the O-terms by inequalities, so that it should be possible (although, presumably quite laborious), to prove (11) for all n.

8. Application of the theorem. Using the definition of $f(z)$ and the general Stirling formula, one may replace the previously indicated formula for $\log f(z)$ by the sharper equality

$$\begin{aligned}\log f(z) = \tfrac{1}{2}z^{1/2}\log(z^{1/2}/2\pi e) &+ \tfrac{7}{8}\log z \\ &+ \tfrac{1}{4}\log \pi/2 + J(z^{1/2}/2 + \tfrac{1}{4}) + Q(z),\end{aligned} \tag{13}$$

where (see [1, p. 165] or [9, p. 218]).

$$J(Z) = \frac{1}{\pi}\int_0^\infty \frac{Z}{u^2 + Z^2}\log\frac{1}{1 - e^{-2\pi u}}\,du = \sum_{\nu=1}^{m}\frac{B_{2\nu}}{2\nu(2\nu - 1)}Z^{1-2\nu} + O(Z^{-1-2m}),$$

$Q(z) = \log \zeta(z^{1/2} + \frac{1}{2})$, and B_ν are the Bernoulli numbers in the even subscript notation. Defining

$$J_1(z) = z\frac{d}{dz}J(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4})$$

and, inductively, for $\nu > 1$,

$$J_\nu(z) = z \frac{d}{dz} J_{\nu-1}(z),$$

(by using the formulas (44), (45), (46) on pp. 166–167 of [1]) one verifies for $|\arg z| \le \pi - \delta$ $(\delta > 0)$ that

$$J_1(z) = -\tfrac{1}{2}B_2 z^{-1/2}(1 - z^{-1/2}) + O(z^{-3/2}),$$
$$J_2(z) = \tfrac{1}{4}B_2 z^{-1/2}(1 - 2z^{-1/2}) + O(z^{-3/2})$$

and, in general,

$$J_\nu(z) = (-1)^\nu B_2 z^{-1/2}(2^{-\nu} - (2z^{1/2})^{-1}) + O(z^{-3/2})$$
$$= \frac{(-1)^\nu}{6} z^{-1/2}(2^{-\nu} - \tfrac{1}{2}z^{-1/2}) + O(z^{-3/2}).$$

Hence, if $a_1(z) = z(f'(z)/f(z))$ and, for $\nu \ge 2$, $a_\nu(z) = za'_{\nu-1}(z)$, one obtains from (13) that, for any $\delta > 0$, one has uniformly in $|\arg z| < \pi - \delta$:

$$(14)\quad \begin{aligned} a_1(z) &= \tfrac{1}{4}z^{1/2} \log \frac{z^{1/2}}{2\pi} + \tfrac{7}{8} - \tfrac{1}{12}(z^{-1/2} - z^{-1}) + O(z^{-3/2}); \\ a_2(z) &= \tfrac{1}{8}z^{1/2} \log \frac{ez^{1/2}}{2\pi} + \tfrac{1}{24}(z^{-1/2} - 2z^{-1}) + O(z^{-3/2}); \end{aligned}$$

and, in general,

$$a_\nu(z) = 2^{-\nu-1}z^{1/2} \log \frac{e^{\nu-1}z^{1/2}}{2\pi} + \frac{(-1)^\nu}{6} z^{-1/2}(2^{-\nu} - \tfrac{1}{2}z^{-1/2}) + O(z^{-3/2})$$

for $\nu \ge 2$.

In order to apply the theorem, we set $|z| = x$ and select $\delta = x^{-1/6}\log^{-1/2} x$; we also observe that $R = \infty$ and that for every fixed $\nu > 1$, $a_1(x) > a_\nu(x)$, if x is sufficiently large. Hence we may take $a(x) = \tilde{a}(x) = a_1(x)$, and if $f(z) \in F$ then $f(z) \in F_1$. Next one verifies that for $x \to \infty$,

$$(15)\quad \delta^2 a_2 \sim \tfrac{1}{16}x^{1/6} \to \infty, \quad \delta^3 a \sim \tfrac{1}{8}\log^{-1/2}x \to 0, \quad \delta^2 x^{1/2} = x^{1/6}\log^{-1}x \to \infty.$$

We now check that all remaining assumptions of the theorem are satisfied by $f(z)$. Indeed, for $z = xe^{i\theta}$ and $\delta < |\theta| < \pi - \delta$, one obtains from (13) and the remark that in this interval

$$|Q(z)| \sim |2^{-(z^{1/2}+1/2)}| < \exp\{-\tfrac{1}{2}(\log 2)x^{1/3}\log^{-1/2} x\},$$

that

$$\log|f(z)| - \log f(x)$$
$$< \tfrac{1}{2}\cos\frac{\delta}{2}\cdot x^{1/2}\log\frac{x^{1/2}}{2\pi e} - \tfrac{1}{8}\delta^2 x^{1/2} + \tfrac{7}{8}\log x + \tfrac{1}{4}\log\frac{\pi}{2}$$

$$+ O(x^{-1/2}) - \left(\tfrac{1}{2}x^{1/2}\log\frac{x^{1/2}}{2\pi e} + \tfrac{7}{8}\log x + \tfrac{1}{4}\log\frac{\pi}{2} + O(x^{-1/2})\right)$$

$$= -\tfrac{1}{2}\left(1 - \cos\frac{\delta}{2}\right)x^{1/2}\log\frac{x^{1/2}}{2\pi e} - \tfrac{1}{8}\delta^2 x^{1/2} + O(x^{-1/2})$$

$$= -\tfrac{1}{16}\delta^2 x^{1/2}\log\frac{x^{1/2}}{2\pi e} - \tfrac{1}{8}\delta^2 x^{1/2} + \frac{1}{2^9\cdot 3}x^{-1/6}\log^{-1}x + O(x^{-1/2})$$

$$= -\tfrac{1}{16}\delta^2 x^{1/2}\log\frac{ex^{1/2}}{2\pi} + O(x^{-1/6}\log^{-1}x) < -\tfrac{1}{32}x^{1/6},$$

by the definition of $\delta(x)$ and (15).

Hence, setting $\lambda(x) = \exp(-\tfrac{1}{32}x^{1/6})$, $f^{-1}(x)|f(xe^{i\theta})| < \lambda(x)$ holds uniformly in $\delta \le |\theta| \le \pi - \delta$ for $x \to \infty$. The same inequality holds, however, also for $\pi - \delta \le |\theta| \le \pi$. Indeed, in this case one still has $\mathrm{Re}\,(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4}) > \tfrac{1}{4}$, so that, by Stirling's formula,

$$|\Gamma(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4})| < (2\pi)^{1/2}\exp\{-\tfrac{1}{4}\pi x^{1/2}(1 - \tfrac{1}{8}\delta^2)\}x^{\delta x^{1/2}/8},$$

or, using the definition of δ,

$$|\Gamma(\tfrac{1}{2}z^{1/2} + \tfrac{1}{4})| < (2\pi)^{1/2}\exp\{-\tfrac{1}{4}\pi x^{1/2} + O(x^{1/6})\}.$$

Also, $\mathrm{Re}\,(z^{1/2} + \tfrac{1}{2}) \ge \tfrac{1}{2}$, $\mathrm{Im}\,(z^{1/2} + \tfrac{1}{2}) \sim x^{1/2}\sin\theta/2 \le x^{1/2}$, so that (see [8])

$$0 \le |\zeta(z^{1/2} + \tfrac{1}{2})| < x^{1/12}, \quad \text{and} \quad -\infty \le \mathrm{Re}\,Q(z) < \tfrac{1}{12}\log x;$$

hence,

$$-\infty \le \log|f(xe^{i\theta})| < -\tfrac{1}{4}\pi x^{1/2} + O(x^{1/6}) + O(\log x) < -\tfrac{1}{5}\pi x^{1/2},$$

say. Therefore,

$$\log|f(xe^{i\theta})| - \log f(x) < -x^{1/2}\left(\log\frac{x^{1/2}}{2\pi e} + \frac{\pi}{5}\right)$$

$$< -x^{1/2}\log\frac{x^{1/2}}{2\pi e} < -\tfrac{1}{32}x^{1/2} = \log\lambda(x)$$

holds uniformly also for $\pi - \delta \le |\theta| \le \pi$. This finishes the verification that $(z) \in F$; hence, that $f(z) \in F_1$. We may, therefore, use (4) for the computation of the coefficients α_n.

Consequently, we replace x by the root $r = r(n)$ of (1), which on account of (14), verifies

$$(16) \quad r^{1/2}\log(r^{1/2}/2\pi) = 4(n - \tfrac{7}{8}) + (r^{-1/2} - r^{-1})/3 + O(r^{-3/2}).$$

We note the following consequences of (16):

(i) $(r^{1/2}\log(r^{1/2}/2\pi))^{-m} = O(n^{-m})$ for every $m > 0$;

(ii) $\log r - 2\log n \sim -2\log_2 n$;

$$(17)\quad \text{(iii)}\quad \tfrac{1}{4}r^{1/2} = (n - \tfrac{7}{8} + O(r^{-1/2}))\log^{-1}(r^{1/2}/2\pi)$$
$$= (n - \tfrac{7}{8} + o(1))(\log n + O(\log_2 n))^{-1}$$
$$= n \log^{-1} n(1 + O((\log_2 n)(\log^{-1} n))).$$

Next, we select $m = 2$ and compute the quantities that enter in the definition of $\varphi_2(r)$, namely

$$(\delta a_2)^{-1} \exp(-\tfrac{1}{2}\delta^2 a_2) = O((r^{1/3}\log^{1/2} r)^{-1} \exp(-\tfrac{1}{32}r^{1/6}));$$
$$\lambda(r) = \exp(-\tfrac{1}{32}r^{1/6});$$
$$a_2^{-17/2}a_1^5 = O(r^{-7/4}\log^{-7/2} r);$$
$$a_2^{-15/2}a_3^4 a_4 = O(r^{-5/4}\log^{-5/2} r);$$
$$a_2^{-17/2}a_3^6 = O(r^{-5/4}\log^{-5/2} r);$$
$$a_2^{-19/6}a_1^6 = O(r^{-7/4}\log^{-7/2} r).$$

The largest of these is $O(r^{-5/4}\log^{-5/2} r)$, so that, by (14), (17) and (2),

$$(18)\qquad \varphi_2(r) = O(n^{-2}).$$

By (4),

$$(19)\qquad \alpha_n = r^{-n}(2\pi a_2(r))^{-1/2} f(r)\{1 + S\},$$

where

$$S = \pi^{-1/2}\textstyle\sum_{\nu=2}^{6}(2a_2^{-1}(r))^{\nu} A_{2\nu}(r)\Gamma(\nu + \tfrac{1}{2}) + O(n^{-2}).$$

Computing directly $\sum_{\nu=4}^{6}(2a_2^{-1})^{\nu}\Gamma(\nu + \frac{1}{2})A_{2\nu}(r)$, it follows from (14) and (17), that this is $O(n^{-2})$. So, e.g., for $\nu = 4$,

$$A_8(r) = \frac{a_8}{8!} + \frac{a_5\, a_3}{5!\,3!} + \frac{a_4^2}{4!\,4!\,2!} = O(r \log^2 r),$$
$$(2a_2^{-1})^4 A_8(r) = O((r^{1/2}\log r)^{-4}\cdot r\log^2 r) = O(n^{-2})$$

and similarly for $\nu = 5$ and $\nu = 6$. The remaining terms of the sum are

$$(a_4\, a_2^{-2}/8) - (5a_3^2\, a_2^{-3}/24) - (a_6\, a_2^{-3}/48).$$

Of these, using (14),

$$a_6\, a_2^{-3} = 4(r \log^2(er^{1/2}/2\pi))^{-1}(1 + o(1)) = O(r^{-1}\log^{-2} r) = O(n^{-2}),$$

and it only remains to find $a_4\, a_2^{-2}$ and $a_3^2\, a_2^{-3}$. By (14),

$$\frac{a_4}{a_2^2} = \frac{\frac{1}{32}\left(r^{1/2}\log\dfrac{er^{1/2}}{2\pi} + 2r^{1/2} + r^{-1/2}/3 - 8/3r + O(r^{-3/2})\right)}{\frac{1}{64}\left(r^{1/2}\log\dfrac{er^{1/2}}{2\pi} + r^{-1/2}/3 - 2/3r + O(r^{-3/2})\right)^2}$$

$$= 2(r^{1/2}\log(er^{1/2}/2\pi))^{-1}$$
$$\times \{1 + 2\log^{-1}(er^{1/2}/2\pi) + (3r\log(er^{1/2}/2\pi))^{-1} - 8(3r^{3/2}\log(er^{1/2}/2\pi))^{-1} + O(r^{-2}\log^{-1} r)\}$$
$$\times \{1 + (3r\log(er^{1/2}/2\pi))^{-1} - 2(3r^{3/2}\log(er^{1/2}/2\pi))^{-1} + O(r^{-2}\log^{-1} r)\}^{-2}$$
$$= 2(r^{1/2}\log(er^{1/2}/2\pi))^{-1}$$
$$\times \{1 + 2\log^{-1}(er^{1/2}/2\pi) - (3r\log(er^{1/2}/2\pi))^{-1} + O(r^{-1}\log^{-2} r)\}.$$

By (17), the last bracket equals $1 + 2(\log(er^{1/2}/2\pi))^{-1} + O(n^{-2}\log n)$. By the same procedure, one finds that

$$a_3^2 a_2^{-3} = 2(r^{1/2}\log(er^{1/2}/2\pi))^{-1}$$
$$\cdot\{1 + 2(\log(er^{1/2}/2\pi))^{-1} + (\log(er^{1/2}/2\pi))^{-2} + O(n^{-2}\log n)\};$$

the sum S in (19) becomes, therefore,

$$S = (4r^{1/2}\log(er^{1/2}/2\pi))^{-1}\{1 + 2(\log(er^{1/2}/2\pi))^{-1} + O(n^{-2}\log n)\}$$
$$- 5(12r^{1/2}\log(er^{1/2}/2\pi))^{-1}\{1 + 2(\log(er^{1/2}/2\pi))^{-1}$$
$$+ (\log(er^{1/2}/2\pi))^{-2} + O(n^{-2}\log n)\} + O(n^{-2})$$
$$= -(6r^{1/2}\log(er^{1/2}/2\pi))^{-1}\{1 + 2(\log(er^{1/2}/2\pi))^{-1}$$
$$+ (\log(er^{1/2}/2\pi))^{-2}\} + O((n\log^2 n)^{-1})$$
$$= -\tfrac{1}{24}(\tfrac{1}{4}r^{1/2}\log(r^{1/2}/2\pi) + \tfrac{1}{4}r^{1/2})^{-1}\{1 + 2(\tfrac{1}{2}\log r$$
$$+ \log(e/2\pi)\}^{-1}) + O((n\log^2 n)^{-1}).$$

By (16) and (17),

$$S = -\tfrac{1}{24}\frac{1 + 2(\log n + O(\log_2 n))^{-1}}{n(1 - 7/8n + O(n^{-2})) + (n/\log n)(1 + O(\log_2 n/\log n))} + O((n\log^2 n)^{-1})$$
$$= -(1/24n)(1 - \log^{-1} n + O(\log_2 n/\log^2 n))$$
$$\cdot(1 + 2\log^{-1} n + O(\log_2 n/\log^2 n) + O((n\log^2 n)^{-1})$$
$$= -(24n)^{-1} + O((n\log n)^{-1})$$

and (19) becomes

$$\alpha_n = r^{-n}(2\pi a_2(r))^{-1/2}f(r)(1 - (24n)^{-1} + O((n\log n)^{-1})). \tag{20}$$

9. Proof of (12). In what follows, the root of $a_1(r) = x$ will be denoted by r_x or $r(x)$, whichever is more convenient. Replacing in

$$D_n = n\alpha_n^2 - (n+1)\alpha_{n-1}\alpha_{n+1} = n\alpha_n^2(1 - (1 + n^{-1})\alpha_n^{-2}\alpha_{n+1}\alpha_{n-1})$$

the coefficients α_j by their values given by (20), we obtain after obvious simplifications,

$$D_n = n\alpha_n^2\{1 - (1 + n^{-1})h(n)[1 + O((n \log n)^{-1})]\}$$

with

$$h(n) = r_n^{2n} r_{n-1}^{-(n-1)} r_{n+1}^{-(n+1)} f^{-2}(r_n) f(r_{n+1}) f(r_{n-1}) \times (a_2^{-2}(r_n) a_2(r_{n+1}) a_2(r_{n-1}))^{-1/2}.$$

Let

$$g(x) = \log f(r(x)) - \tfrac{1}{2} \log a_2(r(x)) - x \log r(x); \tag{21}$$

then, if we denote the successive finite differences by $\Delta^k g(i)$, one has

$$\Delta^2 g(n-1) = g(n+1) - 2g(n) + g(n-1)$$

and $h(n) = \exp\{\Delta^2 g(n-1)\}$. Consequently,

$$D_n = n\alpha_n^2\{1 - (1 + n^{-1}) \exp\{\Delta^2 g(n-1)\}[1 + O((n \log n)^{-1})]\}. \tag{22}$$

By (21), $g(x)$ is twice differentiable; hence (see e.g. [6. p. 57 (19.5)]), $\Delta^2 g(n-1) = g''(n + \eta)$, $|\eta| < 1$. Let $f' = df/dr$, $a_\nu' = da_\nu/dr$, $r' = dr/dx$; then $da_1/dx = a_1' r'$ and, differentiating

$$a_1(r) = x \tag{1'}$$

we obtain $a_1' r' = 1$, whence

$$r'/r = (ra_1')^{-1} = a_2^{-1}. \tag{23}$$

Differentiating (21), we now obtain, using (23) and (1′):

$$\begin{aligned} g'(x) &= (f'/f)r' - \tfrac{1}{2}(a_2'/a_2)r' - \log r - x(r'/r) \\ &= (a_1 - \tfrac{1}{2}ra_2' a_2^{-1} - x)(r'/r) - \log r \\ &= -\tfrac{1}{2}a_3 a_2^{-1}(r'/r) - \log r \\ &= -\tfrac{1}{2}a_3 a_2^{-2} - \log r. \end{aligned}$$

Differentiating once more,

$$\begin{aligned} g''(x) &= \tfrac{1}{2}a_2^{-3}(2a_3 a_2' - a_2 a_3')r' - (r'/r) \\ &= \{\tfrac{1}{2}a_2^{-3}(2a_3 \cdot ra_2' - a_2 \cdot ra_3') - 1\}(r'/r) \\ &= -(r'/r)\{1 - \tfrac{1}{2}(2a_3^2 - a_2 a_4)a_2^{-3}\} \\ &= -(r'/r)(1 + O((r^{1/2} \log r)^{-1})) \\ &= -a_2^{-1}(1 + O(n^{-1})) \end{aligned}$$

by (17) and (23). Using (14), (17) and (1′),

$$2a_2(r(x)) = a_1(r(x)) - \tfrac{7}{8} + \tfrac{1}{4}r^{1/2}(x) + O(r^{-1/2}(x))$$

$$= x + \tfrac{1}{4}r^{1/2}(x) - \tfrac{7}{8} + O(r^{-1/2}(x))$$
$$= x + (x \log^{-1} x)(1 + O(\log_2 x \cdot \log^{-1} x)),$$

so that

$$-a_2^{-1} = -2x^{-1}(1 + \log^{-1} x + O(\log_2 x \cdot \log^{-2} x))^{-1}$$
$$= -2x^{-1}(1 - \log^{-1} x + O(\log_2 x \cdot \log^{-2} x))$$

and

$$\begin{aligned} \Delta^2 g(n-1) &= g''(n+\eta) \\ &= -2(n+\eta)^{-1}(1 - \log^{-1}(n+\eta) \\ &\qquad + O(\log_2(n+\eta)\cdot\log^{-2}(n+\eta)))(1 + O(n^{-1})) \\ &= -2n^{-1}(1 - \log^{-1} n + O(\log_2 n \cdot \log^{-2} n)), \end{aligned}$$

because $|\eta| < 1$. It follows that

$$\exp\{\Delta^2 g(n-1)\} = 1 - 2n^{-1} + O((n \log n)^{-1})$$

and (22) becomes

$$\begin{aligned} D_n &= n\alpha_n^2\{1 - (1 + n^{-1})[1 - 2n^{-1} + O((n\log n)^{-1})] \\ &\qquad \cdot [1 + O((n \log n)^{-1})]\} \\ &= n\alpha_n^2\{1 - [1 - n^{-1} + O((n\log n)^{-1})][1 + O((n\log n)^{-1})]\} \\ &= \alpha_n^2(1 + O(\log^{-1} n)); \end{aligned}$$

this finishes the proof of (12).

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