A TRANSPLANTATION THEOREM BETWEEN ULTRASPHERICAL SERIES[1]

BY

RICHARD ASKEY AND STEPHEN WAINGER

1. Introduction

In the introduction we shall describe our results for Legendre and cosine series. Analogous results hold for ultraspherical series but in the interest of simplicity we state them here only in the most important special case.

$P_n(x)$ is the Legendre polynomial of degree n. The functions

$$(n + \tfrac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2}$$

are orthonormal functions on $(0, \pi)$. They also have the known asymptotic formula [16, Th. 8.21.5]

$$(n + \tfrac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2} = A\cos[(n + \tfrac{1}{2})\theta - \pi/4] + O(1/(n\sin\theta)),$$

$0 < \theta < \pi$. Classically this has been used to set up equiconvergence theorems between Legendre series and cosine series, but only for

$$0 < \varepsilon \leqq \theta \leqq \pi - \varepsilon < \pi.$$

While it isn't possible to get uniform equiconvergence theorems for $0 \leqq \theta \leqq \pi$, it is possible to get a theorem that uses all θ, $0 \leqq \theta \leqq \pi$.

Let $f(\theta)$ be a function in $L^{p,\alpha}(0, \pi)$ where $L^{p,\alpha}$ is the class of measurable functions for which

$$\|f\|_{p,\alpha} = \left[\int_0^\pi |f(\theta)|^p (\sin\theta)^{\alpha p}\, d\theta\right]^{1/p}$$

is finite. In all that follows we will have

$$1 < p < \infty \quad \text{and} \quad -1/p < \alpha < 1 - 1/p.$$

These are the familiar conditions that are necessary to have the Hilbert transform a bounded operator. Also, if $f \in L^{p,\alpha}$ then $f \in L^{1,0}$, so we may talk about its Fourier series. Let

$$a_n = \frac{1}{\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta. \tag{1}$$

Then

$$f(\theta) \sim a_0/2 + \textstyle\sum_{n=1}^\infty a_n \cos n\theta.$$

Since $(n + \frac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2}$ behaves about like $\cos n\theta$ we set

$$T_r f(\theta) = \textstyle\sum_{n=0}^\infty a_n r^n (n + \frac{1}{2})^{1/2} P_n(\cos\theta)(\sin\theta)^{1/2}$$

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with a_n defined by (1). Then our main theorem is that $\lim_{r\to 1} T_r f(\theta)$ exists a.e. and in $L^{p,\alpha}$ norm. If we call this function $Tf(\theta)$, then

$$\| Tf \|_{p,\alpha} \leqq A \| f \|_{p,\alpha}$$

where A depends on p and α but not on $f \in L^{p,\alpha}$.

In order to obtain any results for Legendre series we need the transplantation theorem in the opposite direction also. Let $f(\theta)$ be as above and define

$$b_n = \int_0^\pi f(\theta) \left(n + \frac{1}{2}\right)^{1/2} P_n (\cos \theta) (\sin \theta)^{1/2} \, d\theta.$$

Then if we set

$$S_r f(\theta) = b_0/2 + \textstyle\sum_{n=1}^\infty b_n r^n \cos n\theta$$

we have

$$\lim_{r\to 1} S_r f(\theta) = Sf(\theta) \quad \text{a.e. and in } L^{p,\alpha} \text{ norm,}$$

and

$$\| Sf \|_{p,\alpha} \leqq A \| f \|_{p,\alpha} .$$

The first theorem of this type is due to D. Guy [5] and is a transplantation theorem for Hankel transforms.

Before we mention some of the applications of these results, let us give an indication as to how these theorems are proven. We have not been able to give a proof which uses just the asymptotic formula. However, there is another connection between $P_n(\cos \theta)$ and $\cos n\theta$ given by Mehler's formula [4, p. 182 (43)]

$$P_n (\cos \theta) = 2^{1/2}\pi^{-1} \int_0^\theta (\cos \varphi - \cos \theta)^{-1/2} \cos \left(n + \frac{1}{2}\right) \varphi \, d\varphi.$$

Using this in the series

$$\textstyle\sum a_n(n + \frac{1}{2})^{1/2} P_n(\cos \theta)(\sin \theta)^{1/2}$$

we obtain

$$(2) \quad \int_0^\theta \left[\sum a_n \left(n + \frac{1}{2}\right)^{1/2} \cos \left(n + \frac{1}{2}\right) \varphi \right] (\cos \varphi - \cos \theta)^{1/2} (\sin \theta)^{1/2} \, d\varphi.$$

Now the series $\sum a_n(n + \frac{1}{2})^{1/2} \cos(n + \frac{1}{2})\varphi$ is closely related to the fractional derivative of order one-half of $\sum a_n \cos n\varphi$. The integral (2) is also closely related to the fractional integral of order one-half. Our proof consists in unscrambling these two operators.

Since $\cos n\theta$ is essentially the ultraspherical polynomial of order 0, we have described a transformation between ultraspherical series. In Section 3 we state and prove a transplantation theorem between ultraspherical series for the parameters λ, $0 < \lambda < 1$. In the next section we state a closely related result of B. Muckenhoupt and E. Stein [10] which shows how to transplant between λ and $\lambda + 1$. Their work allows us to extend our theorem for all $\lambda > 0$.

A special case of Muckenhoupt and Stein's work is the usual conjugate function theorem of M. Riesz. Thus it is natural to expect that a Hilbert transform will arise in our work. When thought about in the context of spherical harmonics what we have done is to set up a mapping between zonal functions on spheres of different dimensions. This raises the interesting question of whether such mappings can be set up for more general functions than zonal functions. This seems to be a problem of a completely different order of magnitude than the one we solve.

Some of our applications follow.

COROLLARY 1. *Let* $\{t_n\}$ *be a sequence of real numbers such that*

$$| t_N | \leqq A \quad \text{and} \quad \sum_{2^N+1}^{2^{N+1}} | t_n - t_{n-1} | \leqq A, \quad N = 1, 2, \cdots .$$

Then if $f \in L^{p,\alpha}$, $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, *and*

$$f(\theta) \sim \sum a_n(n + \tfrac{1}{2})^{1/2}P_n(\cos\theta)(\sin\theta)^{1/2},$$

i.e.,

$$a_n = \int_0^\pi f(\theta)\left(n + \frac{1}{2}\right)^{1/2} P_n\,(\cos\theta)\,(\sin\theta)^{1/2}\,d\theta,$$

then

$$\sum t_n\,a_n(n + \tfrac{1}{2})^{1/2}P_n(\cos\theta)(\sin\theta)^{1/2}$$

is the Legendre expansion of a function $Tf(\theta) \in L^{p,\alpha}$ *and*

$$\| Tf \|_{p,\alpha} \leqq A\,\| f \|_{p,\alpha}\,.$$

This is the analogue of a well known result of Marcinkiewicz for Fourier series, [8], [19]. For weighted L^2 spaces it was proven by Hirschman [9] and the first proof for L^p spaces is due to Muckenhoupt and Stein [10].

Specializing t_n to be one for $n \leqq N$ and zero for $n > N$, we get a new proof of Pollard's result on mean convergence [11].

A different type of example is the analogue of the Hardy-Littlewood theorem for series with monotone coefficients.

COROLLARY 2. *If* $f(\theta) \sim \sum a_n(n + \frac{1}{2})P_n(\cos\theta)$, a_n *is monotone decreasing and* $n^{1/2}a_n \to 0$ *then*

$$\int_0^\pi | f(\theta) |^p\,(\sin\theta)^{\alpha p}\sin\theta\,d\theta < \infty$$

if and only if

$$\sum_{n=0}^{\infty} a_n^p(n + \tfrac{1}{2})^{(2-\alpha)p-3} < \infty, \quad 1 < p < \infty, (p - 4)/2 < \alpha p < (3p - 4)/2.$$

These and other applications will be treated in detail in Section 4.

In conclusion we would like to thank Professor Bochner for suggesting to one of us that a theorem of this type might be true and Professor Stein for allowing us to see a manuscript of [10] before publication.

2. Preliminary material

We use a number of facts about ultraspherical polynomials, trigonometric series, and trigonometric functions. For the sake of easy reference, we compile these facts. They are either known or easy to prove.

The ultraspherical polynomial, $P_n^\lambda(x)$, of index λ, degree n, is defined by

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} P_n^\lambda(x)t^n.$$

For fixed λ they satisfy

$$(1) \qquad \int_{-1}^{1} P_n^\lambda(x)P_m^\lambda(x)(1 - x^2)^{\lambda-1/2}\,dx = \frac{(2\lambda)_n\Gamma(\frac{1}{2})\Gamma(\lambda + \frac{1}{2})}{n!\,(\lambda + n)\Gamma(\lambda)}\,\delta_{n,m}$$

where $(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha)$ and $\delta_{n,m}$ is one if $n = m$; otherwise, it is zero. We will usually consider $P_n^\lambda(\cos\theta)$. These functions are orthogonal with respect to $(\sin\theta)^{2\lambda}\,d\theta$. In fact, we will usually consider the orthonormal functions

$$t_n^\lambda(\sin\theta)^\lambda P_n^\lambda(\cos\theta)$$

where

$$t_n^\lambda = \left[\frac{n!\,(n + \lambda)\Gamma(\lambda)}{(2\lambda)_n\Gamma(\frac{1}{2})\Gamma(\lambda + \frac{1}{2})}\right]^{1/2}.$$

Observe that

$$t_n^\lambda = A(\lambda)n^{1-\lambda} + O(n^{-\lambda})$$

and if necessary the O term may be replaced by

$$Bn^{-\lambda} + Cn^{-\lambda-1} + O(n^{-\lambda-2}).$$

This follows from known estimates for $\Gamma(n + a)/\Gamma(n + b)$.

We need the following asymptotic formula for $P_n^\lambda(\cos\theta)$.

LEMMA 1. *For* $\delta \leqq \theta \leqq \pi - \delta$, $\delta > 0$, $0 < \lambda < 1$, *we have*

$$P_n^\lambda(\cos\theta) = \frac{c\Gamma(n + 2\lambda)}{\Gamma(n + \lambda + 1)}\,\frac{\cos\left\{(n + \lambda)\theta - \pi\frac{\lambda}{2}\right\}}{(\sin\theta)^\lambda} + O\left(\frac{n^{\lambda-2}}{(\sin\theta)^{\lambda+1}}\right).$$

See [16, p. 195, Th. 8.21.11].

We also need two estimates for $P_n^\lambda(\cos\theta)$.

LEMMA 2. *For* $\lambda > 0$,

$$|P_n^\lambda(\cos\theta)| \leqq P_n^\lambda(1) = \Gamma(n + 2\lambda)/\Gamma(n + 1)\Gamma(2\lambda),$$

and

$$(\sin\theta)^\lambda\,|P_n^\lambda(\cos\theta)| \leqq An^{\lambda-1},$$

[16, Th. 7.33.1 and formula (7.33.6)].

We also use one form of Mehler's formula:

$$P_n^\lambda(\cos\theta) = \frac{2^\lambda\Gamma(\lambda+\frac{1}{2})(2\lambda)_n}{\pi^{1/2}n!\,\Gamma(\lambda)}(\sin\theta)^{1-2\lambda}\int_0^\theta \frac{\cos(n+\lambda)\varphi\,d\varphi}{[\cos\varphi-\cos\theta]^{1-\lambda}}$$

for $\lambda > 0$, [4, p. 177].

From the theory of Fourier series we need the following lemma.

LEMMA 3. *Let $0 < \alpha < 1$. Then if*

$$f(\theta) = \begin{cases} \sum_{n=1}^\infty n^{-\alpha}\cos n\theta \\ \sum_{n=1}^\infty n^{-\alpha}\sin n\theta, \end{cases}$$

the series converge uniformly in $\delta \leqq |\theta| \leqq 2\pi - \delta$, $\delta > 0$. The sum is

$$C_\alpha|\theta|^{\alpha-1}\begin{Bmatrix} 1 \\ \operatorname{sgn}\theta \end{Bmatrix} + g(\theta)$$

where $g(\theta)$ is infinitely differentiable in $|\theta| \leqq 2\pi - \delta$. Also for any $\delta > 0$, the Abel means of the series for $f(\theta)$ and the series for $f'(\theta)$ converge boundedly to $f(\theta)$ and the derivative of $f(\theta)$ respectively in $\delta \leqq |\theta| \leqq 2\pi - \delta$.

This follows from the results in §1 of [17].

We also need two elementary lemmas which can easily be established by the reader.

LEMMA 4. *For $0 < \alpha < 1$,*

$$\frac{|\theta|^\alpha - |\varphi|^\alpha}{\theta-\varphi} = O\left\{\frac{1}{[|\theta|+|\varphi|]^{1-\alpha}}\right\}.$$

LEMMA 5. *Let $0 \leqq u \leqq \theta \leqq \pi/2$, $0 < \alpha < 1$. Then*

(1) $$|[\cos(\theta-u)-\cos\theta]^{-\alpha} - [u\sin\theta]^{-\alpha}| = O(u^{1-\alpha}/\theta^{1+\alpha}),$$

(2) $$\left|\frac{\partial}{\partial\theta}\{[\cos(\theta-u)-\cos\theta]^{-\alpha} - [u\sin\theta]^{-\alpha}\}\right| = O(u^{1-\alpha}/\theta^{2+\alpha}),$$

(3) $$\left|\frac{\partial}{\partial u}\{[\cos(\theta-u)-\cos\theta]^{-\alpha} - [u\sin\theta]^{-\alpha}\}\right| = O(u^{-\alpha}/\theta^{1+\alpha}).$$

In addition to the asymptotic formula of $P_n^\lambda(\cos\theta)$ in terms of $\cos n\theta$, we need a formula of Hilb type which gives us $P_n^\lambda(\cos\theta)$ in terms of

$$J_{\lambda-1/2}((n+\lambda)\theta),$$

where $J_\alpha(x)$ is the Bessel function of order α.

LEMMA 6. *For $0 \leqq \theta \leqq \pi/2$,*

$$\begin{aligned} t_n(\sin\theta)^\lambda P_n^\lambda(\cos\theta) &= A\theta^{1/2}(n+\lambda)^{1/2}J_{\lambda-1/2}((n+\lambda)\theta) \\ &\quad + A[\theta\cos\theta-\sin\theta]\theta^{-2}(\sin\theta)^{-1}\theta^{3/2}(n+\lambda)^{-1/2} \\ &\quad \cdot J_{\lambda-3/2}((n+\lambda)\theta) + R_2, \end{aligned}$$

where

$$R^2 = O(\theta^2 n^{-2}) \quad \text{if } n\theta \geqq c$$
$$= O(n^{-4}) \quad \text{if } n\theta \leqq c.$$

See [15].

About Bessel functions, we need the Mehler-Sonine formula

$$J_{\lambda-1/2}(x) = \frac{2}{\Gamma(1-\lambda)\Gamma\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^{\lambda-1/2}} \int_1^\infty \frac{\sin xt\, dt}{(t^2-1)^\lambda},$$

$0 < \lambda < 1$ [18, p. 170].

We also need the estimate

$$| J_\alpha(x) | \leqq Ax^\alpha,$$

$0 < x \leqq 1$, α real [18, p. 43]
and the asymptotic formula

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2} [\cos (x - \alpha\pi/2 - \pi/4) + O(1/x)]$$

[18, p. 199].

In addition to the classical theorems of Hardy and M. Riesz on the integrals

$$\frac{1}{x}\int_0^x f(t)\, dt, \qquad \int_x^\pi f(t)t^{-1}\, dt \quad \text{and} \quad \int_0^\pi f(t)/(x-t)\, dt$$

we need these theorems in their $L^{p,\alpha}$ form [6], [7]. We also need the weighted norm form of the Hardy-Littlewood theorem on fractional integration [13]. For convenience we state it here.

LEMMA 7. *If*

$$f_\lambda(x) = \int_0^\infty f(t)x^{-\alpha}|x-t|^{-\beta}t^{-\gamma}\, dt,$$

$\alpha + \beta + \gamma = 1$, $\alpha < 1/p$, $\gamma < 1 - 1/p$, $\alpha + \gamma > 0$, *then*

$$f_\lambda \in L^p(0, \infty) \quad \text{if} \quad f \in L^p(0, \infty), \qquad 1 < p < \infty.$$

In our applications we have an integral of the form

$$f_\lambda(x) = \int_{x/2}^{2x} f(t)x^{-\alpha}|x-t|^{-\beta}t^{-\gamma}\, dt.$$

Since in this range of integration $x \geqq |x - t|$, $t \geqq |x - t|$, and α and γ for us will be positive, we may dominate $f_\lambda(x)$ by

$$|f_\lambda(x)| \leqq \int_{x/2}^{2x} |f(t)|x^{-\varepsilon}|x-t|^{-1+2\varepsilon}t^{-\varepsilon}\, dt$$

for some small ε. Thus we may ignore the conditions $\alpha < 1/p$, $\gamma < 1 - 1/p$.

3. The main theorems

THEOREM 1. *Let*

$$f(\theta)(\sin\theta)^{\alpha} \in L^{p}(0, \infty),$$

$1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$, *and*

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n \cos n\theta,$$

i.e.,

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta \, d\theta.$$

We set

$$f_r^{\lambda}(\theta) = \sum_{n=0}^{\infty} a_n r^n t_n^{\lambda}(\sin\theta)^{\lambda} P_n^{\lambda}(\cos\theta)$$

for $\lambda > 0$. *Then*

$$\lim_{r\to 1} f_r^{\lambda}(\theta) = f^{\lambda}(\theta)$$

exists for almost all θ. *Also*

$$\lim_{r\to 1} \| [f_r^{\lambda}(\theta) - f^{\lambda}(\theta)](\sin\theta)^{\alpha} \|_p = 0,$$

$f^{\lambda}(\theta)(\sin\theta)^{\alpha} \in L^p$, *and*

$$\| f^{\lambda}(\theta)(\sin\theta)^{\alpha} \|_p \leqq A(\alpha, p) \| f(\theta)(\sin\theta)^{\alpha} \|_p$$

where $A(\alpha, p)$ *is independent of* f.

Our next theorem is the dual of this and the two theorems together allow us to go back and forth between ultraspherical series and Fourier series.

THEOREM 2. *Let*

$$g^{\lambda}(\theta)(\sin\theta)^{\alpha} \in L^{p}(0, \pi),$$

$1 < p < \infty$, $-1/p < \alpha < 1 - 1/p, \lambda > 0$. *Then if*

$$g^{\lambda}(\theta) \sim \sum_{n=0}^{\infty} b_n t_n^{\lambda}(\sin\theta)^{\lambda} P_n^{\lambda}(\cos\theta),$$

i.e.,

$$b_n = t_n^{\lambda} \int_0^{\pi} g^{\lambda}(\theta)\,(\sin\theta)^{\lambda} P_n^{\lambda}\,(\cos\theta)\,d\theta,$$

we set

$$g_r(\theta) = \sum_{n=0}^{\infty} b_n r^n \cos n\theta.$$

Then

$$\lim_{r\to 1-} g_r(\theta) = g(\theta)$$

exists for almost all θ. *Also* $g(\theta)(\sin\theta)^{\alpha} \in L^p$,

$$\lim_{r\to 1-} \| [g_r(\theta) - g(\theta)](\sin\theta)^{\alpha} \|_p = 0,$$

and

$$\| g(\theta)(\sin\theta)^{\alpha} \|_p \leqq A(\alpha, p) \| g^{\lambda}(\theta)(\sin\theta)^{\alpha} \|_p$$

where $A(\alpha, p)$ *is independent of* f.

We first prove Theorem 1 for the case $0 < \lambda < 1$. Without loss of generality we may assume that $\int_0^{\pi} f(\theta)\,d\theta = 0$. We also assume for the moment that $f(\theta) \in C^2$. We will remove this restriction later on. We also assume for the moment that $0 \leq \theta \leqq \pi/2$. That $f^\lambda(\theta)$ exists almost everywhere follows from the asymptotic formula for $P_n^\lambda(\cos\theta)$. We have

$$\begin{aligned} f_r^\lambda(\theta) &= \frac{2}{\pi} \sum_{n=0}^{\infty} t_n^\lambda r^n (\sin\theta)^\lambda P_n^\lambda(\cos\theta) \int_0^\pi f(\varphi) \cos n\varphi \, d\varphi \\ &= \frac{2}{\pi} \int_0^{\theta/6} f(\varphi) \left[\sum_{n=1}^{\infty} r^n t_n^\lambda P_n^\lambda(\cos\theta) \cos n\varphi \right] (\sin\theta)^\lambda \, d\varphi \\ &\qquad + \frac{2}{\pi} \int_{\theta/6}^{\pi} f(\varphi) \left[\sum_{n=1}^{\infty} r^n t_n^\lambda P_n^\lambda(\cos\theta) \cos n\varphi \right] (\sin\theta)^\lambda \, d\varphi \\ &= \frac{2}{\pi} [I_1 + I_2]. \end{aligned}$$

The points φ with φ near θ cause most of the difficulty so we handle I_2 first. Different methods are used to take care of I_1.

By Lemma 2 we have that $\sum_{n=1}^{\infty} r^n t_n^\lambda (\sin\theta)^\lambda P_n^\lambda(\cos\theta) \cos n\varphi$ converges uniformly for each $r < 1$ as do all of its formal derivatives with respect to φ. Thus we may differentiate the series term by term. Integrating by parts twice we see that

$$\begin{aligned} I_2 = &-(\sin\theta)^\lambda \int_{\theta/6}^{\pi} \frac{\partial^2 f(\varphi)}{\partial\varphi^2} \left[\sum_{n=1}^{\infty} r^n n^{-2} t_n^\lambda P_n^\lambda(\cos\theta) \cos n\varphi \right] d\varphi \\ &+ (\sin\theta)^\lambda \left.\frac{\partial f(\varphi)}{\partial\varphi}\right|_{\varphi=\pi} \sum_{n=1}^{\infty} (-1)^n r^n n^{-2} t_n^\lambda P_n^\lambda(\cos\theta) \\ &- (\sin\theta)^\lambda \left.\frac{\partial f(\varphi)}{\partial\varphi}\right|_{\varphi=\theta/6} \sum_{n=1}^{\infty} r^n n^{-2} t_n^\lambda P_n^\lambda(\cos\theta) \cos n\theta/6 \\ &- (\sin\theta)^\lambda f(\theta/6) \sum_{n=1}^{\infty} r^n n^{-1} t_n^\lambda P_n^\lambda(\cos\theta) \sin n\theta/6. \end{aligned}$$

Let $J(\theta) = \lim_{r\to1} I_2$. Using dominated convergence and Lemmas 1 and 3, we see that

$$\begin{aligned} J(\theta) = &-(\sin\theta)^\lambda \int_{\theta/6}^{\pi} \frac{\partial^2 f(\varphi)}{\partial\varphi^2} \left[\sum_{n=1}^{\infty} n^{-2} t_n^\lambda P_n^\lambda(\cos\theta) \cos n\varphi \right] d\varphi \\ &+ (\sin\theta)^\lambda \left.\frac{\partial f(\varphi)}{\partial\varphi}\right|_{\varphi=\pi} \sum_{n=1}^{\infty} (-1)^n n^{-2} t_n^\lambda P_n^\lambda(\cos\theta) \\ &- (\sin\theta)^\lambda \left.\frac{\partial f(\varphi)}{\partial\varphi}\right|_{\varphi=\theta/6} \sum_{n=1}^{\infty} n^{-2} t_n^\lambda P_n^\lambda(\cos\theta) \cos n\theta/6 \\ &- (\sin\theta)^\lambda f(\theta/6) \sum_{n=1}^{\infty} n^{-1} t_n^\lambda P_n^\lambda(\cos\theta) \sin n\theta/6 \end{aligned}$$

for $\theta \neq 0, \pi$.

We write the first term on the right as

$$-(\sin\theta)^{\lambda}\lim_{\varepsilon\to 0}\int_{\substack{|\varphi-\theta|\geq\varepsilon\\ \theta/6\leq\varphi\leq\pi}}\frac{\partial^2 f(\varphi)}{\partial\varphi^2}\sum_{n=1}^{\infty} n^{-2}t_n^{\lambda} P_n^{\lambda}(\cos\theta)\cos n\varphi\, d\varphi.$$

From Lemmas 1 and 3, and for $0 < \theta \leqq \pi/2$, $\theta/6 \leqq \varphi \leqq \pi$, and $|\varphi - \theta| \geqq \varepsilon$, we have that $\sum_{n=1}^{\infty} n^{-2}t_n^{\lambda} P_n^{\lambda}(\cos\theta)\cos n\varphi$ is an infinitely differentiable function of φ. Integrating by parts twice we see that

$$\begin{aligned} J(\theta) = &-(\sin\theta)^{\lambda}\lim_{\varepsilon\to 0}\int_{\substack{|\varphi-\theta|\geq\varepsilon\\ \theta/6\leq\varphi\leq\pi}} f(\varphi)\frac{\partial^2}{\partial\varphi^2}\left[\sum_{n=1}^{\infty} n^{-2}t_n^{\lambda} P_n^{\lambda}(\cos\theta)\cos n\varphi\right]d\varphi \\ &+(\sin\theta)^{\lambda}\frac{\partial f(\varphi)}{\partial\varphi}\bigg|_{\varphi=\theta}\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} n^{-2}t_n^{\lambda} P_n^{\lambda}(\cos\theta) \\ &\qquad\cdot[\cos n(\theta+\varepsilon) - \cos n(\theta-\varepsilon)] \\ &+(\sin\theta)^{\lambda} f(\theta)\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} n^{-1}t_n^{\lambda} P_n^{\lambda}(\cos\theta) \\ &\qquad\cdot[\sin n(\theta+\varepsilon) - \sin n(\theta-\varepsilon)] \\ = &\ M(\theta) + A(\theta) + B(\theta). \end{aligned}$$

We have used Lemmas 1 and 3 which show that

$$\sum_{n=1}^{\infty} n^{-2}t_n^{\lambda} P_n^{\lambda}(\cos\theta)\cos n\varphi$$

may be differentiated term by term with respect to φ and also that $f(\varphi)$ and $\partial f(\varphi)/\partial\varphi$ are continuous.

That $A(\theta) \equiv 0$ follows immediately from Lemma 2. To find $B(\theta)$, we use Lemma 1 to obtain

$$B(\theta) = f(\theta)\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} n^{-1}\cos[(n+\lambda)\theta - \lambda\pi/2]\cos n\theta\sin n\varepsilon$$

for $\theta \neq 0, \pi$. Then a simple calculation shows that

$$\begin{aligned} B(\theta) = &\ f(\theta)/2\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} n^{-1}\cos\left[\left(\theta-\frac{\pi}{2}\right)\lambda\right]\sin n\varepsilon \\ &+ f(\theta)/2\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} n^{-1} \\ &\qquad\cdot[\cos(\theta-\pi/2)\lambda\cos 2n\theta - \sin(\theta-\pi/2)\lambda\sin 2n\theta]\sin n\varepsilon. \end{aligned}$$

The second sum approaches zero at $\varepsilon \to 0$ because the convergence is uniform for $\varepsilon < \theta/2$. But

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{\infty} n^{-1}\sin n\varepsilon = \pi/2$$

so we have

$$B(\theta) = (\pi/4)f(\theta)\cos[(\theta-\pi/2)\lambda].$$

Now to the major difficulty of this paper, $M(\theta)$. Using Mehler's formula, we obtain

$$M(\theta) = -A(\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int_{\substack{|\varphi-\theta|\geq\varepsilon \\ \theta/6\leq\varphi\leq\pi}} f(\varphi) \frac{\partial^2}{\partial\varphi^2} \sum_{n=1}^{\infty} n^{-2\lambda} t_n^{\lambda} \frac{\Gamma(n+2\lambda)}{\Gamma(n+1)} \cos n\varphi \cdot \int_0^\theta \frac{\cos(n+\lambda)\psi \, d\psi \, d\varphi}{[\cos\psi - \cos\theta]^{1-\lambda}}$$

$$= -A(\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int f(\varphi) \frac{\partial^2}{\partial\varphi^2} \sum_{n=1}^{\infty} [n^{\lambda-2} + An^{\lambda-3} + B_n n^{\lambda-4}] \cos n\varphi \cdot \int_0^\theta \frac{\cos(n+\lambda)\psi \, d\psi \, d\varphi}{[\cos\psi - \cos\theta]^{1-\lambda}}$$

A will denote an arbitrary constant which may vary from one occurrence to the next. B_n is a bounded sequence. The second and third terms contain series which converge more rapidly than the first term and so are easier to handle. We confine ourselves to the first term. When the limits on an integral are not stated it will be assumed to be over $\theta/6 \leqq \varphi \leqq \pi$, $|\theta - \varphi| \geqq \varepsilon$. Calling the first term $AN(\theta)$ we have

$$N(\theta) = (\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int f(\varphi) \frac{\partial^2}{\partial\varphi^2} \int_0^\theta [\cos\psi - \cos\theta]^{\lambda-1} \cdot \sum_{n=1}^{\infty} n^{\lambda-2} \cos n\varphi \cos(n+\lambda)\psi \, d\psi \, d\varphi$$

$$= \sum_{i=1}^{4} (\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int f(\varphi) \frac{\partial^2}{\partial\varphi^2} \int_0^\theta [\cos\psi - \cos\theta]^{1-\lambda} \cdot \sum_{n=1}^{\infty} n^{\lambda-2} Q_i(n,\varphi,\psi) [\cos\psi - \cos\theta]^{\lambda-1} \, d\psi \, d\varphi$$

$$= f_1 + f_2 + f_3 + f_4 .$$

where

$$\begin{aligned} Q_i(n,\varphi,\psi) &= \cos n(\varphi+\psi)\cos\lambda\psi && \text{for } i = 1 \\ &= \cos n(\varphi-\psi)\cos\lambda\psi && \text{for } i = 2 \\ &= -\sin n(\varphi+\psi)\sin\lambda\psi && \text{for } i = 3 \\ &= \sin n(\varphi-\psi)\sin\lambda\psi && \text{for } i = 4. \end{aligned}$$

f_1 and f_3 cause little trouble since the functions that the series represent are twice differentiable in the range of integration. f_2 and f_4 are treated by similar methods. We treat f_2.

Using Lemma 3, we get

$$f_2(\theta) = (\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int f(\varphi) \frac{\partial^2}{\partial\varphi^2} \int_0^\theta [\cos\psi - \cos\theta]^{\lambda-1} \cos\lambda\psi \, |\varphi-\psi|^{1-\lambda} \, d\psi \, d\varphi$$

$$+ (\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int f(\varphi) \frac{\partial^2}{\partial\varphi^2} \int_0^\theta [\cos\varphi - \cos\theta]^{\lambda-1} \cos\lambda\psi l(\varphi-\psi) \, d\psi \, d\varphi$$

where $l(t) \in C^\infty$ for $|t| \leqq 2\pi - \delta, \delta > 0$. Again the second integral causes no trouble so we only treat the first. Calling it $M_1(\theta)$ we see that

$$M_1(\theta) = (1-\lambda)(\sin\theta)^{1-\lambda} \lim_{\varepsilon\to 0} \int f(\varphi)$$
$$\cdot \frac{\partial}{\partial\varphi}\int_0^\theta [\cos\psi - \cos\theta]^{\lambda-1} \cos\lambda\psi \, |\varphi - \varphi|^{-\lambda} \operatorname{sgn}(\varphi-\psi)\, d\psi\, d\varphi$$

by dominated convergence, using Lemma 4. We write this as

$$M_1(\theta) = (1-\lambda) \lim_{\varepsilon\to 0} \int f(\varphi) K(\theta,\varphi)\, d\varphi$$

where

$$K(\theta,\varphi) = (\sin\theta)^{1-\lambda} \frac{\partial}{\partial\varphi}\int_0^\theta [\cos\psi - \cos\theta]^{\lambda-1}\cos\lambda\psi\,|\varphi-\psi|^{-\lambda}\operatorname{sgn}(\varphi-\psi)\,d\psi.$$

Let $\psi = \theta - u$. We see that

$$K(\theta,\varphi) = \frac{\partial}{\partial\varphi}\int_0^\theta u^{\lambda-1}\cos\lambda(\theta-u)\,|\varphi-\theta+u|^{-\lambda}\operatorname{sgn}(\varphi-\theta+u)\,du$$
$$+ (\sin\theta)^{1-\lambda}\frac{\partial}{\partial\varphi}\int_0^\theta \{[\cos(\theta-u) - \cos\theta]^{\lambda-1} - (u\sin\theta)^{\lambda-1}\}$$
$$\cdot \cos\lambda(\theta-u)\,|\varphi-\theta+u|^{-\lambda}\operatorname{sgn}(\varphi-\theta+u)\,du$$
$$= K_1 + E_1.$$

K_1 is the dominate term and we estimate it first. We consider two cases, $\theta/6 \leqq \varphi < \theta$ and $\theta < \varphi \leqq \pi$. Considering the second first, we set $u = (\varphi - \theta)t$ and get

$$K_1 = \frac{\partial}{\partial\varphi}\int_0^{\theta/(\varphi-\theta)} t^{\lambda-1}\cos\lambda[\theta - t(\varphi-\theta)](1+t)^{-\lambda}\,dt$$
$$= \left(\frac{\theta}{\varphi}\right)^\lambda \frac{1}{\theta-\varphi} + \lambda\int_0^{\theta/(\varphi-\theta)} \sin\lambda[\theta - t(\varphi-\theta)]\,dt$$
$$+ \lambda\int_0^{\theta/(\varphi-\theta)}\left[\left(\frac{t}{1+t}\right)^\lambda - 1\right]\sin\lambda[\theta - t(\varphi-\theta)]\,dt$$
$$= \left(\frac{\theta}{\varphi}\right)^\lambda \frac{1}{\theta-\varphi} + \frac{1}{\varphi-\theta} - \frac{\cos\lambda\theta}{\varphi-\theta} + L_1.$$

If $\varphi \geqq 2\theta$, $|L_1| \leqq C$. If $\theta < \varphi < 2\theta$ we have

$$L_1 = \lambda\int_0^1 + \lambda\int_0^{\theta/(\varphi-\theta)} = L_2 + L_3.$$

But $|L_2| \leqq C$ and it is easily seen that $|L_3| \leqq A \log\theta/(\varphi-\theta)$. Thus

$$K_1 = \frac{\theta^\lambda - \varphi^\lambda}{\varphi^\lambda[\theta - \varphi]} + \frac{\cos \lambda\theta}{\theta - \varphi} + O\left[\log \frac{\theta}{\theta - \varphi} + 1\right]$$

$$= \frac{\cos \lambda\theta}{\theta - \varphi} + O\left[\frac{1}{(\theta + \varphi)^{1-\lambda}\varphi^\lambda} + \log \frac{\theta}{\varphi - \theta} + 1\right]$$

by Lemma 4.

Now we consider $\theta/6 \leqq \varphi < \theta$.

$$K_1 = -\frac{\partial}{\partial\varphi}\int_0^{\theta-\varphi} u^{\lambda-1} \cos \lambda(\theta - u) \mid \varphi - \theta + u \mid^{-\lambda} du$$

$$+ \frac{\partial}{\partial\varphi}\int_{\theta-\varphi}^{\theta} u^{\lambda-1} \cos \lambda(\theta - u) \mid \varphi - \theta + u \mid^{-\lambda} du$$

$$= -\frac{\partial}{\partial\varphi}\int_0^1 t^{\lambda-1}(1 - t)^{-\lambda} \cos \lambda[\theta - t(\theta - \varphi)]\, dt$$

$$+ \frac{\partial}{\partial\varphi}\int_1^{\theta/(\theta-\varphi)} t^{\lambda-1}(t - 1)^{-\lambda} \cos \lambda[\theta - t(\theta - \varphi)]\, dt$$

$$= L_4 + L_5 .$$

Clearly $\mid L_4 \mid \leqq$ C.

$$L_5 = \left(\frac{\theta}{\varphi}\right)^\lambda \frac{1}{\theta - \varphi} - \lambda \int_1^{\theta/(\theta-\varphi)} \left(\frac{t}{t - 1}\right)^\lambda \sin \lambda[\theta - t(\theta - \varphi)]\, dt$$

$$= \left(\frac{\theta}{\varphi}\right)^\lambda \frac{1}{\theta - \varphi} - L_6 .$$

For $\theta/6 \leqq \varphi \leqq \theta/2$, $\mid L_6 \mid \leqq C$. For $\theta/2 \leqq \varphi < \theta$ we have

$$L_6 = \lambda \int_2^{\theta/(\theta-\varphi)} \left(\frac{t}{t - 1}\right)^\lambda \sin \lambda[\theta - t(\theta - \varphi)]\, dt + O(1)$$

$$= \lambda \int_2^{\theta/(\theta-\varphi)} \sin \lambda[\theta - t(\theta - \varphi)]\, dt$$

$$+ O\left[\int_2^{\theta/(\theta-\varphi)} \left[\left(\frac{t}{t - 1}\right)^\lambda - 1\right] dt\right] + O(1)$$

$$= \frac{1}{\theta - \varphi} - \frac{\cos \lambda[2\varphi - \theta]}{\theta - \varphi} + O\left[\log \frac{\theta}{\theta - \varphi}\right] + O(1).$$

Thus

$$L_5 = \frac{\theta^\lambda - \varphi^\lambda}{\varphi^\lambda[\theta - \varphi]} + \frac{\cos \lambda[2\varphi - \theta]}{\theta - \varphi} + O\left[\log \frac{\theta}{\theta - \varphi}\right] + O(1)$$

$$= \frac{\cos \lambda[2\varphi - \theta]}{\theta - \varphi} + O\left[\frac{1}{\varphi^\lambda[\varphi + \theta]^{1-\lambda}} + \log \frac{\theta}{\theta - \varphi} + 1\right]$$

$$= \frac{\cos \lambda\theta}{\theta - \varphi} + O\left[\frac{1}{\varphi^\lambda[\varphi + \theta]^{1-\lambda}} + \log \frac{\theta}{\theta - \varphi} + 1\right]$$

since $\mid \cos \lambda\, (2\varphi - \theta) - \cos \lambda\theta \mid = O(\mid \theta - \varphi \mid)$.

Now we go back and consider the error term E_1. First consider the case $\theta/6 \leqq \varphi < \theta$. Set

$$s(u) = \int_{\theta-\varphi}^{u} \operatorname{sgn}(\varphi - \theta + t) \mid \varphi - \theta + t \mid^{-\lambda} dt$$
$$= (1 - \lambda)^{-1} \mid u - (\theta - \varphi) \mid^{1-\lambda}.$$

We have

$$\int_0^\theta \{[\cos(\theta - u) - \cos\theta]^{\lambda-1} - [u \sin\theta]^{\lambda-1}\}$$
$$\cdot \cos\lambda(\theta - u) \,|\varphi - \theta + u|^{-\lambda} \operatorname{sgn}(\varphi - \theta + u)\, du$$
$$= \lim_{\varepsilon\to 0} \int_\varepsilon^\theta \{\qquad\} \cos\lambda(\theta - u)\, ds(u)$$
$$= [(1 - \cos\theta)^{\lambda-1} - (\theta\sin\theta)^{\lambda-1}](1 - \lambda)^{-1}\varphi^{1-\lambda}$$
$$- (1 - \lambda)^{-1} \lim\nolimits_{\varepsilon\to 0}[(\cos(\theta - \varepsilon) - \cos\theta)^{\lambda-1} - (\varepsilon\sin\theta)^{\lambda-1}]$$
$$\cdot \,|\epsilon - (\theta - \varphi)|^{1-\lambda} \cos\lambda(\theta - \epsilon)$$
$$- \lim_{\varepsilon\to 0} \int_\varepsilon^\theta (1 - \lambda)^{-1} \,|u - (\theta - \varphi)\,|^{1-\lambda}$$
$$\cdot \frac{\partial}{\partial u}[([\cos(\theta - u) - \cos\theta]^{\lambda-1} - (u\sin\theta)^{\lambda-1})\cos\lambda(\theta - u)]\, du$$
$$= [(1 - \cos\theta)^{\lambda-1} - (\theta\sin\theta)^{\lambda-1}](1 - \lambda)^{-1}\varphi^{1-\lambda}$$
$$- \int_0^\theta (1 - \lambda)^{-1} \,|u - (\theta - \varphi)\,|^{1-\lambda}$$
$$\cdot \frac{\partial}{\partial u}[([\cos(\theta - u) - \cos\theta]^{\lambda-1} - (u\sin\theta)^{\lambda-1}) \cos\lambda(\theta - u)]\, du$$

by Lemma 5.

Thus

$$E_1 = (\sin\theta)^{1-\lambda}\varphi^{-\lambda}[(1 - \cos\theta)^{\lambda-1} - (\theta\sin\theta)^{\lambda-1}]$$
$$- (\sin\theta)^{1-\lambda} \int_0^\theta \operatorname{sgn}[u - (\theta - \varphi)] \,|u - (\theta - \varphi\,|^{-\lambda}$$
$$\cdot \frac{\partial}{\partial u}[[(\cos(\theta - u) - \cos\theta)^{\lambda-1} - (u\sin\theta)^{\lambda-1}] \cos\lambda\,(\theta - u)]\, du$$
$$= O((\theta/\varphi)^\lambda\theta^{-1}) + O[(\sin\theta)^{1-\lambda} \int_0^\theta \operatorname{sgn}[u - (\theta - \varphi)] \,|u - (\theta - \varphi)\,|^{-\lambda}$$
$$\cdot\, [u^{\lambda-1}\theta^{\lambda-2} + u^\lambda\theta^{\lambda-2}\sin\theta]\, du$$
$$= O(\theta^{-1}) + O(\theta^{-\lambda}(\theta - \varphi)^{\lambda-1}),$$

where we have used Lemma 5 three times.

The case $\theta < \varphi < \pi$ is easier. We consider two cases: $\theta < \varphi \leqq 2\theta$ and $2\theta \leqq \varphi < \pi$. For the first we have

$$E_1 = A\,(\sin\theta)^{1-\lambda} \int_0^\theta ([\cos(\theta - u) - \cos\theta]^{\lambda-1} - (u\sin\theta)^{\lambda-1}) \cdot\cos\lambda(\theta - u)[\varphi - \theta + u]^{-\lambda-1}\,du$$

$$= O\left(\theta^{1-\lambda}\int_0^\theta u^\lambda\theta^{\lambda-2}[\varphi - \theta + u]^{-\lambda-1}\,du\right) = O\left(\theta^{-1}\int_0^\theta u^\lambda[\varphi - \theta + u]^{\lambda-1}\,du\right)$$

$$= O(\theta^{\lambda-1}(\varphi - \theta)^{-\lambda} + \theta^{-1}\theta^\lambda\varphi^{-\lambda}) = O(\theta^{\lambda-1}(\varphi - \theta)^{-\lambda} + \theta^{-1}).$$

For $2\theta \leqq \varphi < \pi$ we have

$$E_1 = O\left(\theta^{1-\lambda}\int_0^\theta u^\lambda\theta^{\lambda-2}[\varphi - \theta + u]^{-\lambda-1}\,du\right)$$

$$= O\left(\theta^{-1}\varphi^{-\lambda-1}\int_0^\theta u^\lambda\,du\right) = O(\varphi^{-1}).$$

We now treat I_1. Recall that

$$I_1 = \int_0^{\theta/6} f(\varphi)\sum_{n=1}^\infty r^n t_n^\lambda\,(\sin\theta)^\lambda P_n^\lambda\,(\cos\theta)\cos n\varphi\,d\varphi.$$

Using Lemma 6, we get for the sum under the integral

$$A\sum_{n=1}^\infty r^n\theta^{1/2}(n + \lambda)^{1/2}J_{\lambda-1/2}[(n + \lambda)\theta]\cos n\varphi$$
$$+ A\sum_{n=1}^\infty r^n f_1(\theta)\theta^{3/2}(n + \lambda)^{-1/2}J_{\lambda-3/2}[(n + \lambda)\theta]\cos n\varphi$$
$$+ A\sum_{n=1}^\infty r^n s_2(n, \theta)$$
$$= U + V + W.$$

In the above equation

$$f_1(\theta) = [\theta\cos\theta - \sin\theta]\theta^{-2}(\sin\theta)^{-1},$$

and

$$|\,s_2(n, \theta)\,| = O(n^{-2})$$

uniformly in θ. W is therefore clearly bounded. By the estimates for Bessel functions (Lemma 6)

$$V = O\{\sum_{n=1}^{[1/\theta]} r^n\theta^{3/2}(n + \lambda)^{-1/2}(n + \lambda)^{\lambda-3/2}\theta^{\lambda-3/2}\}$$
$$+ O\{|\sum_{n=[1/\theta]+1}^\infty r^n\theta^{3/2}(n + \lambda)^{-1/2}[(n + \lambda)\theta]^{-1/2}$$
$$\cdot\cos[(n + \lambda)\theta - (\lambda - 3/2)(\pi/2) - \pi/4]\cos n\varphi\,|\}$$
$$+ O\{\sum_{n=[1/\theta]+1}^\infty r^n\theta^{3/2}(n + \lambda)^{-1/2}[(n + \lambda)\theta]^{-3/2}\}.$$

It is easy to see that the first and third of the three sums above are $O(\theta)$. In the second sum one sums by parts, summing

$$r^n\cos[(n + \lambda)\theta - (\lambda - 3/2)\pi/2 - \pi/4]\cos n\varphi$$

and taking differences of the powers of n. An easy estimate then shows that this sum is

$$O\{\theta \log (1/\theta)\} = O(1).$$

In U we use the Mehler-Sonine formula (Lemma 6) to get

$$U = A \int_1^\infty (t^2 - 1)^{-\lambda} \sum_{n=1}^\infty r^n \theta^{1-\lambda}(n + \lambda)^{1-\lambda} \sin [(n + \lambda)\theta t] \cos n\varphi \, dt.$$

We divide the range of integration into two parts, $1 \leqq t \leqq 2$ and $t \geqq 2$.

For the first we get

$$A \int_1^2 (t^2 - 1)^{-\lambda} \sum_{n=1}^\infty r^n \theta^{1-\lambda}(n + \lambda)^{1-\lambda} \sin n(\theta t - \varphi) \cos \lambda\theta t \, dt$$

plus similar terms. Treating just the first term we obtain the estimate

$$O\left[\theta^{-\lambda+1} \int_1^2 (t^2 - 1)^{-\lambda}(\theta t - \varphi)^{\lambda-2} \, dt\right] = O(\theta^{-1})$$

by Lemma 3, and the fact that $\varphi \leqq \theta/6$ and $0 < \lambda < 1$. Calling the second integral AJ, we have

$$J = \int_2^\infty h(t) \sum_{n=1}^\infty r^n \theta^{1-\lambda}(n + \lambda)^{1-\lambda} \sin (n + \lambda)\theta t \cos n\varphi \, dt$$

where $h(t) = (t^2 - 1)^{-\lambda}$. Observe that

$$h'(t) = O(t^{-2\lambda-1}) \quad \text{and} \quad h''(t) = O(t^{-2\lambda-2})$$

as $t \to \infty$. Integrating by parts we get

$$J = -h(2) \sum_{n=1}^\infty r^n \theta^{-\lambda}(n + \lambda)^{-\lambda} \cos 2\theta(n + \lambda) \cos n\varphi$$
$$+ \int_2^\infty h'(t) \sum_{n=1}^\infty r^n \theta^{-\lambda}(n + \lambda)^{-\lambda} \cos (n + \lambda)\theta t \cos n\varphi \, dt.$$

The first term is $O(\theta^{-1})$ by an argument similar to that given above. In the second term we split the sum into two parts, $1 \leqq n \leqq 1/\theta$ and $n > 1/\theta$. For $\sum_{n=1}^{1/\theta}$ we obtain

$$O\left(\int_2^\infty h'(t) \sum_{n=1}^{1/\theta} r^n \theta^{-\lambda}(n + \lambda)^{-\lambda} \, dt\right) = O(\theta^{-1}).$$

The other term is handled by an integration by parts which gives

$$O[h'(2) \sum_{1/\theta}^\infty \theta^{-\lambda-1}(n + \lambda)^{-\lambda-1}]$$
$$+ O\left[\int_2^\infty h''(t) \sum_{1/\theta}^\infty \theta^{-\lambda-1}(n + \lambda)^{-\lambda-1} \, dt\right] = O(\theta^{-1}).$$

Thus $J = O(\theta^{-1})$ which is the estimate we need to show

$$I_1 = O\left(\frac{1}{\theta} \int_0^{\theta/6} |f(\varphi)| \, d\varphi\right).$$

Thus we have shown that

$$f^{\lambda}(\theta) = O(f(\theta)) + O\left(\frac{1}{\theta}\int_0^{\theta} f(\varphi)\,d\varphi\right) + \cos\lambda\theta\int_{\theta/6}^{\pi} f(\varphi)/(\theta-\varphi)\,d\varphi$$

$$+ O\left[\int_{\theta/6}^{\pi}\frac{f(\varphi)}{(\theta+\varphi)^{1-\lambda}\varphi^{\lambda}}\,d\varphi\right] + O\left[\int_{\theta/6}^{\pi} f(\varphi)\log\frac{\theta}{\varphi-\theta}\,d\varphi\right]$$

$$+ O\left(\int_{\theta/6}^{\theta}\theta^{-\lambda}(\theta-\varphi)^{\lambda-1}f(\varphi)\,d\varphi\right)$$

$$+ O\left(\int_{\theta}^{2\theta}\theta^{\lambda-1}(\varphi-\theta)^{-\lambda}f(\varphi)\,d\varphi\right) + O\left(\frac{1}{\theta}\int_{\theta}^{2\theta} f(\varphi)\,d\varphi\right)$$

$$+ O\left(\int_{2\theta}^{\pi}\frac{f(\varphi)}{\varphi}\,d\varphi\right).$$

The first term is clearly a bounded operator in $L^{p,\alpha}$. The second, fourth, eighth and ninth are bounded in $L^{p,\alpha}$, by Hardy's inequality. Since $\lambda < 1$ and $0 < \theta \leqq \pi/2$, $(\cos\lambda\theta)^{-1}$ is bounded and so the third term gives a bounded operator in $L^{p,\alpha}$ by the Hardy-Littlewood generalization of M. Riesz's theorem. The fifth term is bounded in $L^{p,\alpha}$ by a simple application of Hölder's inequality. The sixth and seventh terms are weighted fractional integrals and they are bounded by Lemma 7.

Next we must remove the restriction that $0 \leqq \theta \leqq \pi/2$. This follows from the fact that

$$t_n^{\lambda}(\sin(\pi-\theta))^{\lambda}P_n^{\lambda}(\cos(\pi-\theta)) = (-1)^n t_n^{\lambda}(\sin\theta)^{\lambda}P_n^{\lambda}(\cos\theta)$$

and

$$\cos n(\pi-\theta) = (-1)^n\cos n\theta.$$

We also need to remove the restriction that $f \in C^2$. We have shown that

$$\|f^{\lambda}\|_{p,\alpha} \leqq A\,\|f\|_{p,\alpha}.$$

Since the $f \in C^2$ are dense in $L^{p,\alpha}$ we may extend the operator to a bounded linear operator T^{λ}. For $f \in L^{p,\alpha}$ we define

$$\bar{T}f = \lim_{r\to 1}\sum a_n r^n t_n^{\lambda}P_n^{\lambda}(\cos\theta)(\sin\theta)^{\lambda}.$$

The fact that $\bar{T}f$ exists for almost every θ and is integrable on compact proper subintervals of $(0, \pi)$ follows from the asymptotic formulas (Lemma 1). To complete the proof of Theorem 1 for $0 < \lambda < 1$ we must show that $T^{\lambda}f = \bar{T}f$ almost everywhere. For this it suffices to show

$$\int_0^{\pi}\bar{T}f(\theta)g(\theta)\,d\theta = \int_0^{\pi}T^{\lambda}f(\theta)g(\theta)\,d\theta$$

for $g \in C^{\infty}$ and vanishing in a neighborhood of 0 and of π. We know that

$$\bar{T}f_n(\theta) = T^{\lambda}f_n(\theta)$$

for $f_n \in C^2$. Let $f_n \in C^2$ and $f_n \to f$ in $L^{p,\alpha}$. Since T^λ is continuous in $L^{p,\alpha}$ it follows that

$$\lim_{n\to\infty} \int_0^\pi T^\lambda f_n(\theta)g(\theta)\, d\theta = \int_0^\pi T^\lambda f(\theta)g(\theta)\, d\theta.$$

From the asymptotic formulas for P_n^λ, Lemma 1, we see that

$$\left[\int_\varepsilon^{\pi-\varepsilon} |\bar{T}f(\theta)|^p (\sin\theta)^{\alpha p}\, d\theta\right]^{1/p} \leqq A_{p,\varepsilon}\|f\|_{p,\alpha}.$$

Since g vanishes outside a neighborhood of 0 and π this implies

$$\lim_{n\to\infty} \int_0^\pi \bar{T}f_n(\theta)g(\theta)\, d\theta = \int_0^\pi \bar{T}f(\theta)g(\theta)\, d\theta.$$

Thus

$$\int_0^\pi \bar{T}f(\theta)g(\theta)\, d\theta = \int_0^\pi T^\lambda f(\theta)g(\theta)\, d\theta.$$

Before we complete the proof of Theorem 1, i.e., extend the theorem to all $\lambda > 0$ instead of just $0 < \lambda < 1$, we give a proof of Theorem 2 for $0 < \lambda < 1$.

This we prove by a standard duality argument. Let

$$g(\theta) \sim \sum b_n t_n^\lambda P_n^\lambda(\cos\theta)(\sin\theta)^\lambda.$$

We need only consider a dense subset of $L^{p,\alpha}$, e.g., the bounded C^∞ functions. Then $n^\alpha b_n = O(1)$ for all $\alpha > 0$. So the series

$$g_\lambda(\theta) = \sum b_n \cos n\theta$$

converges to a C^∞ function. We wish to show that

$$\| g_\lambda \|_{p,\alpha} \leqq A \| g \|_{p,\alpha}.$$

We choose a function $f \in L^{p',-\alpha}$ where $1/p + 1/p' = 1$. This space is the dual space to $L^{p,\alpha}$. We choose $f \sim \sum a_n \cos n\theta$ so that

$$\|f\|_{p',-\alpha}\|g_\lambda\|_{p,\alpha} = \int_0^\pi f(\theta)g_\lambda(\theta)\, d\theta = A\sum a_n b_n = A\int_0^\pi f^\lambda(\theta)g(\theta)\, d\theta$$

where $f^\lambda(\theta)$ is the function defined in Theorem 1. Then by Theorem 1 and Hölder's inequality we get

$$\| f \|_{p',-\alpha} \| g_\lambda \|_{p,\alpha} \leqq A \| f \|_{p',-\alpha} \| g \|_{p,\alpha},$$

or the operator taking g into g_λ is bounded in $L^{p,\alpha}$.

To extend our theorems to $\lambda > 1$ we use an idea of Muckenhoupt and Stein [10]. To the series

$$f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta)$$

they associate the series

$$\tilde{f}(r,\theta) = 2\lambda \sum a_n(n+2\lambda)^{-1} r^n \sin\theta\, P_{n-1}^{\lambda+1}(\cos\theta).$$

They show that

$$\int_0^\pi |\tilde{f}(r, \theta)|^p (\sin \theta)^{2\lambda} d\theta \leqq A_p \int_0^\pi |f(\theta)|^p (\sin \theta)^{2\lambda} d\theta,$$

$1 < p < \infty$ where A_p does not depend on f or r. They use the function $P_{n-1}^{\lambda+1}(\cos \theta)$ instead of $P_n^{\lambda+1}(\cos \theta)$ because it is the function which arises naturally when trying to obtain an H^p theory for ultraspherical expansions. In our work we do not have this option, since $P_{n-\alpha}^{\lambda+\alpha}(\cos \theta)$ is not a polynomial for $0 < \alpha < 1$, so we use $P_n^{\lambda+\alpha}(\cos \theta)$. Also their theorem is a transplantation theorem for a different series, and a different measure. Recall that we essentially transplant between

$$\sum a_n P_n^\lambda(\cos \theta)(\sin \theta)^\lambda$$

and

$$\sum a_n\, n^{-1} P_n^{\lambda+1}(\cos \theta)(\sin \theta)^{\lambda+1}$$

with the measure $d\theta$ and they transplant between

$$\sum a_n\, P_n^\lambda(\cos \theta)$$

and

$$\sum a_n\, n^{-1} P_{n-1}^{\lambda+1}(\cos \theta)\sin \theta$$

with the measure $(\sin \theta)^{2\lambda} d\theta$. For p in the critical range,

$$(2\lambda + 1)/(\lambda + 1) < p < (2\lambda + 1)/\lambda,$$

it is possible to go from one of these results to the other; but for other p it is not possible to get one result from the statement of the other theorem. Since our result between λ and $\lambda + 1$ is still unproven we would like to be able to get it from their work. This is possible using the following inequalities which are derived in [10].

LEMMA 8. *If*

$$Q(r, \theta, \varphi) = \sum_{n=1}^{\infty} \frac{2\lambda}{n + 2\lambda} r^n (t_n^\lambda)^2 \sin \theta P_{n-1}^{\lambda+1} (\cos \theta) P_n^\lambda (\cos \varphi)$$

then for $0 \leqq \theta \leqq \pi$, $0 \leqq \varphi \leqq \pi/2$, $\lambda > 0$, *we have*

$$Q(r, \theta, \varphi) = O\left((\sin \varphi)^{-2\lambda-1}\right) \qquad \text{if} \quad 2\theta < \varphi,$$

$$Q(r, \theta, \varphi) = O\left((\sin \theta)^{-2\lambda-1}\right) \qquad \text{if} \quad \varphi < \theta/2,$$

$$Q(r, \theta, \varphi) = \frac{c_\lambda r^\lambda (\sin \theta)^{-\lambda} (\sin \varphi)^{-\lambda} \sin (\theta - \varphi)}{1 - 2r \cos (\theta - \varphi) + r^2} + O\left[(\sin \theta)^{-2\lambda-1}\left(1 + \log^+ \frac{\sin \theta \sin \varphi}{1 - \cos (\theta - \varphi)}\right)\right] \quad \text{if} \quad \theta/2 \leqq \varphi \leqq 2\theta.$$

As a preliminary step to completing the proofs of Theorem 1 and 2, we prove

THEOREM 3. *Let* $f(\theta) \in L^{p,\alpha}$, $1 < p < \infty$, $-1/p < \alpha < 1 - 1/p$. *We* *define*

$$a_n = t_n^\lambda \int_0^\pi f(\theta) P_n^\lambda (\cos\theta) (\sin\theta)^\lambda \, d\theta,$$

and set

$$T_r f(\theta) = \sum_{n=0}^{\infty} a_n r^n t_{n-1}^{\lambda+1} P_{n-1}^{\lambda+1}(\cos\theta)(\sin\theta)^{\lambda+1}.$$

Then

$$\| T_r f \|_{p,\alpha} \leqq A \| f \|_{p,\alpha}, \tag{1}$$

$$\lim_{r\to 1} T_r f(\theta) = Tf(\theta) \tag{2}$$

a.e. and in $L^{p,\alpha}$ *norm, and*

$$\| Tf \|_{p,\alpha} \leqq A \| f \|_{p,\alpha}.$$

Conversely if

$$b_n = t_n^{\lambda+1} \int_0^\pi f(\theta) P_n^{\lambda+1} (\cos\theta) (\sin\theta)^{\lambda+1} \, d\theta$$

and if we set

$$\tilde{T}_r f(\theta) = \sum a_n t_{n+1}^\lambda P_{n+1}^\lambda(\cos\theta)(\sin\theta)^\lambda$$

we have

$$\| \tilde{T}_r f(\theta) \|_{p,\alpha} \leqq A \| f \|_{p,\alpha},$$

$\lim_{r\to 1} \tilde{T}_r f(\theta) = \tilde{T}f(\theta)$ *a.e. and in* $L^{p,\alpha}$ *norm, and*

$$\| \tilde{T}f \|_{p,\alpha} \leqq A \| f \|_{p,\alpha}.$$

Proof. We would like to show that

$$T_r f(\theta) = \int_0^\pi f(\varphi) Q(r, \theta, \varphi) (\sin\theta)^\lambda (\sin\varphi)^\lambda \, d\varphi. \tag{3}$$

This is not quite true since we only have

$$t_{n-1}^{\lambda+1} = \frac{2\lambda}{n + 2\lambda} t_n^\lambda \left[1 + O\left(\frac{1}{n}\right)\right] \tag{4}$$

instead of equality without the factor $1 + O(1/n)$. However the term involving $O(1/n)$ is enough better than the main term that we may disregard it. A sketch of a proof is that $1 + O(1/n)$ in (4) can be replaced by

$$1 + a_1/n + a_2/n^2 + \cdots + a_k/n^k + O(n^{-k-1})$$

where k is sufficiently large. There are estimates for

$$\sum \frac{1}{n^{1+p}} r^n (t_n^\lambda)^2 \sin\theta P_{n-1}^{\lambda+1} (\cos\theta) P_n^\lambda (\cos\varphi)$$

which are similar to those for $Q(r, \theta, \varphi)$ but better by a power of θ or φ. For this reason we ignore these terms and assume that (3) holds. Then the estimates given by Muckenhoupt and Stein for $Q(r, \theta, \varphi)$ suffice to prove the first part of Theorem 3. The second half is done by duality.

To complete the proof of Theorems 1 and 2 we choose $\lambda > 0$ and let $[\lambda]$ denote the greatest integer less than or equal to λ. Applying Theorem 3 $[\lambda]$ times to $\sum a_n t_n^\lambda P_n^\lambda(\cos\theta)(\sin\theta)^\lambda$ leads us to the series

$$\sum_{n=0}^\infty a_n t_{n+[\lambda]}^{\lambda-[\lambda]} P_{n+[\lambda]}^{\lambda-[\lambda]}(\cos\theta)(\sin\theta)^{\lambda-[\lambda]}.$$

This is the ultraspherical expansion of a function $f^{[\lambda]}(\theta) \in L^{p,\alpha}$, and by Theorem 2 there is a $g(\theta) \in L^{p,\alpha}$ such that

$$g(\theta) \sim \sum_{n=0}^\infty a_n \cos(n + [\lambda])\theta = \sum_{n=0}^\infty b_{n+[\lambda]} \cos(n+[\lambda])\theta.$$

But the mapping between

$$h(\theta) \sim \sum_{n=0}^\infty a_n \cos n\theta$$

and

$$k(\theta) \sim \sum_{n=0}^\infty a_n \cos(n+1)\theta = \sum_{n=0}^\infty b_{n+1} \cos(n+1)\theta$$

is bounded in $L^{p,\alpha}$ so we obtain a bounded operator from

$$\sum a_n t_n^\lambda P_n^\lambda(\cos\theta)(\sin\theta)^\lambda$$

to

$$\sum a_n \cos n\theta.$$

This completes Theorem 2 and a duality argument takes care of Theorem 1.

4. Applications

Our first application is to obtain an analogue of the Marcinkiewicz multiplier theorem for ultraspherical expansions. One form of it follows immediately from Theorems 1 and 2, but for many applications it is important to have the theorem for expansions in terms of $P_n^\lambda(\cos\theta)$ instead of $P_n^\lambda(\cos\theta)(\sin\theta)^\lambda$. We will give the argument that is needed to take care of this point in detail in this application and then just state results for further applications.

Let $\int_0^\pi |f(\theta)|(\sin\theta)^{2\lambda} d\theta$ be finite and define

$$c_n = \int_0^\pi f(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\, d\theta.$$

We write

$$f(\theta) \sim \sum_{n=0}^\infty c_n t_n^2 P_n^\lambda(\cos\theta).$$

$\mathfrak{L}_\alpha^p$ will be the functions f such that

$$N_\alpha^p[f] = \left[\int_0^\pi |f(\theta)|^p (\sin\theta)^{\alpha p} (\sin\theta)^{2\lambda}\, d\theta\right]^{1/p}$$

is finite. We say that a sequence s_n is an $\mathfrak{L}_\alpha^p$ multiplier if given $f \in \mathfrak{L}_\alpha^p$ there is a function $Tf \in \mathfrak{L}_\alpha^p$ such that

$$s_n c_n = \int_0^\pi Tf(\theta) P_n^\lambda(\cos\theta)(\sin\theta)^{2\lambda}\, d\theta$$

and

$$N_\alpha^p[Tf] \leqq A N_\alpha^p[f].$$

The analogue of the theorem of Marcinkiewicz is as follows.

THEOREM 4. *Let* $\{s_n\}$ *be a sequence of real numbers satisfying*
(a) $|s_n| \leqq C \qquad n = 0, 1, \cdots.$
(b) $\sum_{2^n+1}^{2^{n+1}} |s_{k+1} - s_k| \leqq C \qquad n = 0, 1, \cdots.$
Then $\{s_n\}$ *is an* $\mathcal{L}_\alpha^p$ *multiplier sequence for*

$$1 < p < \infty, \qquad (1 - 2/p)\lambda - 1/p < \alpha < 1 - 1/p + (1 - 2/p)\lambda.$$

In particular $\{s_n\}$ *is an* $\mathcal{L}_0^p$ *multiplier for*

$$(1 + 2\lambda)/(1 + \lambda) < p < (1 + 2\lambda)/\lambda.$$

For the sequence $s_n = 1, n \leqq N; s_n = 0, n > N$ this gives a new proof of Pollard's mean convergence theorem and gives some insight into why the numbers $2 + 1/\lambda$ and $(1 + 2\lambda)/(1 + \lambda)$ occur in his work.

We consider the series

$$(\sin\theta)^{-\lambda} \sum_{n=0}^{\infty} s_n c_n t_n^2 P_n^\lambda(\cos\theta)(\sin\theta)^\lambda.$$

By Theorem 1 this is $(\sin\theta)^{-\lambda}h(\theta)$ with $h(\theta) \in L^{p,\beta}$ if $\sum s_n c_n t_n \cos n\theta \in L^{p,\beta}$. By the Marcinkiewicz theorem this is so if $\sum c_n t_n \cos n\theta \in L^{p,\beta}$. But this is in $L^{p,\beta}$ if

$$f(\theta) = (\sin\theta)^{-\lambda} \sum c_n t_n^2 P_n^\lambda(\cos\theta)(\sin\theta)^\lambda$$

is in $\mathcal{L}_\alpha^p$ for $\alpha p + (2 - p)\lambda = \beta p$.

But we have $-1 < \beta p < p - 1$ so we must have

$$(p - 2)\lambda - 1 < \alpha p < p - 1 + (p - 2)\lambda,$$

which is the condition given above.

We say that a series $\sum a_n$ is lacunary if $a_n = 0$ for $n \neq n_1, n_2, \cdots$, with $n_{k+1}/n_k \geqq \lambda > 1$. The following information is known about lacunary cosine series. If a lacunary cosine series, $\sum a_n \cos n\theta$, is summable on a set of positive measure, then $\sum a_n \cos n\theta$ converges to a function in L^p for every $p < \infty$. Using the asymptotic formula for $P_n^\lambda(\cos\theta)$ it is then easy to show that a lacunary ultraspherical expansion of an $\mathcal{L}_0^1$ function is in $\mathcal{L}_0^2$. Using Theorems 1 and 2 it is then possible to prove that an $\mathcal{L}_0^1$ function is in $\mathcal{L}_0^p$ for any $p < (1 + 2\lambda)/(1 + \lambda)$ and is in $\mathcal{L}_\alpha^p$ for p larger than $(1 + 2\lambda)/(1 + \lambda)$ if α is chosen appropriately. However, it is not necessarily in $\mathcal{L}_0^p$ for

$$p = (1 + 2\lambda)/(1 + \lambda).$$

This follows since the $\mathcal{L}_0^1$ norm of $t_n P_n^\lambda(\cos\theta)$ is bounded and the $\mathcal{L}_0^{1+2\lambda/1+\lambda}$ norm goes to infinity like a power of $\log n$. Use Lemma 6 and the asymptotic formula in $J_\alpha(x)$ which follows Lemma 6. This observation is due to E. Stein. In his thesis, D. Rider [12] has observed that for expansions on the sphere, the usual type of lacunary theorem fails. If f is integrable on the sphere and its expansion is lacunary, then it does not necessarily belong to any L^p space on the sphere for $p > 1$. This shows that the expansions of zonal functions are not as typical of spherical harmonic expansions as one might hope.

Finally let us state the analogue of a theorem of Hardy and Littlewood. For Fourier series they have proven the following theorem.

THEOREM A. *Let* $f(\theta) \in L^1(0, \pi)$ *and define* a_n *by*

$$a_n = \frac{1}{\pi} \int_0^\pi f(\theta) \cos n\theta \, d\theta$$

Then if $a_{n+1} \leqq a_n$, $f(\theta) \in L^p(0, \pi)$ *if and only if* $\sum a_n^p n^{p-2}$ *is finite*, $1 < p < \infty$.

We have generalized this theorem to quasi-monotone coefficients [1], so that we can get a stronger theorem for ultraspherical expansions than follows from Theorem A. Also we state our theorem for weighted norms but this could easily have been done in Theorem A.

THEOREM B. *If* a_n *is defined as in Theorem* A *and if*

$$(n + 1)^{-k} a_{n+1} \leqq n^{-k} a_n$$

for some k *and* $a_n \to 0$ *then*

$$\int_0^\pi |f(\theta)|^p (\sin \theta)^{\alpha p} < \infty$$

if and only if

$$\sum_{n=0}^\infty [a(n)]^p (n + 1)^{p-\alpha p-2} < \infty, \quad 1 < p < \infty, -1 < \alpha p < p - 1.$$

The numbers $(n + 1)$ could be replaced by any similar sequence. More importantly $\sin \theta$ could be replaced by θ. We state the theorem in this form because of the form of our transplantation theorem. Actually, as we state, the theorem we only need to assume $(n + 2)^{-k} a_{n+2} \leqq n^{-k} a_n$. From Theorem B and Theorems 1 and 2, we obtain Theorem 5 by the same argument as in Theorem 4.

THEOREM 5. *Let* $f(\theta) \in \mathfrak{L}_0^1(0, \pi)$ *and define* a_n *by*

$$a_n = (t_n^\lambda)^2 (n + \lambda)^{-1} \int_0^\pi f(\theta) P_n^\lambda (\cos \theta) (\sin \theta)^{2\lambda} \, d\theta$$

so that

$$f(\theta) \sim \sum_{n=0}^\infty a_n (n + \lambda) P_n^\lambda (\cos \theta).$$

Then if $a_{n+1} \leqq a_n$ *and* $n^\lambda a_n \to 0$ *we have* $f \in \mathfrak{L}_\alpha^p(0, \pi)$ *if and only if*

$$\sum_{n=0}^\infty a_n^p \{(n + \lambda)(1 + 2\lambda - \alpha)p - 2(1 + \lambda)\} \text{ is finite,}$$

$1 < p < \infty, \lambda p - (2\lambda + 1) < \alpha p < (1 + \lambda)p - (2\lambda + 1).$

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UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN
CORNELL UNIVERSITY
ITHACA, NEW YORK