ON THE EXTENSION AND THE SOLUTION OF NONLINEAR OPERATOR EQUATIONS

BY

W. V. Petryshyn

1. Introduction

In a series of recent papers Browder [1], [2], [3], [5] and Minty [13], [14] (see also Vainberg and Kachurovsky [22] and Kachurovsky [8]) have developed the theory of nonlinear functional equations $Pu = f$ in Hilbert and reflexive Banach spaces involving monotone operators[1] P satisfying certain very mild continuity conditions which guarantees the existence and the uniqueness of the solution for every f in a given space. In a number of papers Browder uses then this theory in the investigation of nonlinear elliptic and parabolic boundary value problems. In [23] Zarantonello derived similar results for continuous bounded nonlinear operators P in Hilbert space H which satisfy in H the weaker condition

$$\text{(i)} \qquad |(Pu - Pv, u - v)| \geqq c \, \| u - v \|^2, \qquad c > 0.$$

This result was in turn considerably extended and generalized by Browder [4], [6] to operators P in reflexive Banach spaces with P satisfying much weaker conditions.

In [16] the author developed a procedure for the construction of solvable extensions L_0 for the so called non-K-p.d.[1] densely defined unbounded linear operators L such that $L_0 \supset L$ and L_0 has a bounded inverse defined on all of H.

The purpose of Section 3 of this paper is to extend the above construction to densely defined nonlinear operators in Hilbert space. Our main result of this section (Theorem 1 below) depends significantly on the recent theorem of Browder [4]. In this section we also consider the problem, though from a different point of view, discussed by Kato [9] and Browder [5].

While in Section 3 we consider the existence and the uniqueness of ordinary or generalized[1] solutions of nonlinear equations, in Sections 4 and 5 we consider the problem of actually obtaining these solutions or their approximations. Thus in Section 4 we prove the convergence of a simple iterative method for the solution of strongly H_0-monotonic[1] operator equations. For potential operator equations similar procedure was recently investigated by Vainberg [21] and Simeonov [19]. The former author also studies iterative procedures for the solution of equations in Banach spaces with everywhere defined monotone operators. A similar iterative scheme with variable parameters was proposed by Zarantonello [23].

Received January 20, 1965.

[1] For the precise definitions of the concepts mentioned in the introduction and the statements of the corresponding results see Sections 2 and 3.

In Section 5 we discuss the applicability of the projective method, which is practically realized by the method of moments, for the approximate solution of nonlinear operator equations satisfying our general conditions. Similar results for Ritz method in the solution of essentially potential operator equations were recently derived by Mikhlin [15], Langenbach [11], Hagen-Torn and Mikhlin [7], and others [12], [10]. In our investigation of the projection method we follow the argument of Browder [2].

In Section 6 we apply our results of Sections 3 and 4 to the investigation and the approximate iterative solution of a nonlinear elliptic boundary value problem of second order.

2. Preliminaries

Let H denote a complex Hilbert space with the inner product $(\ ,\ )$ and norm $\|\ \|$. A linear operator T defined on a dense domain $D_T \subset H$ will be called *K-positive definite* (K-p.d.) if there is a closeable linear operator K with $D_K \supseteq D_T$ mapping D_T onto a dense subset KD_T of H and two constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that for all $u \in D_T$

$$(1) \qquad (Tu, Ku) \geq \alpha_1 \| u \|^2, \qquad \| Ku \|^2 \leq \alpha_2 (Tu, Ku).$$

It is known [16], [17] that T so defined has a bounded inverse T^{-1}, is K-symmetric[2], i.e.,

$$(2) \qquad (Tu, Kv) = (Ku, Tv), \qquad u, v \in D_T,$$

and is closeable; furthermore, if H_0 denotes the completion of D_T in the metric

$$(3) \qquad [u, v] = (Tu, Kv), \qquad | u | = [u, u]^{1/2},$$

then H_0 can be regarded as a subset of H, K can be extended to a bounded operator K_0 (as a mapping of all of H_0 into H) so that $K \subset K_0 \subset \bar{K}$, where $\bar{K}$ is the closure of K in H, and T has a closed K_0-p.d. and K_0-symmetric extension T_0 such that $T_0 \supset T$ and T_0 has a bounded inverse T_0^{-1} defined on the range $R_{T_0} = H$. Moreover, the inequality (1) remains valid for all $u \in H_0$ in the form

$$(4) \qquad | u |^2 \geq \alpha_1 \| u \|^2, \qquad \| K_0 u \|^2 \leq \alpha_2 | u |^2.$$

In [17] the author has extended the above results to unbounded linear non-K-p.d. operators by proving that if L is a linear operator defined on $D_L = D_T$ and such that for all u and v in D_L

$$(5) \qquad |(Lu, Ku)| \geq \eta_1 | u |^2, \qquad \eta_1 > 0$$

$$(6) \qquad |(Lu, Kv)| \leq \eta_2 | u | \, | v |, \qquad \eta_2 > 0$$

[2] For examples and theory of bounded and unbounded K-p. d. operators in H see Petryshyn [16], [17]. The results obtained below are also valid for real H provided T is also assumed to be K-symmetric.

then L has a solvable extension L_0 such that L_0 is closed, $L_0 \supset L$, L_0 has a bounded inverse L_0^{-1} defined on $R_{L_0} = H$, and L_0 has the representation $L_0 = T_0 W_0$, where W_0 is a certain extension of $T_0^{-1}L$ in H_0.

3. Extensions of nonlinear operators

In this section we extend the above construction of solvable extensions of linear non-K-p.d. operators to densely defined nonlinear operators. At the same time, we extend to the operators considered here some of the results involving the notions of monotone, demicontinous, locally bounded, and demiclosed nonlinear operators introduced and thoroughly studied by Browder, Minty, Zarantonello, and Kato.

Let P be a nonlinear operator transforming a dense domain $D_P \subset H$ into H and let T be a linear K-p.d. operator defined on $D_T = D_P$. In analogy to the concepts introduced by the above authors we say that P is *H_0-demicontinuous* if $\{u_n\} \subset D_P$, $u \in D_P$, and $u \to u$ strongly in H_0 imply $Pu_n \to Pu$ weakly in H; P is *H_0-locally bounded* if Pu_n is bounded in H whenever $\{u_n\} \subset D_P$ is a Cauchy sequence in H_0; P is *H_0-demiclosed* if $\{u_n\} \subset D_P$, $u_n \to u$ strongly in H_0, and $Pu_n \to g$ weakly in H imply $u \in D_P$ and $Pu = g$; P is *strongly H_0-monotonic* on D_P if for all u and v in D_P

$$\text{Re}\,(Pu - Pv, K(u - v)) \geqq \gamma \mid u - v \mid^2, \qquad \gamma > 0. \tag{7}$$

Evidently, if $K = T = I$ on D_P, then our definitions are identical with those considered in [1]–[6], [13], [9], [23].

THEOREM 1. *Let T be K-p.d. and P be a nonlinear mapping of $D_P = D_T$ into H. If for some positive constant $\eta > 0$*

$$|(Pu - Pv, K(u - v))| \geq \eta \mid u - v \mid^2, \qquad u, v \in D_P \tag{8}$$

and

$$(Pu_n - Pu_m, K_0 h) \to 0 \quad (n, m \to \infty), \qquad h \in H_0, \tag{9}$$

whenever $\{u_n\} \subset D_P$ is a Cauchy sequence in H_0, then P has an extension P_0 such that $P_0 \supset P$, P_0 is a one-to-one mapping of D_{P_0} onto H, P_0 is given by

$$P_0 = T_0 W_0, \tag{10}$$

where W_0 is a certain extension of $T_0^{-1}P$ in H_0, and P_0 is H_0-demiclosed. Furthermore, P_0 is unique.

Proof. Let T_0 be the K_0-p.d. extension of T constructed by Theorem 1 in [17] and let W be an operator in H_0 with domain $D_W = D_P \subset H_0$ and range $R_W \subset H_0$ defined by $W \equiv T_0^{-1}P$. Note that, in view of (9), (3), and the definition of W,

$$[Wu_n - Wu_m, h] \to 0 \quad (n, m \to \infty), \qquad h \in H_0, \tag{9_0}$$

whenever $\mid u_n - u_m \mid \to 0$ $(n, m \to \infty)$ with $u_n \in D_P$, i.e., W maps every strongly Cauchy sequence $\{u_n\}$ $(\subset D_P)$ in H_0 into a weakly Cauchy sequence $\{Wu_n\}$ in H_0.

Let us now extend W by weak closure to $\hat{W}$ mapping H_0 into H_0 as follows: if $u \in D_W$, then we put $\hat{W}u = Wu$; if $u \in \bar{D}_W = H_0$, then there is a sequence $\{u_n\}$ in D_W such that $u_n \to u$ strongly in H_0 and, consequently, $\{Wu_n\}$ is a weakly convergent sequence in H_0. Since H_0 is weakly complete, there is a unique element u^* in H_0 such that $u^* = \text{weak} \lim_n Wu_n$. Note that any two sequences $\{u_n'\}$ and $\{u_n''\}$ in D_P with the same limit u in H_0 must have weak $\lim_n Wu_n' =$ weak $\lim_n Wu_n''$ since otherwise the sequence of Wu's would have no limit. Thus, u^* depends only on u. We may therefore take $\hat{W}u = u^* = \text{weak} \lim_n Wu_n$. (No contradiction with the previous definition of $\hat{W}$ on D_W is possible for, if $u \in D_W$, we may take $u_n = u$ for each n.)

Thus it follows from the construction of $\hat{W}$ that it is a demicontinuous mapping of H_0 into H_0. Furthermore, $\hat{W}$ is such that for all u and v in H_0.

$$|[\hat{W}u - \hat{W}v, u - v]| \geq \eta \, | u - v |^2. \tag{11}$$

To see this, let u and v be any elements in H_0 and $\{u_n\}$ and $\{v_n\}$ be sequences in D_W so that $| u_n - u | \to 0$ and $| v_n - v | \to 0$, as $n \to \infty$. Then, by demicontinuity of $\hat{W}$ in H_0, $\{Wu_n - Wv_n\} \to \hat{W}u - \hat{W}v$ weakly in H_0. Hence, the passage to the limit in the inequality

$$|[\hat{W}u_n - \hat{W}v_n, u_n - v_n]| \geq \eta \, | u_n - v_n |^2$$

(which, in view of (8), is valid for all elements in D_W) yields the validity of (11) for all u and v in H_0.

Since $\hat{W}$ is a demicontinuous mapping of H_0 into H_0 satisfying the inequality (11), Browder's Theorem [4] implies that $\hat{W}$ maps H_0 onto H_0 and has a continuous inverse defined on $H_0 = R_{\hat{W}_0}$.

Thus, we may consider a mapping W_0 in H_0 such that $W \subset W_0 \subset \hat{W}$ with $R_{W_0} = D_{T_0}$. If we now define P_0 on $D_{P_0} = D_{W_0}$ by $P_0 \equiv T_0 W_0$, then it is easy to see that $P_0 \supset P$ and that P_0 is a one-to-one mapping of D_{P_0} onto H. Indeed, for $u \in D_P$ we have $W_0 u = Wu = T_0^{-1}Pu$ and, hence, $P_0 u = T_0 W_o u = Pu$, i.e., $P_0 \supset P$; furthermore, since $R_{W_0} = D_{T_0}$ and T_0 maps R_{W_0} onto H, P_0 maps D_{P_0} onto H; finally if $P_0 u = f$ and $P_0 v = f$, then the definition of P_0 and (11) imply that

$$0 = |(P_0 u - P_0 v, K_0(u - v))| = |[W_0 u - W_0 v, u - v]| \geq \eta \, | u - v |^2$$

from which we derive the equality $u = v$.

To prove the other assertion of Theorem 1 note that if $\{u_n\} \subset D_P$ with $u_n \to u_0$ strongly in H_0 and $P_0 u_n \to f$ weakly in H, then by demicontinuity of $\hat{W}$ in H_0, the continuity of T_0^{-1} in H, and the structure of P_0 we find that

$$\hat{W}u_n \to \hat{W}u_0$$

weakly in H_0 and

$$T_0^{-1}P_0 u_n = W_0 u_n \to T_0^{-1}f$$

weakly in H, i.e., $[\hat{W}u_n, h] \to [\hat{W}u_0, h]$ for every h in H_0 and $(P_0 u_n, z) \to (f, z)$ for every z in H and, in particular, for every $z = Kh$ with $h \in D_P$. Since $\hat{W} = W_0 = W$ on D_P and $P_0 = T_0 W_0$ we find that $[\hat{W}u_0, h] = [T_0^{-1}f, h]$ for every $h \in D_P$. Since D_P is dense in H_0, $\hat{W}u_0 = T_0^{-1}f$. Hence $\hat{W}u_0 \in D_{T_0}$, i.e.,

$$u_0 \in D_{W_0} = D_{P_0} \quad \text{and} \quad P_0 u_0 = T_0 W_0 u_0 = f;$$

hence, P_0 is H_0-demiclosed.

Finally, to prove the uniqueness of P_0 note first that $(P_0 u, K_0 v)$ is continuous in u on H_0 for each fixed v in H_0. This follows from the demicontinuity of $\hat{W}$ and the equation $(P_0 u, K_0 v) = [\hat{W}u, v]$ which holds for each u in D_P and v in H_0. Since the latter equation would be valid for any P_0 satisfying the conditions of our Theorem 1, it is easy to verify that these conditions determine P_0 uniquely.

Corollary 1. *If T is K-p.d. and $P = T + S$ is such that $D_S \supseteq D_T$,*

$$(8_1) \qquad |(Pu - Pv, K(u - v))| \geq \eta_1 |u - v|^2, \qquad \eta_1 > 0, \quad u, v \in D_T$$

and

$$(9_1) \qquad (Su_n - Su_m, K_0 h) \to 0 \quad (n, m \to \infty), \qquad h \in H_0,$$

whenever $\{u_n\}$, $u_n \in D_P$, is a Cauchy sequence in H_0, then

$$(10_1) \qquad P_0 = T_0(I + R_0),$$

where R_0 is a certain extension of $R \equiv T_0^{-1}S$ in H_0.

Proof. The conditions (8_1) and (9_1) imply that $P = T + S$ satisfies (8) and (9) with $\eta = \eta_1$. Hence, by Theorem 1, P has a solvable extension $P_0 = T_0 W_0$, where $W_0 \supset W = T_0^{-1}P$ is the restriction of $\hat{W}$ such that $R_{W_0} = D_{T_0}$. Since

$$W = T_0^{-1}(T + S) = T_0^{-1}T + T^{-1}S = I + R$$

on D_T and, by (9_1), the operator $R = T_0^{-1}S$ (defined on $D_T \subset H_0$) has the demicontinuous extension $\hat{R} = \hat{W} - I$ with $R_0 = I - W_0$. This implies the validity of (10_1).

In applications, as for example in elastico-plasticity, it often happens that instead of (9) it is easier to verify a stronger condition for which the assertions of Theorem 1 remain valid. In fact, the following corollary is an immediate consequence of Theorem 1.

Corollary 2. *Let T be K-p.d. and P be a nonlinear mapping of $D_P = D_T$ into H such that*

$$(8_2) \quad |(Pu - Pv, K(u - v))| \geq \eta |u - v|^2, \qquad \eta > 0,\ u, v \in D_P,$$

$$(9_2) \qquad |(Pu - Pv, K_0 h)| \leqq \theta |u - v| |h|, \quad \theta > 0,\ u, v \in D_P,\ h \in D_{T_0};$$

then P has an H_0-demiclosed extension P_0 such that $P_0 \supset P$, P_0 is a one-to-one mapping of D_{P_0} onto H, and

$$(10_2) \qquad P_0 = T_0 W_0 ,$$

where W_0 is a certain extension of $W = T_0^{-1}P$ in H_0.

Remark 1. Let us remark that in view of our stronger condition (9_2) the operator $W = T_0^{-1}P$ satisfies actually the Lipschitzian condition on the subset D_T of H_0. Indeed, if u and v are arbitrary elements of D_T and $h = Wu - Wv$, then by (9_2)

$$| h |^2 = [Wu - Wv, h] = (Pu - Pv, Kh) \leq \theta | u - v | | h |$$

and, consequently, W satisfies the Lipschitz condition

$$| Wu - Wv | \leq \theta | u - v |.$$

Hence there exists a unique Lipschitzian extension $\tilde{W}$ of W to all of H_0 such that $\tilde{W}u = Wu$ for $u \in D_T$ and

$$| \tilde{W}u - \tilde{W}v | \leq \theta | u - v | \quad \text{and} \quad |[\tilde{W}u - \tilde{W}v, u - v]| \geq \eta | u - v |^2$$

for all $u, v \in H_0$. In this case we can apply the result of Zarantonello [23] to show that $\tilde{W}$ maps H_0 onto H_0 and thus use the mapping $\tilde{W}$ in our construction of P_0. This we will do in the next two corollaries.

Let us also remark that in this case it is not necessary for the restrictive condition (9_2) to hold for all $h \in D_{T_0}$. Indeed, it follows from the proof of the Lipschitzian property of W that it is sufficient for (9_2) to hold only for all $h \in D_{T_0}$ of the form $h = T_0^{-1}(Wu - Wv)$ with $u, v \in D_{T_0}$.

The following two corollaries determine the useful conditions under which $D_{P_0} = D_{T_0}$.

COROLLARY 3. *If T is K-p.d. and $P = T + S$ is such that*

$$(8_3) \qquad |(Pu - Pv, K(u - v))| \geq \eta_1 | u - v |^2, \quad \eta_1 > 0, u, v \in D_P$$

$$(9_3) \qquad \| Su - Sv \| \leq \theta_1 | u - v |, \quad \theta_1 > 0, u, v \in D_P$$

then $D_{P_0} = D_{T_0}$ and

$$(10_3) \qquad P_0 = T_0 + S_0 ,$$

where S_0 is an extension of S in H_0.

Proof. It is easy to prove that, in view of (9_3), $P = T + S$ satisfies also the condition (9_2) with $\theta = 1 + \theta_1\sqrt{\alpha_2}$. Hence $P_0 = T_0(I + N_0)$, where N_0 is the restriction of $\tilde{N} = (T_0^{-1}S)^{\sim} = T_0^{-1}\tilde{S}$ with $\tilde{S}$ being extension of S to H_0 (which, in view of (9_3), certainly exists). Now, $\tilde{W}u \in D_{T_0}$ if and only if $u \in D_{T_0}$. This follows from the fact that $\tilde{W} = I + T_0^{-1}\tilde{S}$ and $T_0^{-1}\tilde{S}u \in D_{T_0}$ for all u in H_0. Thus, $D_{W_0} = D_{T_0}$; hence $D_{P_0} = D_{T_0}$ and $P_0 = T_0 + S_0$, where we have put $S_0 = T_0 N_0$.

COROLLARY 4. *Let T be K-p.d. and K be closed in $D_K = D_T$. If P satisfies the conditions of Corollary* 2 (*or even the weaker conditions of Theorem* 1), *then $P_0 = P$, i.e., P is a one-to-one mapping of D_P onto H.*

Proof. In view of our additional hypothesis on K, Theorem 2 in [17] implies that $T_0 = T$, $K_0 = K$, and $H_0 = D_T$. Hence $\tilde{W} = W_0 = W$ (or $\hat{W} = W_0 = W$) and, by Corollary 2 (or by Theorem 1), $P_0 = P$.

Remark 2. If $P = L$, where L is a linear mapping of D_L ($=D_T$) into H, then the conditions and the assertions of Corollaries 2, 3, and 4 reduce to the corresponding conditions and assertions of Theorem 3, Corollary 4, and Theorem 4 in [17], respectively. The assertion of Corollary 1 with the stronger condition $|(Su - Sv, K_0 h)| \leq \theta_2 \, | u - v | \, | h |$ reduces to Corollary 3 in [17].

The following theorem and corollary establish a two-way connection between the range and the H_0-demicontinuity of an H_0-locally bounded operator satisfying the condition (8).

THEOREM 2. *Let T be K-p.d., K be closed with $D_K = D_T$, and P satisfy the inequality* (8). *If there is a constant $M > 0$ such that for every Cauchy sequence $\{u_n\}$ in H_0 and every $h \in H_0$*

$$|(Pu_n, K_0 h)| \leq M \, | h |, \tag{12}$$

then P maps D_P onto H if and only if P is H_0-demicontinuus.

Proof. (*Necessity*). Let us first note that, in view of our conditions on K, Theorem 2 in [17] implies that $T_0 = T$, $K_0 = K$, and $H_0 = D_T$. Let W be the operator in H_0 defined by $W \equiv T^{-1}P$.

If we assume that P maps D_P ($=H_0$) onto H, then W maps H_0 onto H_0 since T^{-1} maps H onto H_0. Let $\{u_n\}$ be a Cauchy sequence in H_0. Since H_0 is complete, there is $u_0 \in H_0$ such that $u_n \to u_0$ strongly in H_0 and, in view of (12), $|[Wu_n, h]| \leq M \, | h |$ for every $h \in H_0$. Hence $\{Wu_n\}$ is itself a bounded sequence in H_0. Since W maps every Cauchy sequence $\{u_n\}$ in H_0 into a bounded sequence $\{Wu_n\}$ in H_0 and the latter is weakly precompact in H_0, it suffices to show that there is a subsequence of $\{Wu_n\}$ converging weakly to Wu_0 in H_0. Now, let $\{Wu_{n_k}\}$ be a subsequence of $\{Wu_n\}$ which converges weakly in H_0 to some element, say p, in H_0. Hence, in view of (8) and the fact that $D_P = H_0$, for every v in H_0 we have the inequality

$$|[Wu_{n_k} - Wv, u_{n_k} - v]| \geq \eta \, | u_{n_k} - v |^2. \tag{13}$$

Passing to the lumit in (13) as $n_k \to \infty$ we get the inequality

$$|[p - Wv, u_0 - v]| \geq \eta \, | u_0 - v |^2 \tag{13_0}$$

valid for each v in H_0. Applying the Schwarz inequality to (13_0) we get

$$\eta \, | u_0 - v |^2 \leq | p - Wv | \, | u_0 - v |.$$

This shows that for each v in H_0 we have the inequality $\eta \mid u_0 - v \mid \leq \mid p - Wv \mid$. Since $R_W = H_0$, there exists a $y \in D_W = H_0$ such that $p = Wy$ and $\eta \mid u_0 - v \mid \leq \mid Wy - Wv \mid$ for each $v \in H_0$. If we take $v = y$, then the last inequality implies that $u_0 = y$ and $p = Wu_0$. Thus $Wu_n \to Wu_0$ weakly in H_0, whenever $u_n \to u_0$ strongly in H_0. This and the definition of W and (3) imply that $Pu_n \to Pu_0$ weakly in H, i.e., P is H_0-demicontinuous.

(*Sufficiency*]. Suppose P is H_0-demicontinuous. Then for every $z \in H$, $(Pu_n, z) \to (Pu_0, z)$. Since K has a bounded inverse defined on all of H, for every $z \in H$ there is a unique $h \in D_K = H_0$ such that $z = Kh$. Defining W by $W \equiv T^{-1}P$ we find that W maps H_0 into H_0 and that

$$[Wu_n, h] = (Pu_n, z) \to (Pu_0, z) = [Wu_0, h]$$

for every h in H_0 whenever $u_n \to u_0$ strongly in H_0. Hence W is a demicontinuous mapping of H_0 into H_0 such that

$$|[Wu - Wv, u - v]| \geq \eta \mid u - v \mid^2$$

for all u and v in H_0. Thus, by Browder's Theorem [4], W maps H_0 onto $H_0 = D_T$. Since T maps D_T onto H, this implies that $TW = P$ maps D_P onto H and completes the proof of Theorem 2.

COROLLARY 5. *If P is a locally bounded mapping of H into H such that*

$$(14) \qquad |(Pu - Pv, u - v)| \geq c \, \| u - v \|^2, \qquad u, v \in H,$$

then P is onto H if and only if P is demicontinuous.

Proof. Corollary 5 is a special case of Theorem 2 if in it we take $T = K = I$.

Strongly H_0-monotonic operators. Let us observe in passing that the condition (8) of Theorem 1 or (14) of Corollary 5 is obviously satisfied when the nonlinear operator P is strongly H_0-monotonic, i.e., if there is a constant $\gamma > 0$ such that

$$(15) \qquad \operatorname{Re}(Pu - Pv, K(u - v)) \geq \gamma \mid u - v \mid^2, \quad u, v \in D_P = D_T.$$

Sometimes, in applications, this is the condition which is easier to verify. Hence the theorems and corollaries proved above are valid for strongly H_0-monotonic operators with the corresponding additional conditions. Similarly, instead of (9_2), it is sufficient to assume a slightly weaker condition

$$(16) \quad \mid \operatorname{Re}(Pu - Pv, K_0(u - v)) \mid \leq \beta \mid u - v \mid \mid h \mid, \beta > 0, u, v \in D_P, h \in D_{T_0}$$

valid for all h in D_{T_0} of the form $h = T_0^{-1}(Pu - Pv)$ with $u, v \in D_P$.

Thus it appears to be useful to have some easily verifiable tests for the H_0-monotonicity of an operator. To this end the following lemma appears to be convenient (see also Minty [13]).

LEMMA 1. *If P has the property that for any $x, z \in D_P$ and real t there is a* *constant* $\gamma > 0$ *so that*

$$\left[\frac{d}{dt}\operatorname{Re}(Kh, P(z + th))\right]_{t=0} \geq \gamma |h|^2, \qquad h = x - z, \tag{17}$$

then P *is strongly* H_0*-monotonic on* D_P.

Proof. Let x and y be any elements in D_P and $u = x - y$; let $f(s)$ be the real-valued function defined for $0 \leq s \leq 1$ by $f(s) = \operatorname{Re}(Ku, P(y + su))$. In view of our conditions, it is not hard to see that $f(s)$ is differentiable on $(0, 1)$ and hence by the mean-value theorem there is a ξ such that

$$\begin{aligned} f(1) - f(0) &= \operatorname{Re}(K(x - y), Px - Py) = f'(\xi) \\ &= \left[\frac{d}{ds}\operatorname{Re}(Ku, P(y + su))\right]_{s=\xi}, \qquad 0 < \xi < 1. \end{aligned}$$

That is, letting $z = y + \xi u$, $t = \Delta s/(1 - \xi)$, and noting that $h = (1 - \xi)u = (1 - \xi)(x - y)$ we get

$$\begin{aligned} f'(\xi) &= \lim_{\Delta s \to 0} \frac{\operatorname{Re}(Ku, P(z + \Delta s u) - Pz)}{\Delta s} \\ &= \lim_{t \to 0} (1 - \xi)^{-2} \frac{\operatorname{Re}(Kh, P(z + th) - Pz)}{t} \\ &= (1 - \xi)^{-2}\left[\frac{d}{dt}\operatorname{Re}(Kh, P(z + th))\right]_{t=0}. \end{aligned} \tag{18}$$

On the other hand, since z and x belong to D_P and

$$h = x - z = (1 - \xi)(x - y),$$

(18) and our assumption (17) imply that

$$\operatorname{Re}(K(x - y), Px - Py) \geq \frac{\gamma}{(1 - \xi)^2} |h|^2 = \gamma |x - y|^2$$

for any x and y in D_P. This shows that P is strongly H_0-monotonic.

4. Iterative solution of strongly H_0-monotonic operator equations

Consider the problem of actually finding the solution of the equation

$$Pu = f, \qquad f \in H, \tag{19}$$

where P is a given strongly H_0-monotonic operator for which the inequality (16) is valid for all h in D_{T_0} of the form $h = T_0^{-1}(Pu - Pv)$ with $u, v \in D_P$. Evidently the operator P thus defined satisfies the conditions of Theorem 1. Hence the solvable extension P_0 exists and is given by (10).

In what follows we shall regard the solution $u^* \in D_{P_0}$ of the equation

$$P_0 u = f, \qquad f \in H, \tag{20}$$

as the *generalized solution* of (19). Theorem 1 above guarantees the existence and the uniqueness of generalized solutions of (19) but says nothing about their effective computation. Below we consider a simple iterative method for the approximate solution of equation (19) or (20) which for linear equations with unbounded operators was investigated by the author [17] and for operators satisfying other conditions a similar procedure was studied in [19], [21], [23].

In what follows in this section we shall assume for practical reasons that T is a simple and well-investigated K-p.d. linear mapping of $D_T(=D_P)$ into H so that the equation

$$Tu = g, \qquad g \in H, \tag{21}$$

is relatively easily solvable (at least for a certain set of elements $g \in H$). We additionally assume that $D_{P_0} = D_{T_0}$. It was shown above that this would be the case, for example, when the conditions of Corollary 3 or 4 are satisfied. In practical applications it is not absolutely necessary that this additional condition be satisfied for all we need is that for a given P and f a certain sequence of solutions of equation (21) belongs to D_{P_0}.

The iterative method for the solution of (19) or (20) is based on the following theorem.

THEOREM 3. *Let P be a strongly H_0-monotonic mapping of D_P into H which satisfies the condition* (16) *and let α be a real number such that*

$$0 < \alpha < 2\gamma/\beta^2. \tag{22}$$

(a) *If $u_0 \in D_P$ is an arbitrary initial approximation to the solution u^* of* (20), *then the sequence $\{u_{n+1}\}$ of iterants determined by the process*

$$T_0 u_{n+1} = T_0 u_n - \alpha(P_0 u_n - f), \qquad n = 0, 1, 2, \cdots, \tag{23}$$

converges monotonically in the H_0-metric to the solution of (20). *The error estimate is given by the formula*

$$| u_{n+1} - u^* | \le \frac{p^{n+1} \cdot \sqrt{\alpha_2}}{\gamma} \| P_0 u_0 - f \|, \tag{24}$$

where $p = p(\alpha)$ is a function of α given by

$$p(\alpha) = [1 - 2\gamma\alpha + \beta^2\alpha^2]^{1/2}. \tag{25}$$

(b) *If additionally we assume that K is closed and $D_K = D_T$, then*

$$\| P_0 u_n - f \| \to 0 \quad as \quad n \to \infty,$$

and the error estimate is given by the convenient formula

$$| u_{n+1} - u^* | \le \frac{\sqrt{\alpha_2}}{\gamma} \| P_0 u_{n+1} - f \|. \tag{26}$$

Proof. Let u^* be the solution of (20). Then

$$T_0 u^* = T_0 u^* - \alpha(P_0 u^* - f) \tag{27}$$

and, if $\{u_{n+1}\}$ is a sequence of iterants determined by (23), the subtraction of (27) from (23) yields the equality

$$T_0(u_{n+1} - u^*) = T_0(u_n - u^*) - \alpha(P_0 u_n - P_0 u^*). \tag{28}$$

Let e_n denote the error vector $e_n = u_n - u^*$. Then, in view of the K_0-symmetry of T_0, (28) yields the equality

$$\begin{aligned}(T_0 e_{n+1}, K_0 e_{n+1}) &= (T_0 e_n - \alpha(P_0 u_n - P_0 u^*),\\ &\qquad K_0 e_n - \alpha K_0 T_0^{-1}(P_0 u_n - P_0 u^*))\\ &= (T_0 e_n, K_0 e_n) - 2\alpha \operatorname{Re}(K_0 e_n, P_0 u_n - P_0 u^*)\\ &\qquad + \alpha^2(P_0 u_n - P_0 u^*, K_0 T_0^{-1}(P_0 u_n - P_0 u^*))\end{aligned}$$

which for $h = T_0^{-1}(P_0 u_n - P_0 u^*)$ gives the relation

$$\begin{aligned}| e_{n+1} |^2 = | e_n |^2 - 2\alpha \operatorname{Re}(K_0(u_n - u^*), P_0 u_n - P_0 u^*)\\ + \alpha^2(P_0 u_n - P_0 u^*, K_0 h)\end{aligned}$$

Hence, in virtue of our conditions (15) and (16) we get

$$| e_{n+1} |^2 \leq | e_n |^2 - 2\alpha\gamma | e_n |^2 + \alpha^2\beta^2 | e_n | = p^2 | e_n |^2.$$

Since $0 < p < 1$ for any fixed α satisfying the condition (22) and

$$| e_{n+1} | \leq p | e_n | \leq p^2 | e_{n-1} | \leq \cdots \leq p^{n+1} | e_0 | \tag{29}$$

we see that $| e_{n+1} | \to 0$, as $n \to \infty$, i.e., u_{n+1} converges to u^* in H_0.

It is seen from (29) that to obtain the estimate (24) it is only necessary to estimate $| e_0 | = | u_0 - u^* |$. Using (4) and (15) we derive the inequality

$$\begin{aligned}\gamma | u_0 - u^* |^2 &\leq \operatorname{Re}(Pu_0 - P_0 u^*, K(u_0 - u^*))\\ &\leq \| Pu_0 - f \| \, \| Ku_0 - u^* \| \leq \sqrt{\alpha_2} \|Pu_0 - f \| \, \| u_0 - u^*|\end{aligned}$$

from which (24) follows.

To prove Theorem 3(b) note that under the additional condition on K, Theorem 2 in [17] and Corollary 4 imply that $K_0 = K$, $T_0 = T$, $H_0 = D_T$, $R_T = H$, and $P_0 = P$ with $R_P = H$. Furthermore, there is a constant $\tilde{\theta} > 0$ such that for all $u \in D_T$

$$\tilde{\theta} \| Tu \| \leq | u | \leq \sqrt{\alpha_2} \| Tu \|. \tag{30}$$

Hence, by Theorem 3(a) and the relations (23) and (30),

$$\| Pu_n - f \| = \alpha^{-1} \| T(u_{n+1} - u_n)\| \to 0 \quad \text{as} \quad n \to \infty.$$

The error estimate (26) follows from (1) and (15) because

$$\gamma \mid u_{n+1} - u^* \mid^2 \leq \text{Re}\,(Pu_{n+1} - Pu^*, K(u_{n+1} - u^*))$$
$$\leq \sqrt{\alpha_2}\, \| Pu_{n+1} - f \| \mid u_{n+1} - u^* \mid.$$

Thus the proof of Theorem 3 is complete.

Remark 3. Theorem 3 allows us to replace the problem of solving nonlinear equation (19) by the problem of solving a sequence of simple linear equations (21) in such a way that the generalized solution of (19) is given as the limit of the H_0-convergent sequence $\{u_{n+1}\}$ determined by the process (23). Thus each iteration with the nonlinear equation requires the solution of a simple linear equation so that the solution of the linear equation lies in the domain of definition of the nonlinear operator. Furthermore, the usefulness of the scheme (23), when applied to the approximate solution of various types of nonlinear differential equations, consists in the fact that there is a great freedom in the choice of the linear operator T and that K need not be the same as T.

Sometimes it may be convenient first to calculate T_0^{-1} and then compute u_{n+1} from the scheme

$$(23_0) \qquad u_{n+1} = u_n - \alpha T_0^{-1}(P_0 u_n - f), \qquad n = 0, 1, 2, \cdots.$$

In case P is of the form $P = T + S$, where S satisfies the conditions of Corollary 3 or Remark 2, the scheme (23) becomes

$$(23_1) \qquad Tu_{n+1} = (1 - \alpha)Tu_n - \alpha(Su_n - f), \quad n = 0, 1, 2, \cdots.$$

Finally, let us remark that the best value of α, i.e., the value of α for which $p(\alpha)$ assumes its least value, is

$$(31) \qquad \tilde{\alpha} = \gamma/\beta^2$$

for which

$$(31)^{\sim} \qquad p(\tilde{\alpha}) = 1 - \gamma^2/\beta^2.$$

In this case the error estimate (24) is given by a convenient formula

$$(24)^{\sim} \qquad | u_{n+1} - u^* | \leq \frac{\sqrt{\alpha_2}}{\gamma}\left(1 - \frac{\gamma^2}{\beta^2}\right)^{n+1} \| Pu_0 - f \|.$$

5. The projection method in the solution of nonlinear equations

It is known [18], [16], [17], that among the procedures in the solution of linear equations the projection method plays an important role in the family of direct methods such as the method of Ritz, Galerkin, least squares, moments, Murray, etc., not only because of its geometrical basis and unifying property but also because it extends their applicability to a larger class of linear equations.

In this section we discuss the applicability of the projection method to the solution of

$$Pu = f, \qquad f \in H, \tag{32}$$

where the nonlinear operator P satisfies the conditions of Theorem 1. It will be shown that one of the practical realizations of the projection method is the generalized method of moments or the Galerkin method which for some special operators P is formally identical with the method of Galerkin and Ritz discussed by a number of authors [10], [7], [15], [11], [19]. Our investigation of the projection method is based on our results in Section 3 and Browder's Lemma 3 in [2] which we generalize to the class of operators satisfying the conditions of Theorem 1.

Let us first note that, in view of Theorem 1, solving equation (32) is equivalent to solving equation

$$W_0 u = f_0 , \tag{33}$$

where $f_0 = T_0^{-1} f$ and W_0 is the demicontinuous extension of $W = T_0^{-1} P$ in H_0 so that $W \subset W_0 \subset \hat{W}$ with $D_{W_0} = D_{P_0}$ and $R_{W_0} = D_{T_0}$. If $\{H_n\} \subset D_P$ is a sequence of finite-dimensional subspaces of H_0 which is projectionally complete in H_0 (i.e., $\{H_n\}$ is such that $| g - \Pi_n g | \to 0$ $(n \to \infty)$ for every $g \in H_0$, where Π_n denotes the orthogonal projection of H_0 onto H_n), then according to to the projection method the approximate solution u_n $(\in H_n)$ of (32) or (33) is determined by the condition

$$\Pi_n W_0 u_n = \Pi_n f_0 . \tag{34}$$

It seems at first that the practical realization of the method (34) is very difficult if not impossible for, in its form (34), it requires the advance knowledge of W_0 and T_0^{-1}. However, if we choose a sequence $\{\varphi_i\}$, $\varphi_i \in D_P$, of linearly independent elements which is complete in H_0 and which for the sake of simplicity we assume to be orthonormal in H_0, then taking H_n as the span of $\{\varphi_1, \cdots, \varphi_n\}$ we see that $\{H_n\}$ so determined is projectionally complete in H_0, every solution $u_n \in H_n$ of equation (34) is of the form

$$u_n = \textstyle\sum_{i=1}^n a_i \varphi_i , \tag{35}$$

and the equation (34) can be written in the form

$$\textstyle\sum_{i=1}^n [W_0 u_n , \varphi_i]\varphi_i = \sum_{i=1}^n [f_0 , \varphi_i]\varphi_i . \tag{36}$$

Since H_n is a subset of D_P and $\{\varphi_i\}$ is linearly independent, Theorem 1 implies that, in view of (36), equation (34) is equivalent to the algebraic system of nonlinear equations

$$(Pu_n , K\varphi_j) = (f, K\varphi_j), \qquad 1 \leq j \leq n. \tag{37}$$

We summarize the above discussion in the following lemma.

LEMMA 2. *An element* u_n (ϵH_n) *given by* (35) *is a solution of equation* (34) *if and only if* $\{a_1, \cdots, a_n\}$ *satisfies the algebraic system* (37).

Using essentially the arguments of Browder [2] we now prove the following lemma which we will utilize in the proof of Theorem 4 below.

LEMMA 3. *Let* P *be a nonlinear operator satisfying the conditions of Theorem* 1. *If* $\{H_n\} \subset D_P$ *is a sequence of finite-dimensional subspaces which is projectionally complete in* H_0 *and* $\{u_n\}$ *is a sequence in* D_P *such that* $u_n \epsilon H_n$, $u_n \rightarrow u_0$ *weakly in* H_0 *and* $\Pi_n W_0 u_n \rightarrow g_0$ *strongly in* H_0 *with* $g_0 \epsilon D_{T_0}$, *then* $u^* \epsilon D_{W_0}$ *and* $W_0 u_0 = g_0$.

Proof. Let j be a fixed integer and u be any element in H_j. Since $\Pi_j u = u$, $\Pi_n \Pi_j = \Pi_j$ for $n > j$, $\Pi_n u_n = u_n$ for $u_n \epsilon H_n$, $u_n \rightarrow u_0$ weakly in H_0, and $\Pi_n W_0 u_n \rightarrow g_0$ strongly in H_0 we have the equality

$$[u_n - \Pi_j u, W_0 u_n - W_0(\Pi_j u)] = [u_n - \Pi_j u, \Pi_n W_0 u_n] - [u_n - \Pi_j u, W(\Pi_j u)]$$

from which, on passage to the limit as $n \rightarrow \infty$, we obtain for all $u \epsilon H_j$ the relation

$$(40) \quad \begin{aligned} &[u_n - \Pi_j u, W_0 u_n - W_0(\Pi_j u)] \\ &\rightarrow [u_0 - \Pi_j u, g_0] - [u_0 - \Pi_j u, W(\Pi_j u)] = [u_0 - \Pi_j u, g_0 - W_0(\Pi_j u)]. \end{aligned}$$

Since, by (8) and (10), for each n we have

$$(41) \qquad |[u_n - \Pi_j u, W_0 u_n - W_0(\Pi_j u)]| \geq \eta \mid u_n - \Pi_j u \mid^2$$

and the sequence $\{u_n - \Pi_j u\}$, which converges weakly to $\{u_0 - \Pi_j u\}$, has the property that $\mid u_0 - \Pi_j u \mid \leq \liminf \mid u_n - \Pi_j u \mid$ we derive from this and the relations (40) and (41) the inequality

$$(42) \qquad \eta \mid u_0 - u \mid^2 \leq |[u_0 - u, g_0 - W_0 u]|$$

valid for all $u = \Pi_j u \epsilon H_j$. Since j is arbitrary, (42) is true for all u in a dense set $\bigcup_j H_j \subset H_0$. This and the demicontinuity of W_0 implies that (42) also holds for all $u \epsilon D_{W_0}$. Thus, applying the Schwarz inequality to (42) we get

$$(43) \qquad \eta \mid u_0 - u \mid \leq \mid g_0 - W_0 u \mid.$$

As $g_0 \epsilon D_{T_0} = R_{W_0}$, there exists a unique $v \epsilon D_{W_0}$ such that $g_0 = W_0 v$ and, in virtue of (43), $\eta \mid u_0 - u \mid \leq \mid W_0 v - W_0 u \mid$ for all $u \epsilon D_{W_0}$. If we take $u = v$, the last inequality implies that $u_0 = v$ and, consequently, $u_0 \epsilon D_{W_0}$ and $g_0 = W u_0$.

Remark 4. Before we state and prove Theorem 4 which justifies the applicability of the projection method or the generalized moments method to the solution of (32), let us first note that there is no loss in generality in assuming that $P(0) = 0$. Indeed, if $P(0) \neq 0$, then instead of (32) it is only necessary to solve the equivalent equation $Qu = g$, where $Qu \equiv Pu - P(0)$ and $g = f - P(0)$ with the operator Q satisfying all the conditions of Theorem 1 including the condition $Q(0) = 0$; furthermore, in this case the equations (37) and $(Qu_n, K\varphi_j) = (g, K\varphi_j), 1 \leq j \leq n$, are essentially the same. Thus, indeed we can and will assume in what follows that $P(0) = 0$.

THEOREM 4. *if T is K-p.d., P satisfies the conditions of Theorem* 1, *and $\{H_n\} \subset D_P$ is a projectionally complete sequence of finite-dimensional subspaces in H_0 which is determined by $\{\varphi_1, \cdots, \varphi_n\}$ for $n = 1, 2, 3, \cdots$, then*

(a) *For each $f \in H$, equation* (32) *has a unique* (*possibly generalized*) *solution u^* such that $P_0 u^* = f$.*

(b) *For each $f \in H$, the approximate equation* (34) (*or the system* (37)) *has a unique solution $u_n \in H_n$ given by* (35).

(c) *The sequence $\{u_n\}$ determined by equation* (34) *converges weakly in H_0 to the solution u^* of* (32).

(d) *If additionally we assume that $\{W_0 u_n\}$ is bounded in H_0 whenever $\{u_n\}$ is bounded in H_0, then the sequence $\{u_n\}$ converges in H_0 also strongly to u^*.*

(e) *If instead of the additional condition in* (d) *we assume that P satisfies the stronger conditions of Corollary* 2 *and that K is closed with $D_K = D_T$, then $u_n \to u^*$ strongly in H_0, $Pu_n \to f$ strongly in H, and the following simple error estimate is valid*

$$|u_n - u^*| \leq \frac{\sqrt{\alpha_2}}{\eta} \| Pu_n - f \|. \tag{44}$$

Proof. (a) The validity of assertion (a) follows from Theorem 1 according to which to each $f \in H$ there exists a unique generalized solution u^* of equation (32) such that $P_0 u^* = f$.

(b) To prove (b) let W_n be the mapping of H_n into H_n given by $W_n x = \Pi_n W_0 x$ for each $x \in H_n$. Hence, for x and y in H_n, we have

$$|[W_n x - W_n y, x - y]| = |[W_0 x - W_0 y, \Pi_n(x - y)]|$$
$$= |[W_0 x - W_0 y, x - y]| \geq \eta \, | x - y |^2;$$

furthermore, W_n is a demicontinuous mapping of H_n into H_n (in fact, since H_n is finite-dimensional, W_n is continuous). Thus, by Corollary 4, W_n is a one-to-one mapping of H_n onto H_n, i.e., there is a unique solution $u_n \in H_n$ such that equation (34) is satisfied or, in view of Lemma 2, the system (37) is uniquely solvable for $\{a_1, \cdots, a_n\}$.

(c) Taking the absolute value of the H_0-inner product of the equation (34) with u_n and using the condition that $P(0) = 0$ and the inequality (11) we get

$$\eta \, | u_n |^2 = \eta \, | \Pi_n u_n |^2 \leq |[\Pi_n W_0 u_n, u_n]| = |[\Pi_n f_0, u_n]| = |[f_0, u_n]| \leq | f_0 | \, | u_n.$$

Hence for all n, $| u_n | \leq | f_0 |/\eta$. Thus we may choose a weakly convergent subsequence of $\{u_n\}$ which we can assume to be the original sequence itself.

Consequently, u_n converges weakly to some element u_0 in H_0 and $\Pi_n W_0 u_n$, being equal to $\Pi_n f_0$, converges strongly in H_0 to $f_0 \in D_{T_0}$. Hence, by Lemma 3, $u_0 \in D_{W_0}$ and $W_0 u_0 = f_0$. Finally, Theorem 1 implies that $P_0 u_0 = f$, i.e., u_0 is a solution (possibly generalized) of (32). Since, for a given $f \in H$, the solution is unique, we must have $u_0 = u^*$.

(d) Since $\Pi_n u_n = u_n$ and u_n satisfies the equation (34) we have

$$
\begin{aligned}
[W_0 u_n - W_0 u^*, u_n - u^*] &= [W_0 u_n - f_0, u_n - u^*] \\
&= [W_0 u_n, u_n] - [f_0, u_n] - [W_0 u_n, u^*] \\
&\quad + [f_0, u^*] \\
&= [\Pi_n f_0, u_n] - [f_0, u_n] - [W_0 u_n, u^*] \\
&\quad + [f_0, u^*] \\
&= [f_0, u^*] - [W_0 u_n, u^*].
\end{aligned}
\tag{45}
$$

Since, by additional condition, W_0 maps bounded sets in H_0 into bounded sets in H_0, $|u_n| \leq (|f_0|/\eta)$, $\{H_n\}$ is projectionally complete in H_0, and $\Pi_n W_0 u_n$ converges strongly to f_0, we see that

$$
[W_0 u_n, u^*] = [W_0 u_n, \Pi_n u^*] + [W_0 u_n, (I - \Pi_n)u^*] \to [f_0, u^*] \tag{45_0}
$$

as $n \to \infty$. Consequently, the relations (45) and (45_0) and the inequality

$$
\eta \,|\, u_n - u^* \,|^2 \leq |[W_0 u_n - W_0 u^*, u_n - u^*]|
$$

imply that $|u_n - u^*| \to 0$, as $n \to \infty$.

(e) In view of our stronger conditions, Remark 1 implies that W_0 is Lipschitzian and, consequently, maps bounded sets into bounded sets of H_0. Thus, by (d), $u_n \to u^*$ strongly in H_0 and $W_0 u_n \to f_0 = W_0 u^*$ strongly in H_0. This, the structure (10_2) of P_0, and the inequality (30) which is valid under present conditions, imply that $P_0 u_n \to f$ strongly in H. The error estimate (44) follows from (8) and (30). This completes the proof of Theorem 4.

Remark 5. Let us observe that if we choose K to be $K = I$, then the projection method is practically realized by the Galerkin method while if $K = T$, then it is realized by the ordinary method of moments. Thus Theorem 4 establishes also the applicability of these methods to the approximate solution of equation (32).

6. Applications to elliptic nonlinear equations

As an application of Corollary 2 and Theorem 3 we consider the Dirichlet boundary value problem for an elliptic nonlinear partial differential equation of second order. Let us add that some of the problems in elastico-plasticity [11], [12] are described by differential equations of the type considered below.

Let Q be a bounded region the n-space R^n with a smooth boundary Γ. Let L_2 be the Hilbert space of real-valued square-integrable functions $u(x)$, $x = (x_1, x_2, \cdots, x_n)$, defined on $\bar{Q} = Q + \Gamma$ with the inner product and norm

$$(46) \qquad (u, v) = \int_Q uv\, dx, \qquad \| u \| = \left(\int_Q u^2\, dx \right)^{1/2}.$$

Let $C_0^2(\bar{Q})$ denote the set of all $u(x) \in L_2$ which are twice continuously differentiable on $\bar{Q}$ and satisfy the boundary conditions

$$(47) \qquad u \,|_\Gamma = 0.$$

Let P be the nonlinear partial differential operator of second order defined for all $u \in D_P = C_0^2(\bar{Q})$ by the expression

$$(48) \qquad Pu = -\sum_{i=1}^{n} \frac{\partial a_i(x_j, p_j)}{\partial x_i} + b(x_j, u), \qquad p_j = \frac{\partial u}{\partial x_j},$$

such that the following conditions are satisfied:

(i) P is elliptic, i.e.,

$$\sum_{ik,=1}^{n} \frac{\partial a_i}{\partial p_k} \xi_i \xi_k \geq m \left(\sum_{i=1}^{n} \xi_i^2 \right), \qquad m > 0,$$

(ii) there exists three constants l, $C > 0$, $D > 0$ such that $| \partial a_i / \partial p_k | \leq C$, $| \partial b / \partial u | \leq D$, and $\partial b / \partial u$ is bounded below by l so that $\eta \equiv m + l/d > 0$ if $l < 0$ and $\eta \equiv m$ if $l \geq 0$, where $d > 0$ is a constant determined by the Friedrichs inequality

$$(49) \qquad \int_Q \sum_{i=1}^{n} \left(\frac{\partial h}{\partial x_i} \right)^2 dx \geq d \int_Q h^2\, dx, \qquad h \in C_0^1(\bar{Q}).$$

Our problem is to solve the boundary-value problem

$$(50) \qquad -\sum_{i=1}^{n} \frac{\partial a_i(x_j, p_j)}{\partial x_i} + b(x_j, u) = f(x_j), \qquad u \,|_\Gamma = 0,$$

where $f(x)$ is a given function in L_2, or equivalently, the equation

$$(51) \qquad Pu = f, \qquad f \in L_2.$$

If we chose the operators K and T to be such that $K = I$ and T is defined for all $u \in D_T = D_P = C_0^2(\bar{Q})$ by

$$(52) \qquad Tu = -\Delta u = -\textstyle\sum_{i=1}^{n} \partial^2 u / \partial x_i^2,$$

then, as is known [20], T is symmetric and positive definite on D_T, i.e.,

$$(53) \qquad (Tu, u) = (-\Delta u, u) = \int_Q \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i} \right)^2 dx \geq \tilde{\alpha} \| u \|^2, \qquad \tilde{\alpha} > 0.$$

Furthermore, the space H_0 obtained as a completion of $C_0^2(\bar{Q}) = D_T$ in the metric

$$(54) \qquad [u, v] = (Tu, v) = (-\Delta u, v), \qquad |u| = [u, u]^{1/2}$$

is equivalent to the space $\mathring{W}_2^1(Q) \subset W_2^1(Q) \subset L_2$ [20] and the operator T has a self-adjoint positive definite extension, which we shall also denote by T or by $-\Delta$, mapping its domain $\mathring{W}_2^2 = W_2^2 \cap \mathring{W}_2^1$ onto L_2.[3] Thus the problem

$$(55) \qquad Tu = -\Delta u = g$$

has a unique solution $u \in \mathring{W}_2^2$ for every $g \in L_2$.

Remark 6. It is important from the practical point of view to note that if the region $\bar{Q}$ is a unit sphere in R^n (or if $\bar{Q}$ admits a transformation into a unit sphere with a nonvanishing Jacobian), then whenever $g \in C^1(\bar{Q})$, (55) has in $\bar{Q}$ a twice continuously differentiable solution $u \in \mathring{W}_2^1$. Furthermore, if g is a polynomial, then the solution u of (55) is also a polynomial.

Let us now verify that under conditions (i) and (ii) the operator P defined by (48) satisfies the conditions of Corollary 2. Indeed, for every $h \in D_P$, we have

$$(Pu, h) = \sum_{i=1}^{n} \int_Q a_i(x_j, p_j) \frac{\partial h}{\partial x_i} dx + \int_Q b(x_j, u) h \, dx, \qquad u \in D_P.$$

Consequently, for any u and v in D_P with $g_j \equiv \partial v / \partial x_j$

$$(56) \qquad \begin{aligned} (Pu - Pv, u - v) = \sum_{i=i}^{n} \int_Q [a_i(x_j, p_j) - a_i(x_j, g_j)] \frac{\partial}{\partial x_i}(u - v) \, dx \\ + \int_Q [b(x_j, u) - b(x_j, v)](u - v) \, dx. \end{aligned}$$

In view of our conditions (i) and (ii), we derive from (56) the relations

$$(57) \quad (Pu - Pv, u - v) \geq \eta \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}(u - v) \right)^2 = \eta(T(u - v), u - v)$$

where $\eta = m + l/d > 0$ if $l < 0$ and $\eta = m$ if $l \geqq 0$, and

$$(58) \qquad |(Pu - Pv, h)| \leqq \beta(T(u - v), u - v)^{1/2}(Th, h)^{1/2}, \qquad h \in \mathring{W}_2^2,$$

where $\beta = \beta(C, D, d) > 0$. Thus, by Corollary 2, the operator P has a solvable extension P_0 so that the equation (50) has a unique (possibly generalized) solution $u^* \in D_{P_0} \subset \mathring{W}_2^1$ for every $f \in L_2$.

Furthermore, we can construct the solution u^* by the iterative method (23) as follows: when $u_0 \in D_P$ is an initial approximation to u^*, then the

---

[3] The class of $W_2^2(\bar{Q})$ consists of all functions u which are square-integrable over Q together with their first and second generalized derivatives while the class $\mathring{W}_2^2(\bar{Q})$ consists of functions $u \in W_2^2$ which satisfy the boundary conditions $u|_\Gamma = 0$.

successive approximations u_{n+1} are determined by the formula

$$u_{n+1} = u_n - \alpha z_n \qquad (n = 0, 1, 2, \cdots), \tag{59}$$

where $z_n = -\Delta^{-1}(Pu_n - f)$, i.e., z_n is obtained as the solution of the equation

$$\Delta z = f - Pu_n , \qquad z \mid_\Gamma = 0, \tag{60}$$

and α is any fixed real number satisfying the condition

$$0 < \alpha < 2\eta/\beta^2. \tag{61}$$

It should be noted that when $\bar{Q}$ is a unit sphere the iterative method (59)–(61) is particularly effective when the functions a_i, b, and f are polynomials since, as was observed in Remark 6, in that case all the iterants $\{u_{n+1}\}$ are also polynomials provided the initial approximation u_0 is taken to be a polynomial.

Remark 7. Similar results can be obtained for the differential equation of the type (50) if a_i are also functions of u and b is also a function of p, i.e., $a_i = a_i(x_i, u, p_j)$, $b = b(x_j, u, p_j)$.

References

1. F. E. Browder, *The solvability of nonlinear functional equations*, Duke Math. J., vol. 30 (1963), pp. 557–566.
2. ———, *Nonlinear elliptic boundary value problems*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 862–874.
3. ———, *Variational boundary value problems for quasi-linear elliptic equations of arbitrary order*, Proc. Nat. Acad. Sci., vol. 50 (1963), pp. 31–37.
4. ———, *Remarks on nonlinear functional equations*, Proc. Nat. Acad. Sci., vol. 51 (1964), pp. 985–989.
5. ———, *Continuity properties of monotone nonlinear operators in Banach spaces*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 691–692.
6. ———, *Remarks on nonlinear functional equations II, III*, Illinois J. Math., vol. 9 (1965), pp. 608–616; 617–622.
7. Hagen-Torn and S. G. Mikhlin, *On the solvability of the Ritz nonlinear System*, Dokl. Akad. Nauk SSSR, vol. 138 (1961), pp. 258–260.
8. R. I. Kachurovsky, *On some fixed point principles*, Uch. Zap. Moskov Reg. Pet. Inst., vol. 96 (1960), pp. 251–219.
9. T. Kato, *Demicontinuity, hemicontinuity, and monotonicity*, Bull. Amer. Math. Soc., vol. 70 (1964), pp. 548–550.
10. M. A. Krasnoselsky, *The convergence of Galerkin method for nonlinear equations*, Dokl. Akad. Nauk SSSR, vol. 73 (1950), pp. 1121–1124.
11. A. Langenbach, *Variationsmethoden in der nichtlineares Elastizitäts-und Plastizitätstheorie*, Wiss. Z. Humboldt-Univ. Berlin, Math.-naturwiss. Reihe, vol. 9 (1959–1960), pp. 145–164.
12. ———, *On the application of the method of least squares to nonlinear equations*, Dokl. Akad. Nauk SSSR., vol. 143 (1962), pp. 31–34.
13. G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. vol. 29 (1962), pp. 341–346.
14. ———, *Two theorems on nonlinear functional equations in Hilbert space*, Bull. Amer. Math. Soc., vol. 69 (1963), pp. 691–692.

15. S. G. MIKHLIN, *On the method of Ritz for nonlinear problems*, Dokl. Akad. Nauk SSSR., vol. 142 (1962), pp. 792–793.
16. W. V. PETRYSHYN, *Direct and iterative methods for the solution of linear operator equations in Hilbert space*, Trans. Amer. Math. Soc., vol. 105 (1962), pp. 136–175.
17. ———, *On a class of K-p.d. and non-K-p.d. operators and operator equations*, J. Math. Anal. Appl., vol. 10 (1965), pp. 1–24.
18. N. I. POLSKY, *Projection methods in Applied Mathematics*, Dokl. Akad. Nauk SSSR, vol. 143 (1962), pp. 787–790.
19. S. V. SIMEONOV, *On certain methods in the solution of nonlinear problems in the mechanics of deformed bodies*, Prikl. Mat. Mekh., vol. 28 (1964), pp. 418–429.
20. S. L. SOBOLEV, *Certain applications of functional analysis to mathematical physics*, Leningrad, State Publishing House, 1950.
21. M. M. VAINBERG, *On the convergence of the method of steepest descent*, Sibirski Mat. J., vol. 2 (1961), pp. 201–220.
22. M. M. VAINBERG AND R. I. KACHUROVSKY, *On the variational theory of nonlinear operators and equations*, Dokl. Akad. Nauk SSSR, vol. 129 (1959), pp. 1199–1202.
23. E. ZARANTONELLO, *The closure of the numerical range contains the spectrum*, Bull. Amer. Math. Soc., vol. 70 (1964), pp. 781–787.

THE UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS