SEMI-GROUPS OF SCALAR TYPE OPERATORS AND A THEOREM OF STONE

BY
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1. Introduction

In this paper we show (§4) that if $\{U_t\}$ is a strongly continuous one-parameter group of scalar type operators on a weakly complete Banach space X with spectra contained in the unit circumference and such that their resolutions of the identity are uniformly bounded in norm, then there is a spectral measure E of class X^* on the family of Borel sets of the real line such that

$$x^* U_t x = \int e^{it\lambda}\, dx^* E(\lambda) x, \qquad x \in X,\ x^* \in X^*,\ t \text{ real}.$$

By special consideration of the case where $\{U_t\}$ is a group of unitary operators on a Hilbert space, our work yields (and so generalizes) a well-known theorem of M. H. Stone ([11; pages 173, 174] and [12]). Our work is related in spirit to [8; §5], although we assume weak completeness of X rather than reflexivity, and we obtain as a result rather than assume that the resolutions of the identity for the U_t generate a bounded Boolean algebra of projections.

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In what follows, all spaces are over the complex field, and an operator T in a Banach space X will be a linear transformation (not necessarily continuous) with domain and range contained in X. We shall denote the domain, spectrum, resolvent set, and resolvent (evaluated at λ) of T by $D(T)$, $\sigma(T)$, $\rho(T)$, and $R(\lambda; T)$, respectively. We shall use the symbol I for the identity operator, and the symbol $[X]$ for the algebra of continuous operators on the Banach space X. The set of real numbers will be designased by R_0, and the set of pure-imaginary numbers, $\{it \mid t \in R_0\}$, by J. Our terminology concerning semi-groups and groups of operators will be that of [5; Ch. VIII]. Unless otherwise stated, all semi-groups and groups occurring below will be understood to be strongly continuous.

Frequent use will be made of the operational calculus for unbounded scalar type operators introduced in [1; §3]. This operational calculus is further considered in [6], where, for example, it is shown that a Borel function of an unbounded scalar type operator is of scalar type.

We shall employ the following result [6; XVII. 2.5], which strengthens [4; Theorem 18, conclusion (iv)], and which we list here for ease of reference:

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(1.1) THEOREM. *Let $\mathfrak{A}$ be an algebra of bounded operators on a weakly complete Banach space X, $\mathfrak{A}$ being the image under a continuous homomorphism S of the algebra $C(\mathfrak{M})$ of all continuous complex-valued functions on the compact Hausdorff space $\mathfrak{M}$. Then there is a spectral measure E of class X^* on the Borel sets of $\mathfrak{M}$ such that*

$$S(f) = \int_{\mathfrak{M}} f(\lambda)\, dE(\lambda), \qquad f \in C(\mathfrak{M}).$$

2. On a theorem of Bade

In this section we obtain a version of [2; Theorem 2.3] covering the case of a possibly unbounded limit operator.

(2.1) LEMMA. *Let T be an operator in the Banach space X, and let $\mu \in \rho(T)$. Then T is a scalar type operator if and only if $R(\mu;\, T)$ is of scalar type.*

Proof. Suppose T is of scalar type (in particular, T is closed). Consider the functions $f(\lambda) = \mu - \lambda$ and $g(\lambda) = (\mu - \lambda)^{-1}$. Applying the operational calculus of T, one finds that $g(T)$ is a bounded scalar type operator on X, $f(T) = \mu I - T$, $D(g(T)f(T)) = D(T)$, and $g(T)(\mu I - T)x = x$, for $x \in D(T)$. Thus the scalar type operator $g(T)$ coincides with $R(\mu;\, T)$. Conversely, suppose $R(\mu;\, T)$ is of scalar type. In particular, $R(\mu;\, T)$ is densely defined, continuous, and closed. So $R(\mu;\, T) \in [X]$. Let E denote the resolution of the identity for $R(\mu;\, T)$. Since $R(\mu;\, T)$ is one-to-one, $E(\{0\}) = 0$. Consider the function $h(\lambda) = \lambda$. Since the set of zeros of h has spectral measure 0, $h(R(\mu;\, T)) = R(\mu;\, T)$ has an inverse given by $(1/h)(R(\mu;\, T))$. This inverse is therefore a scalar type operator, and clearly must be $\mu I - T$. It follows that T is of scalar type.

Throughout the remainder of this section our terminology is that of [2; §2]. Before taking up the theorem of this section, however, it is necessary to discuss the fact that [2; Theorem 2.3] is incorrect as it stands. This fact has been communicated to Bade by C. Foias and A. Lebow. Lebow's counter-example is as follows. On the Hilbert space l^2, define the operators T and T_n $(n = 1, 2, \cdots)$ by

$$\begin{aligned} T_n x &= (x_n, x_1, \cdots, x_{n-1}, x_{n+1}, x_{n+2}, \cdots), \\ Tx &= (0, x_1, \cdots, x_{n-1}, x_n, x_{n+1}, \cdots), \end{aligned} \qquad \text{for} \quad x = (x_1, x_2, \cdots, x_n, \cdots).$$

Then $\{T_n\}$ is a sequence of unitary operators converging strongly to T. Since each $\sigma(T_n)$ is contained in the unit circumference (an R-set), and $\sigma(T)$ is the unit disc, [2; Lemma 2.4] fails. Moreover, the adjoint T^* is given by $T^*x = (x_2, x_3, \cdots)$ and this operator is not spectral (see the example of Kakutani in [4; page 326]). Hence T is not a scalar type operator, and so [2; Theorem 2.3] fails. It is straightforward to see that the proof of Bade's theorem is valid if one assumes at the outset that (in the notation of [2]) $\sigma(T) \subseteq V$, or some condition sufficient to insure this inclusion. One such condition, due to Foias, is that the complement of V have no bounded component. The demonstration that this condition insures $\sigma(T) \subseteq V$ is incorporated in the proof of the following theorem.

(2.2) THEOREM. *Let X be a weakly complete Banach space, and let $\{T_\alpha\}$, $\alpha \in A$, be a net of bounded scalar type operators on X with spectra contained in some fixed R-set V. We assume that the resolutions of the identity E_α for T_α are uniformly bounded in norm (i.e., there is a number M such that $\| E_\alpha(\delta) \| \leq M$ for $\alpha \in A$, and δ in the class B of Borel sets of the complex plane). Let T be a closed operator in X such that $\lim_\alpha T_\alpha x = Tx$ for $x \in D(T)$, and let $\sigma(T) \subseteq V$. Then T is a scalar type operator. If E denotes the resolution of the identity for T, then for $\mu \notin V$, $x \in X$, $x^* \in X^*$,*

$$x^*R(\mu; T)x = \int_V (\mu - \lambda)^{-1} \, dx^*E(\lambda)x. \tag{2.3}$$

In order for $\sigma(T)$ to be a subset of V it is necessary and sufficient that each component of the complement of V intersect $\rho(T)$. If X is a Hilbert space, and each T_α is normal, then T is normal.

Proof. Our proof that T is of scalar type is patterned after the demonstration of [6; Theorem XVII. 4.1] for the case of T bounded on X. The proof of [2; Lemma 2.4] shows that for $\lambda \notin V$, $x \in D(T)$,

$$\|(\lambda I - T)x \| \geq \| x \|(4M)^{-1} d(\lambda, V),$$

and, just as in that proof, one has for $\lambda \notin V$, $\alpha \in A$, and $x \in X$, that

$$\| R(\lambda; T_\alpha)\|, \| R(\lambda; T)\| \leq 4M[d(\lambda, V)]^{-1}$$

and

$$\lim_\alpha R(\lambda; T_\alpha)x = R(\lambda; T)x.$$

The standard operational calculus for an arbitrary closed operator with non-void resolvent set (see, e.g., [5; §VII. 9]) will now be used for the operator T. In the notation of this operational calculus, one sees from the foregoing that if f belongs to the subalgebra A of $C_\infty(V)$ generated by the class

$$\{g \mid g(\lambda) = (\mu - \lambda)^{-1}, \mu \notin V\},$$

then $\lim_\alpha f(T_\alpha)x = f(T)x$, for $x \in X$. Moreover, since

$$f(T_\alpha) = \int_V f(\lambda) \, dE_\alpha(\lambda),$$

it follows that

$$\| f(T)\| \leq 4M(\sup_{\lambda \in V} | f(\lambda)|).$$

Since V is an R-set, A is dense in $C_\infty(V)$, and so the homomorphism $f \to f(T)$ of the algebra A into [X] extends to a continuous homomorphism of $C_\infty(V)$. This latter homomorphism, in turn, can be extended to a continuous homomorphism into [X] of $C(V_\infty)$, where V_∞ denotes the one-point compactification of V. By (1.1) and [4; Lemma 6], we conclude that for $\mu \notin V$, $R(\mu; T)$ is a scalar type operator. By (2.1), T is of scalar type. (2.3) follows from the necessity proof of (2.1).

Next we show that if each component of V', the complement of V, intersects $\rho(T)$, then $\sigma(T) \subseteq V$. The argument we shall use is essentially due to Foias. Denote by $\rho_0(T)$ the set of all complex λ such that for some $\varepsilon_\lambda > 0$

$$\|(\lambda I - T)x\| \geq \varepsilon_\lambda \|x\|, \qquad x \in D(T).$$

Denote the complement of $\rho_0(T)$ by $\sigma_0(T)$. Then by [13; Theorem 5.1-D], the boundary of $\sigma(T)$ is contained in $\sigma_0(T)$. The proof of [2; Lemma 2.4] shows that $V' \subseteq \rho_0(T)$. If a point λ of $\sigma(T)$ should lie in a component of V', then we could connect λ to a point of $\rho(T)$ by an arc contained in V'. This arc would contain a boundary point of $\sigma(T)$, which must be in $\sigma_0(T)$, and yet, being in V', must belong to $\rho_0(T)$. This contradiction establishes $\sigma(T) \subseteq V$.

To complete the proof of the theorem, we assume that X is a Hilbert space, and each T_α is normal. The proof of [2; Lemma 2.5] shows that for each $x \in X$, there is a regular measure ρ_x on the Borel sets of V such that for $\mu \notin V$,

$$(2.4) \qquad (R(\mu; T)x, x) = \int_V (\mu - \lambda)^{-1} \, d\rho_x(\lambda).$$

Moreover, ρ_x is a cluster point of the net $\{(E_\alpha(\quad)x, x)\}$ in the weak*-topology of the dual space of $C_\infty(V)$. Since $E_\alpha(\delta)$ is Hermitian for $\alpha \in A$, $\delta \in B$ it follows that ρ_x is positive. By (2.3), (2.4), and the fact that V is an R-set, we have that $\rho_x(\delta) = (E(\delta)x, x)$ for $x \in X$, $\delta \in B$. So the resolution of the identity for T assumes only Hermitian values, and T is normal.

3. Semi-groups with generators of scalar type

(3.1) THEOREM.[2] *Let $\{T_t\}$, $t \geq 0$, be a semi-group of bounded operators on a Banach space X, with infinitesimal generator T. If T is a scalar type operator, then each T_t is of scalar type, and*

$$x^* T_t x = \int e^{t\lambda} \, dx^* E(\lambda) x, \qquad x \in X, x^* \in X^*, t \geq 0,$$

where E denotes the resolution of the identity for T.

Proof. For each $t \geq 0$, let the function f_t be defined on the complex plane by $f_t(\lambda) = e^{t\lambda}$. By [5; Theorem VIII.1.11] there is a real number w such that $\sigma(T)$ is contained in the set $W = \{\lambda \mid \operatorname{re} \lambda \leq w\}$. Since $E(\sigma(T)) = I$ (by [1; Lemma 3.1]), each f_t is E-essentially bounded. It follows that, in terms of the operational calculus of the scalar type operator T, each $f_t(T)$ is a bounded

[2] The referee has informed the author that a proof of Theorem (3.1) was given by Lyle H. Lanier, Jr., in his unpublished dissertation submitted to the University of Illinois in January, 1964.

scalar type operator, and

$$(3.2) \qquad x^*f_t(T)x = \int_W e^{t\mu}\, dx^*E(\mu)x, \qquad x \in X, x^* \in X^*, t \geq 0.$$

To complete the proof we show that $f_t(T) = T_t$ for $t \geq 0$. It is easy to see that $\{f_t(T)\}$, $t \geq 0$, is a weakly continuous semi-group, and hence, by [7; page 306], is strongly continuous. Clearly $\| f_t(T) \| e^{-wt}$ is bounded for $t \geq 0$. Application of (3.2) and interchange of the order of integration give, for $x \in X$, $x^* \in X^*$, re $\lambda > w$,

$$x^*\left[\int_0^\infty e^{-\lambda t} f_t(T)x\, dt\right] = \int_W (\lambda - \mu)^{-1}\, dx^*E(\mu)x = x^*R(\lambda; T)x.$$

By [5; Corollary VIII.1.16], T is the infinitesimal generator of $\{f_t(T)\}$. Hence $f_t(T) = T_t$, for $t \geq 0$.

4. A generalization of Stone's theorem

Throughout this section we assume that $\{U_t\}$, $t \in R_0$, is a group of scalar type operators on a Banach space X such that:

(i) Each U_t has its spectrum contained in the unit circumference

$$\{\lambda \mid |\lambda| = 1\}.$$

(ii) The resolutions of the identity F_t for U_t are uniformly bounded in norm by a constant M.

(4.1) LEMMA. *The generator T of $\{U_t\}$ has purely imaginary spectrum, and for $x \in X$, $x^* \in X^*$,*

$$x^*R(\lambda; T)x = \int_0^\infty e^{-\lambda t} x^* U_t x\, dt, \qquad \textit{if } \text{re } \lambda > 0,$$

$$= -\int_0^\infty e^{\lambda t} x^* U_{-t} x\, dt, \qquad \textit{if } \text{re } \lambda < 0.$$

Proof. Clearly the commutative group $\{U_t\}$ is bounded in norm by $4M$. By [9; proof of Theorem 6] X can be renormed with an equivalent norm which makes each U_t an isometry. We shall assume for purposes of this lemma that this has been done. T generates $\{U_t\}$, $t \geq 0$, and $-T$ generates $\{U_{-t}\}$, $t \geq 0$. Clearly

$$\lim_{t\to\infty} t^{-1} \log \| U_t \| = \lim_{t\to\infty} t^{-1} \log \| U_{-t} \| = 0.$$

Application of [5; Theorem VIII.1.11] completes the proof of this lemma.

We now express the generator T of the group $\{U_t\}$ in the form $T = iA$, where A is likewise a closed operator with domain $D(T)$. By (4.1), $\sigma(A)$ is real. For each t we can also write $U_t = \int_{R_0} e^{i\lambda}\, dG_t(\lambda)$, where G_t is a spectral measure of class X^* on the family B_0 of Borel sets of R_0, satisfying

$\| G_t(\delta) \| \leq M$ for $\delta \in B_0$, and $G_t([0, 2\pi]) = I$. We have $U_t = e^{iA_t}$, where A_t is the scalar type operator given by $\int_{R_0} \lambda \, dG_t(\lambda)$.

(4.2) THEOREM. *Suppose X is weakly complete. Then the generator T of $\{U_t\}$, $t \in R_0$, is of scalar type. There is a spectral measure E of class X^* on the family B_0 of Borel sets of R_0 such that*

$$x^* U_t x = \int_{R_0} e^{it\lambda} \, dx^* E(\lambda) x, \qquad x \in X, x^* \in X^*, t \in R_0. \tag{4.3}$$

The spectral measure E is uniquely determined, and is the restriction to B_0 of the resolution of the identity for the scalar type operator $-iT$. If X is a Hilbert space, and each U_t is unitary, then the values of E are all Hermitian.

Proof. We have for all $x \in D(T)$,

$$Tx = \lim_{t \to 0^+} t^{-1}(e^{iA_t} - I)x, \qquad -Tx = \lim_{t \to 0^+} t^{-1}(e^{-iA_t} - I)x.$$

Subtraction of the second of these equations from the first and division by 2 give:

$$Tx = \lim_{t \to 0^+}(it^{-1} \sin A_t)x.$$

Let $B_t = it^{-1} \sin A_t$ for $t > 0$. Clearly each B_t is a scalar type operator whose resolution of the identity is bounded by M. Also, $\sigma(B_t) \subseteq J$. By (4.1) $\sigma(T) \subseteq J$, and so we have from (2.2) that T is a scalar type operator. Denote by H the resolution of the identity for T. Application of (3.1) to the semi-groups $\{U_t\}$ and $\{U_{-t}\}$, $t \geq 0$, gives

$$x^* U_t x = \int_J e^{t\lambda} \, dx^* H(\lambda) x, \qquad x \in X, x^* \in X^*, t \in R_0.$$

It is now clear that (4.3) holds, with E denoting the restriction to B_0 of the resolution of the identity for $A = -iT$.

Suppose E_0 is also a spectral measure satisfying (4.3). Application of (4.1) gives the result:

$$x^* R(\lambda; T)x = \int_0^\infty dt \int_{R_0} e^{-\lambda t} e^{i\mu t} \, dx^* E_0(\mu) x, \quad x \in X, x^* \in X^*, \text{re } \lambda > 0.$$

After interchanging the order of integration, we obtain:

$$x^* R(\lambda; T)\, x = \int_{R_0} (\lambda - i\mu)^{-1} \, dx^* E_0(\mu) x. \tag{4.4}$$

A similar calculation with re $\lambda < 0$ shows that (4.4) is valid if $\lambda \notin J$. From the fact that E is the restriction to B_0 of the resolution of the identity for $-iT$, it is easy to see that:

$$x^* R(\lambda; T)x = \int_{R_0} (\lambda - i\mu)^{-1} \, dx^* E(\mu) x, \qquad x \in X, x^* \in X^*, \lambda \notin J. \tag{4.5}$$

From (4.4), (4.5), and the fact that the real line is an R-set, we see that the measures $x^*E_0(\ )x$ and $x^*E(\ )x$ coincide for arbitrary x and x^*. Hence $E_0 = E$.

To conclude the proof we observe that if X is a Hilbert space, and each U_t is unitary, then each B_t is normal, and so by (2.2) T is normal. Thus the resolution of the identity for A is Hermitian-valued.

(4.6) Corollary. *An operator $C \in [X]$ commutes with each U_t if and only if C commutes with each value of E.*

The straightforward proof of (4.6) will be omitted.

(4.7) Corollary. *The range of each F_t is contained in the range of E, and so the resolutions of the identity for the operators U_t generate a bounded Boolean algebra of projections in $[X]$.*

Proof. By (4.3) and [4; Lemma 6].

For the sufficiency proof of the next corollary we shall use the notions of bounded generalized Hermitian operator on a Banach space, of semi-inner-product, and of dissipative operator. We shall not take up space here for a discussion of these notions, but we refer the reader to [3; pages 365, 366] and to [10] for such a discussion. We shall also use the fact that, as pointed out in [10; page 681], a scalar type operator T has the property that

$$\{x \in D(T^\infty) \mid \| T^n x \|^{1/n} = o(n)\}$$

is dense in the underlying Banach space.

(4.8) Corollary. *An operator T in a weakly complete Banach space X generates a strongly continuous one-parameter group of scalar type operators with spectra contained in the unit circumference and resolutions of the identity uniformly bounded in norm if and only if T is a scalar type operator with $\sigma(T) \subseteq J$.*

Proof. If T generates such a group, then by (4.1) and (4.2) T has the desired properties. Conversely, if T is a scalar type operator with $\sigma(T) \subseteq J$, Then the resolution of the identity for T can be made into a Hermitian family (in the generalized sense) by equivalent renorming of X (see Lemmas 2.2 and 2.3 of [3], which apply to any bounded Boolean algebra of projections). Thus, after introduction of an appropriate semi-inner-product for X, the operators T and $-T$ will be dissipative. It now follows by [10; Theorem 3.2] that the scalar type operators T and $-T$ are generators of semi-groups. Hence by the Hille-Yosida-Phillips Theorem and [5; Corollary VIII.1.17], T generates a group $\{V_t\}$, $t \in R_0$, of bounded operators on X. By (3.1) each V_t is of scalar type, and there is a spectral measure E of class X^* on B_0 such that (4.3) holds with $\{V_t\}$ in place of $\{U_t\}$. It is clear that each $\sigma(V_t)$ is contained in the unit circumference. Finally, by [4; Lemma 6], the range of the resolution of the identity for each V_t is contained in the range of E.

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