SOME CHARACTERIZATIONS OF C^*-ALGEBRAS

BY
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Introduction[1]

In [13] I. Vidav introduces the following generalized notion for an arbitrary complex Banach algebra A with identity of norm 1: the element $h \in A$ is said to be hermitian if and only if for t real, $\| 1 + ith \| = 1 + o(t)$ as $t \to 0$. This notion of hermiticity coincides with the usual one on C^*-algebras (for a proof of this fact see [9, proof of Theorem 21]). Let H be the set of hermitian elements of the Banach algebra A. In [13] Vidav shows the following theorem: if (a) $A = H + iH$, and (b) for every $h \in H$, h^2 can be expressed in the form $h^2 = u + iv$ with $u, v \in H$ and $uv = vu$, then there is an involution on A and a certain Banach algebra norm equivalent to the given one so that in terms of this involution and this new norm A becomes a C^*-algebra. In the present paper it is shown, first in the commutative case (§2) and then in general (§4), that if A satisfies these hypotheses, then A is, with its original norm and the involution produced by Vidav, a C^*-algebra. Since a C^*-algebra satisfies the conditions (a) and (b), we thus obtain a metric characterization of those Banach algebras which can be made into C^*-algebras by introduction of some suitable involution. The transition from the commutative case to the general case is aided by establishing in Theorem (4.2) a generalization of a well known characterization of C^*-algebras [6; Theorem 11].

Our result concerning the theorem of Vidav enables us to establish in §5 a strengthening of a theorem of Lumer [9, Theorem 21]. This gives a characterization of those Banach *-algebras which are C^* in terms of a local differential condition.

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In what follows, all spaces are over the complex field, all Banach algebras possess an identity of norm 1, and the term operator will signify a bounded linear transformation with range contained in its domain. The algebra of all operators on a Banach space X will be designated by $[X]$. We shall denote the spectrum of the element t of a Banach algebra A by $sp_A(t)$, and the spectral radius by $r(t)$ (in this notation for spectral radius we suppress mention of A, since the spectral radius remains invariant under extension or contraction of A). By the term semi-closed rectangle we shall mean a Cartesian product of real intervals of the form $(a, b] \times (c, d]$.

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1. Preliminaries

In this section we reproduce some machinery from [9] and [13] which will be needed in the sequel. We first discuss definitions and results from [9].

Definition. Let X be a vector space. A semi-inner-product (abbreviated s.i.p.) for X is a mapping $[\ ,\ ]$ of $X \times X$ into the field of complex numbers such that:

(i) $[x + y, z] = [x, z] + [y, z]$ for $x, y, z \in X$.
(ii) $[\lambda x, y] = \lambda[x, y]$ for $x, y \in X$, λ complex.
(iii) $[x, x] > 0$ for $x \neq 0$.
(iv) $|[x, y]|^2 \leq [x, x][y, y]$ for $x, y \in X$.

When a s.i.p. is defined for X, we call X a semi-inner-product space (abbreviated s.i.p.s.).

If X is a s.i.p.s., then $[x, x]^{1/2}$ is a norm on X. On the other hand, every normed linear space can be made into a s.i.p.s. (in general, in infinitely many ways) so that the s.i.p. is consistent with the norm—i.e., $[x, x]^{1/2} = \| x \|$, for each $x \in X$. By virtue of the Hahn-Banach theorem this can be accomplished by choosing for each $x \in X$ exactly one bounded linear functional f_x such that $\| f_x \| = \| x \|$ and $f_x(x) = \| x \|^2$, and then setting $[x, y] = f_y(x)$, for arbitrary $x, y \in X$.

Definition. Given a linear transformation T mapping a s.i.p.s. into itself, we denote by $W(T)$ the set, $\{[Tx, x] \mid [x, x] = 1\}$, and call this set the numerical range of T.

An important fact concerning the notion of numerical range is the following:

(1.1) Let X be a Banach space and T an operator on X. Although in principle there may be many different semi-inner-products consistent with the norm of X, nonetheless if the numerical range of T relative to one such semi-inner-product is real, then the numerical range relative to any such semi-inner-product is real. If this is the case, we call T a hermitian operator.

It is shown in [9; §9] that an operator T on a Banach space X is a hermitian operator if and only if it is hermitian in the sense of Vidav's definition —i.e., if and only if $\| I + itT \| = 1 + o(t)$ for t real, where I is the identity operator. Thus we have at our disposal two equivalent formulations of the notion of hermiticity for operators on Banach spaces.

There is a device for carrying general results concerning hermitian operators over to results concerning hermitian elements of an arbitrary Banach algebra. For, given a Banach algebra A, let L be the left regular representation of A (i.e., $L_a x = ax$, for $a, x \in A$). Then L can be used to represent A isometrically as an algebra of operators on A, and $a \in A$ is hermitian if and only if L_a is a hermitian operator. For example, it is easy to see from the s.i.p. formulation in (1.1) that a real linear combination of hermitian operators is hermitian, and then to obtain the same result for hermitian elements of any Banach algebra. It should be mentioned, however, that the powers of a hermitian operator are not, in general, all hermitian, even on a reflexive space [10, §8].

For greater convenience, we list the theorem of Vidav [13] in more detail than in the introduction.

(1.2) THEOREM. *Let A be a Banach algebra, and let H be the set of hermitian elements of A. If $A = H + iH$, and if for every $h \in H$, h^2 can be expressed in the form $h^2 = u + iv$ with $u, v \in H$ and $uv = vu$, then:*

(i) *For each $x \in A$, the decomposition $x = u + iv$, $u, v \in H$, is unique.*

(ii) *The map * which assigns to each element $x = u + iv$ (where $u, v \in H$) the element $x^* = u - iv$ is an involution on A (note that the self-adjoint elements relative to this involution are precisely the elements of H).*

(iii) *$\| \ \|_0$, defined by $\| x \|_0 = \| x^*x \|^{1/2}$, is a Banach algebra norm on A equivalent to the given norm, and moreover, $\| h \|_0 = \| h \|$ for every $h \in H$.*

(iv) *The algebra A with the involution * and the norm $\| \ \|_0$ is a C^*-algebra.*

When a Banach algebra A satisfies the hypotheses of (1.2), we shall refer to the involution in (ii) as the Vidav involution, and employ the notation * and $\| \ \|_0$.

As is well known, the Gelfand representation of a commutative C^*-algebra B is an isometry onto $C(\mathfrak{M}_B)$, the algebra of all continuous complex-valued functions on the maximal ideal space $\mathfrak{M}_B$ (see, for example, [8, §26A]). Thus we have:

(1.3) If A is a commutative Banach algebra to which (1.2) applies, the Gelfand representation of A is a bicontinuous isomorphism of A onto the algebra $C(\mathfrak{M}_A)$, and for each $x \in A$, $\| x \|_0 = r(x)$.

2. Vidav's theorem in the commutative case

(2.1) THEOREM. *Let A be any Banach algebra, and let $e_1, e_2, \cdots, e_n$ (n arbitrary) be non-zero, hermitian idempotents which are disjoint (i.e., $e_j e_k = 0$ for $j \neq k$). Then for $\lambda_1, \lambda_2, \cdots, \lambda_n$ complex,*

$$\| \textstyle\sum_{j=1}^{n} \lambda_j e_j \| = \max_{1 \leq j \leq n} | \lambda_j |.$$

Proof. We first observe that it suffices to prove the theorem under the additional assumption that $\sum_{j=1}^{n} e_j = 1$. For if this sum is not 1, then one uses

$$\| \textstyle\sum_{j=1}^{n} \lambda_j e_j \| = \| O(1 - \sum_{j=1}^{n} e_j) + \sum_{j=1}^{n} \lambda_j e_j \|.$$

So we assume $\sum_{j=1}^{n} e_j = 1$. The linear span B of $e_1, e_2, \cdots, e_n$ is a subalgebra of A, and being finite-dimensional is closed. Clearly B is commutative and satisfies the hypotheses of (1.2). Let $\lambda_j = | \lambda_j | e^{i\theta_j}$, and put

$x = \sum_{j=1}^{n} \lambda_j e_j$. Then

$$x = [\exp (i \textstyle\sum_{j=1}^{n} \theta_j e_j)] \textstyle\sum_{j=1}^{n} | \lambda_j | e_j ,$$

where "exp" stands for "exponential of." Thus x is expressible in B in the form $x = e^{iu}h$, where u and h are hermitian. From the discussion atop page 126 of [13] we have $\| x \| = \| x \|_0$ ($\| \;\; \|_0$ being defined on B by (1.2)). Hence from (1.3) we have $\| x \| = r(x) = \max_{1 \leq j \leq n} | \lambda_j |$.

We next consider some aspects of scalar type operators. It is shown in [2] that if S is a scalar type operator on a Banach space X, then S has a unique decomposition $S = R + iJ$, where $RJ = JR$, and where, relative to some norm on X equivalent to the given norm, the operators $R^m J^n$ are hermitian for $m, n = 0, 1, 2, \cdots$. The operators R and J are called, respectively, the real and imaginary parts of S, and are, in fact, given by $R = \int \operatorname{Re} \lambda \, dE(\lambda)$, $J = \int \operatorname{Im} \lambda \, dE(\lambda)$, where E denotes the resolution of the identity for S.

In what follows prime superscripts will be used to denote adjoint spaces and adjoint operators.

(2.2) THEOREM. *Let S be a scalar type operator of class X' on the Banach space X, let E be its resolution of the identity, and let R and J be its real and imaginary parts, respectively. Then $E(\sigma)$ is hermitian for every Borel set σ in the complex plane if and only if the operators $R^m J^n$ are all hermitian* $(m, n = 0, 1, 2, \cdots)$.

Proof. If each $E(\sigma)$ is hermitian, then clearly so is each product $R^m J^n = \int (\operatorname{Re} \lambda)^m (\operatorname{Im} \lambda)^n \, dE(\lambda)$, since real linear combinations and uniform limits of hermitian operators are hermitian. Conversely, suppose for each polynomial p in two variables having real coefficients, $p(R, J)$ is hermitian. Choose a s.i.p. consistent with the norm of X. Let δ be a semi-closed rectangle in the complex plane. There is a sequence $\{p_n\}$ of polynomials in two variables with real coefficients tending pointwise to the characteristic function of δ on $sp_{[X]}(S)$ and uniformly bounded on $sp_{[X]}(S)$. Clearly

$$p_n(R, J) = \int p_n(\lambda) \; dE(\lambda).$$

For each $x \in X$ of norm 1, $[\;\;, x]$ is in X', and so

$$\lim_n \, [p_n(R, J)x, x] = \lim_n \int p_n(\lambda) \; d[E(\lambda)x, x] = [E(\delta)x, x].$$

Since $[p_n(R, J)x, x]$ is real, for each n, so is $[E(\delta)x, x]$. Thus $E(\delta)$ is hermitian. Now let C be the semi-ring of subsets of the complex plane consisting of the semi-closed rectangles and the empty set. Then the class D consisting of all sets expressible as a finite union of disjoint sets from C is a ring of subsets of the complex plane, generating the σ-ring of Borel sets. For any $x \in X$ of norm 1, the measure $[E(\ )x, x]$ coincides with the measure $\operatorname{Re}[E(\ )x, x]$ on D. It is now easy to see from [7, Theorem 13.A] that these measures coincide on the class of Borel sets. Hence the numerical range of each $E(\sigma)$ is real, and $E(\sigma)$ is hermitian.

Before proving the next theorem we take note of the well-known fact that if Ω is a compact Hausdorff space, then $C(\Omega)'$ is weakly complete (see, for example, [4, Theorem IV. 6.2 and Theorem IV. 9.9]). Hence if A is a commutative Banach algebra whose Gelfand representation is a bicontinuous isomorphism onto $C(\mathfrak{M}_A)$, then A' is weakly complete. We also observe the following known fact which does not appear explicitly in the literature: if T is a scalar type operator of class Γ on a weakly complete Banach space X, Γ being a total linear manifold in X', then T is automatically of class X'. For, if E is the resolution of the identity of T, and $\{\sigma_n\}$ is a sequence of disjoint Borel sets of the complex plane with union σ, then it follows from [1, Lemma 2.3 and Corollary 2.10] that for $x \in X$, $\sum_{n=1}^{\infty} E(\sigma_n)x$ converges in the norm topology. The value of each $x^* \in \Gamma$ at the sum of this series is the same as at $E(\sigma)x$, and so $\sum_{n=1}^{\infty} E(\sigma_n)$ converges strongly to $E(\sigma)$.

(2.3) THEOREM. *Let A be a commutative Banach algebra, and let H be the set of hermitian elements of A. If $A = H + iH$, then $\| t \| = r(t) = \| t \|_0$ for each $t \in A$, and so A is, with the given norm and the Vidav involution, a C^*-algebra.*

Proof. Let $t \in A$, with $t = r + ij$, where $r, j \in H$. Let L denote the regular representation of A, and set $T = L_t$, $R = L_r$, $J = L_j$. It is clear that the map which assigns to each $a \in A$ the operator L_a' is an isometric isomorphism of A onto a closed subalgebra W of $[A']$. By (1.3) the image of A under the regular representation is an algebra of operators on A equivalent to $C(\mathfrak{M}_A)$. It follows by [3, Theorem 18, conclusion (iii)] that W consists of scalar type operators on A' of class A. By (1.3) and the remarks preceding this theorem, we conclude that every operator belonging to W is of class A''. In particular T' is a scalar type operator of class A''. Since the products $r^m j^n$ $(m, n = 0, 1, 2, \cdots)$ are self-adjoint with respect to the Vidav involution, they belong to H, and hence the operators $R^m J^n$ are hermitian operators on A. It is obvious from the Vidav formulation of hermiticity that an operator is hermitian if and only if its adjoint is. Thus the operators $(R')^m (J')^n$ are hermitian. Clearly the operators R' and J' are, respectively, the real and imaginary parts of T'. By (2.2) the resolution of the identity for T' is hermitian-valued. It is clear from (2.1) that a scalar type operator whose resolution of the identity is hermitian-valued has norm equal to its spectral radius. Hence

$$\| t \| = \| T' \| = \lim_n \| (T')^n \|^{1/n} = \lim_n \| t^n \|^{1/n} = r(t).$$

This completes the proof.

(2.4) COROLLARY. *Let A be a commutative Banach algebra, and let H be the set of hermitian elements of A. The following are equivalent:*

(i) $A = H + iH$.

(ii) *There is an involution on A relative to which A becomes a C^*-algebra.*

(iii) *The algebra A is isometrically isomorphic to an algebra $C(\Omega)$, for some compact Hausdorff space Ω.*

3. Normal elements

DEFINITION. Let A be an arbitrary Banach algebra. An element $x \in A$ will be called a normal element of A if and only if x can be expressed in the form $x = u + iv$, where $uv = vu$ and the products $u^m v^n$ are hermitian for $m, n = 0, 1, 2, \cdots$.

It is worth noting that if an element x is merely expressible in the form $x = u + iv$, u and v hermitian, then such a representation is unique [13, Lemma 2(c)]. We observe that since the powers of a hermitian element are not always hermitian, it is not superfluous in the preceding definition to require more than the hermiticity of u and v together with their commutativity. However, if the Banach algebra A satisfies the hypotheses of (1.2), then an element x is normal in the above sense if and only if x can be expressed in the form $x = u + iv$, where u and v are commuting hermitians, or, alternatively, if and only if $xx^* = x^*x$.

(3.1) LEMMA. *Let A be a Banach algebra, and let B be a subset of A such that every $b \in B$ can be expressed in the form $b = u + iv$, where u and v are hermitian elements of A lying in B. Then every element c of $\bar{B}$ ($^-$denotes closure) can be expressed in the form $c = u_0 + iv_0$, where u_0 and v_0 are hermitian elements of A belonging to $\bar{B}$.*

Proof. The first paragraph in the proof of [2, Theorem (3.1)] is easily adapted to prove the lemma for the case where A is the Banach algebra $[X]$ of operators on an arbitrary Banach space X. The general case where A is an arbitrary Banach algebra follows by using the left regular representation to map A isometrically onto a closed subalgebra of $[A]$.

(3.2) THEOREM. *Let A be an arbitrary Banach algebra, and let x be a normal element of A having the (necessarily unique) decomposition $x = u + iv$, u and v hermitian. Let C be the closed subalgebra generated by 1, u, and v. Then for every $y \in C$, $\| y \| = r(y)$. In particular, if A satisfies the hypotheses of Vidav's theorem, and $xx^* = x^*x$, then $\| x \| = r(x) = \| x \|_0$.*

Proof. Since x is normal, u and v commute and the products $u^m v^n$ are hermitian for $m, n = 0, 1, 2, \cdots$. Let B be the class of polynomials with complex coefficients in u and v. Then B is a commutative subalgebra of A, and $C = \bar{B}$. C is commutative. Clearly Lemma (3.1) applies to B, and it follows that (2.3) applies to the Banach algebra C. To complete the proof we observe that if A satisfies the hypotheses of Vidav's theorem and $xx^* = x^*x$, then by [11, Lemma (4.8.1), conclusion (ii)] applied to A (regarded as a C^*-algebra with * and $\| \ \|_0$), we get $\| x \|_0 = r(x)$.

4. The general case

We shall call a Banach algebra satisfying the hypotheses of Vidav's theorem and equipped with the Vidav involution a V^*-algebra. Clearly a Banach *-algebra is V^* if and only if the set S of elements self-adjoint with respect to its involution coincides with the set H of elements hermitian in the sense of Vidav; moreover, it is easy to see with the aid of [13, Lemma 2(c)] that $S = H$ is equivalent to $S \subseteq H$. In this section we show that every V^*-algebra is C^*.

In [6, Theorem 11] C^*-algebras are characterized as those Banach *-algebras such that $\| x^*x \| = \| x^* \| \| x \|$ for each x. Let A be a Banach *-algebra such that $\| x^*x \| = \| x^* \| \| x \|$ whenever x and x^* commute. In [5], J. Feldman has observed that with only slight modification all but the last step in the proof of [6, Theorem 11] remains valid for A, and the following conclusions result:

(4.1)
(i) A has an equivalent Banach algebra norm $| \ |$ such that, relative to $| \ |$, A is a C^*-algebra.
(ii) $| x |^2 = \| x^*x \|$ for each x.
(iii) $| x | \leq \| x \|$ for each x.
(iv) $\| x \| = | x |$ if x is invertible or if $x^*x = xx^*$.

Recently B. Russo and H. A. Dye have shown [12, Theorem 1] that the unit ball of an arbitrary C^*-algebra B is the closed convex hull (in the norm topology) of the unitary elements of B (i.e., of the elements u such that $u^*u = uu^* = 1$). As an immediate consequence of this fact and (i), (iii), and (iv) of (4.1) we have:

(4.2) THEOREM. *A Banach *-algebra is C^* if and only if*

$$\| x^*x \| = \| x^* \| \| x \|$$

whenever x and x^* commute.

It is clear from (3.2) that in a V^*-algebra $\| x^*x \| = \| x^* \| \| x \|$ whenever x and x^* commute. Hence

(4.3) THEOREM. *A Banach *-algebra is V^* if and only if it is C^*.*

(4.4) COROLLARY. *A Banach algebra A satisfies the hypotheses of Theorem* (1.2) *if and only if there is an involution on A relative to which A becomes, with its given norm, a C^*-algebra.*

We remark that since, in a V^*-algebra, $\| x \|_0^2 = \| x^*x \|$, it is now clear that $\| \ \|_0$ and $\| \ \|$ coincide. This also now follows from the well-known fact that a *-algebra has at most one C^*-norm.

5. C^* as a local differential condition

Suppose A is a Banach *-algebra. In [9, Theorem 21] it is shown that if the condition

$$\| x^*x \| (\| x \| \| x^* \|)^{-1} = 1 + o(r), \qquad r = \| 1 - x \|,$$

holds near 1, then A is, within equivalent renorming, a C^*-algebra. The method of proof employed in [9] is to show that the set S of elements self-adjoint with respect to the given involution is the same as the set H of elements hermitian in the sense of Vidav, and then to employ the theorem of Vidav. Thus such an algebra is V^*, and hence by (4.3) we can state:

(5.1) THEOREM. *A Banach *-algebra is C^* if and only if the condition*

$$\| x^*x \| (\| x \| \| x^* \|)^{-1} = 1 + o(r), \qquad r = \| 1 - x \|,$$

holds near 1.

Added in proof: It has just come to the author's attention that, in essence, Theorem (2.3) was announced earlier by Mr. B. W. Glickfeld in the A. M. S. Notices, vol. 11 (1964), p. 51. His proof, by a different method from the one in this paper, was incorporated in his April 1964 doctoral dissertation at Columbia University. Also, other results of this paper are announced by him in the A. M. S. Notices, vol. 13 (1966), p. 52.

REFERENCES

1. W. G. BADE, *On Boolean algebras of projections and algebras of operators*, Trans. Amer. Math. Soc., vol. 80 (1955), pp. 345–360.
2. E. BERKSON, *A characterization of scalar type operators on reflexive Banach spaces*, Pacific J. Math., vol. 13 (1963), pp. 365–373.
3. N. DUNFORD, *Spectral operators*, Pacific J. Math., vol. 4 (1954), pp. 321–354.
4. N. DUNFORD AND J. T. SCHWARTZ, *Linear operators—Part I: General theory*, New York, Interscience, 1958.
5. J. FELDMAN, Seminar notes, University of California, Berkeley, 1962, dittoed.
6. J. G. GLIMM AND R. V. KADISON, *Unitary operators in C^*- algebras*, Pacific J. Math., vol. 10 (1960), pp. 547–556.
7. P. R. HALMOS, *Measure theory*, New York, Van Nostrand, 1950.
8. L. H. LOOMIS, *An introduction to abstract harmonic analysis*, New York, Van Nostrand, 1953.
9. G. LUMER, *Semi-inner-product spaces*, Trans. Amer. Math. Soc., vol. 100 (1961), pp. 29–43.
10. ———, *Spectral operators, Hermitian operators, and bounded groups*, Acta Sci. Math., vol. 25 (1964), pp. 75–85.
11. C. E. RICKART, *General theory of Banach algebras*, New York, Van Nostrand, 1960.
12. B. RUSSO AND H. A. DYE, *A note on unitary operators in C^*-algebras*, to appear in Duke Math J.
13. I. VIDAV, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Zeitschr., vol. 66 (1956), pp. 121–128.

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